Outline solution to 231 annual exam 2007 q7

7 (a) Find the general solution to

$$y' - \frac{y}{x} = -x$$

(b) A Bernoulli equation is an equation of the form

$$\frac{dy}{dx} + p(x)y - q(x)y^n = 0$$

These equations are solved by substituting $z = y^{1-n}$. Find the general solution to the Bernoulli equation

$$y' + \frac{y}{x} = xy^2.$$

(c) Find the solution to

$$y'' + 2y' + y = 2e^{-x}$$

with $y(0) = 1$ and $y'(0) = 0$.

Solution: We solve this using an integrating factor, it is actually easy to guess this is

$$\left(\frac{y}{x}\right)' = -1 \tag{1}$$

so, integrating,

$$y = -x^2 + Cx \tag{2}$$

To work this out using an integrating factor

$$\lambda = e^{-\int p(x)dx} \tag{3}$$

we have

$$\lambda = e^{-\int (1/x)dx} = \frac{1}{x} \tag{4}$$

Now, for the second part we have

$$y' + \frac{1}{x}y - xy^2 = 0 \tag{5}$$

which is off the Bernoulli form, so we just do the substitution with $n = 2$: $z = y^{-1} = 1/y$

$$-\frac{1}{z^2}z' + \frac{1}{x}\frac{1}{z} - x\frac{1}{z^2} = 0 \tag{6}$$

and multiplying across by $-z^2$

$$z' - \frac{1}{x}z = -x \tag{7}$$

so, from the first part, this mean

$$z = -x^2 + Cx \tag{8}$$

and

$$y = \frac{1}{-x^2 + Cx} \tag{9}$$

For the final part we first of all find the auxillary equation

$$\lambda^2 + 2\lambda + 1 = 0 \tag{10}$$

with solution $\lambda = -1$ twice. Thus, the complementary function is

$$y_c = (C_1 + xC_2)e^{-x} \tag{11}$$

and to find the particular integral we substitute in

$$y_p = Cx^2e^{-x} \tag{12}$$

giving

$$C\left(x^2 - 4x + 2 + 4x - 2x^2 + x^2\right)e^{-x} = 2e^{-x} \tag{13}$$

so $C = 1$ and

$$y = \left(C_1 + C_2x + x^2\right)e^{-x} \tag{14}$$