Annual exam 2007: solution to question 5

5 (a) The Fourier series relies on the orthogonality of the sine and cosine functions: as an example, show

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$$

for integers n and m.

(b) Show that the Fourier series of the function $f(x)$ with period 2π defined by

$$f(t) = x^2 + x$$

for $-\pi < t < \pi$ and $f(x + 2\pi) = f(x)$ is

$$\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx - \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin nx$$

(c) What is

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Soln. The orthgonality of the sines and cosines can be demonstrate using trig idenities, or by changing to complex exponentials, here I will use the former method, with $n \neq m$ and n and m positive:

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin nx \cos mx dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin{(n+m)x} + \sin{(n-m)x}]dx \\
&= -\frac{1}{2(n+m)} \cos{(n+m)x}\Big]_{-\pi}^{\pi} \\
&\quad - \frac{1}{2(n-m)} \cos{(n-m)x}\Big]_{-\pi}^{\pi} \\
&= 0
\end{aligned} \tag{1}$$

Next, consider the function f;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 + x)dx = \frac{2}{3}\pi^2 \tag{2}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2 + x)\cos nx dx$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} \\
&= \frac{4}{n^2}\cos n\pi \\
&= \frac{4}{n^2}(-1)^n \qquad (3)
\end{aligned}
$$

where we have integrated by parts. Similarily

$$
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\sin nx dx = -\frac{2}{n}(-1)^n \qquad (4)
$$

Of course, it is easier to do this problem if you split it into the x part, which only has a sine series and the x^2 part which only has a cosine series. Finally, if $x = \pi$ we expect the righthand side to interpolate the discontinuity,

$$
\begin{aligned}
\lim_{x\to\pi-} &= \pi^2 + \pi \\
\lim_{x\to\pi+} &= \pi^2 - \pi \qquad (5)
\end{aligned}
$$

so

$$
\pi = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos n\pi = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \qquad (6)
$$

giving

$$
\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{6}\pi^2 \qquad (7)
$$

where the subtle point is to use the $\cos n\pi$ to get rid of the $(-1)^n$ and then to remember that a Fourier series interpolates across a discontinuity.