Annual exam 2007: solution to question 5

5 (a) The Fourier series relies on the orthogonality of the sine and cosine functions: as an example, show

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$$

for integers n and m.

(b) Show that the Fourier series of the function $f(x)$ with period 2π defined by

$$f(t) = x^2 + x$$

for $-\pi < t < \pi$ and $f(x + 2\pi) = f(x)$ is

$$\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx - \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin nx$$

(c) What is

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Soln. The orthgonality of the sines and cosines can be demonstrate using trig idenities, or by changing to complex exponentials, here I will use the former method, with $n \neq m$ and n and m positive:

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin nx \cos mx dx &= \frac{1}{2}\int_{-\pi}^{\pi} [\sin (n+m)x + \sin (n-m)x] dx \\
&= -\left.\frac{1}{2(n+m)} \cos (n+m)x\right]_{-\pi}^{\pi} \\
&\quad -\left.\frac{1}{2(n-m)} \cos (n-m)x\right]_{-\pi}^{\pi} \\
&= 0 \qquad (1)
\end{aligned}$$

Next, consider the function f;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x) dx = \frac{2}{3}\pi^2 \qquad (2)$$

and

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x) \cos nx dx \\
&= \frac{1}{\pi}\left[\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx + \frac{1}{n^2} \cos nx\right]_{-\pi}^{\pi} \\
&= \frac{4}{n^2} \cos n\pi \\
&= \frac{4}{n^2}(-1)^n \qquad (3)
\end{aligned}$$

where we have integrated by parts. Similarily

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x) \sin nx dx = -\frac{2}{n}(-1)^n \qquad (4)$$

Of course, it is easier to do this problem if you split it into the x part, which only has a sine series and the x^2 part which only has a cosine series. Finally, if $x = \pi$ we expect the righthand side to interpolate the discontinuity,

$$\begin{aligned}
\lim_{x \to \pi -} &= \pi^2 + \pi \\
\lim_{x \to \pi +} &= \pi^2 - \pi \qquad (5)
\end{aligned}$$

so

$$\pi = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos n\pi = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \qquad (6)$$

giving

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2 \qquad (7)$$

where the subtle point is to use the $\cos n\pi$ to get rid of the $(-1)^n$ and then to remember that a Fourier series interpolates across a discontinuity.