231 Annual 2007: solution to question 8

8 Using the method of Fröbenius, or otherwise, find the general solution to
$$xy'' + 2y' - \alpha xy = 0$$
where $\alpha > 0$.

*Solution:*So, to use the method of Froebenius, we assume there is a solution of the form
$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1}$$
This gives
$$\begin{aligned} xy'' &= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} \\ 2y' &= \sum_{n=0}^{\infty} 2a_n(n+r)x^{n+r-1} \\ xy' &= \sum_{n=0}^{\infty} a_n x^{n+r+1} \end{aligned} \tag{2}$$
which means we need to shift the indices for the first two terms forward two steps by setting $n = m + 2$, after taking terms out of the sum to get the range of the index correct, and renaming m back to n, this gives
$$\begin{aligned} xy'' &= r(r-1)a_0x^{r-1} + (r+1)ra_1x^r \\ &\quad + \sum_{n=0}^{\infty} a_{n+2}(n+r+2)(n+r+1)x^{n+r+1} \\ 2y' &= 2ra_0x^{r-1} + 2(r+1)a_1x^r + \sum_{n=0}^{\infty} 2a_{n+2}(n+r+2)x^{n+r+1} \end{aligned} \tag{3}$$
Thus, the equation becomes
$$r(r+1)a_0x^{r-1}+(r+1)(r+2)a_1x^r+\sum_{n=0}^{\infty}[a_{n+2}(n+r+2)(n+r+3)-\alpha a_n]x^{n+r+1} = 0 \tag{4}$$
The indicial equation, which is a consequence of the assumption that $a_0 \not= 0$ is, therefore
$$r(r+1) = 0 \tag{5}$$

so $r = 0$ and $r = -1$ are the possilities, hence $r = 0$ or $r = -1$, giving general solution

$$y = C_1 y_1 + C_2 y_2 \tag{6}$$

where

$$y_1 = \sum_{n=0}^{\infty} a_n x^n \tag{7}$$

where

$$a_{n+2} = \frac{\alpha}{(n+2)(n+1)} \tag{8}$$

with $a_0 = 1$ and $a_1 = 0$ and, where where

$$y_2 = \frac{1}{x} \sum_{n=0}^{\infty} a_n x^n \tag{9}$$

where

$$a_{n+2} = \frac{\alpha}{(n+1)n} \tag{10}$$

with $a_0 = 1$ and $a_1 = 0$.