Outline solution to 231 annual exam 2007 q1

1. (a) Evaluate

$$\int_0^2 dx \int_{x^2}^{x} dy\, y^2 x$$

and

$$\int_0^{\pi} dy \int_0^{\cos y} dx\, x \sin y$$

(b) By reversing the order of integration, or otherwise, evaluate

$$\int_0^2 dy \int_{y/2}^{1} dx\, e^{x^2}.$$

(c) Use double integration

$$\text{Volume of } D = \int\int_D dxdydz$$

to find the volume of the solid bound by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3 - x$.

*Solution:*For 1(a) the integral are just integrated directly as they are written:

$$\begin{aligned} \int_0^2 dx \int_{x^2}^{x} dy\, y^2 x &= \int_0^2 dx\, \frac{1}{3} y^3 x \Big]_{x^2}^{x} \\ &= \int_0^2 dx \frac{1}{3}(x^4 - x^7) \\ &= \left(\frac{1}{15}x^5 - \frac{1}{24}x^7\right)_0^2 \\ &= \frac{32}{3}\left(\frac{1}{5} - 1\right) = -\frac{256}{15} \end{aligned} \tag{1}$$

and

$$\int_0^{\pi} dy \int_0^{\cos y} dx\, x \sin y = \frac{1}{2}\int_0^{\pi} dy \cos^2 y \sin y \tag{2}$$

and now let $u = \cos y$ so $du = -\sin y dy$ and $y = \pi$ gives $u = -1$ and $y = 0$ gives $u = 1$, so

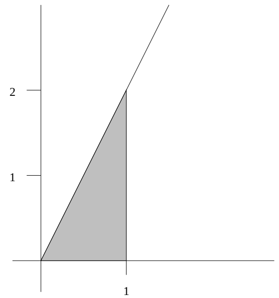
$$\int_0^{\pi} dy \int_0^{\cos y} dx\, x \sin y = \frac{1}{2}\int_{-1}^{1} u^2 du = \frac{1}{6}u^3\Big]_{-1}^{1} = \frac{1}{3} \tag{3}$$

Obviously, this could also have been done using a trignometric identity.

For (b) the trick bit is reversing the order of integration. In the x-direction the integration region goes from the line $y = 2x$ to $x = 1$ and then in the y-direction, from zero to two; see the figure. Now, going in the y direction first, we have y going from zero to $2x$ and x then goes from zero to one; hence

$$\int_0^2 dy \int_{y/2}^{1} dx\, e^{x^2} = \int_0^1 dx \int_0^{2x} dx\, e^{x^2} = \int_0^1 dx 2x e^{x^2} = e - 1 \tag{4}$$

where we have used the u substitution $u = x^2$ to do the x integral.



Finally, for part (c) the thing is to use cylindrical polars. In cylindricals $x = \rho\cos\phi$ so the top of the solid is given by $z = 3 - \rho\cos\phi$ and hence, integrating one to get the volume

$$\begin{aligned} V &= \int_0^{2\pi} d\phi \int_0^3 d\rho\rho \int_0^{3-\rho\cos\phi} dz = \int_0^{2\pi} d\phi \int_0^3 d\rho\rho(3 - \rho\cos\phi) \\ &= \int_0^{2\pi} d\phi \left(\frac{3}{2}\rho^2 - \frac{1}{3}\rho^3 \cos\phi\right)_0^3 \\ &= \int_0^{2\pi} d\phi \left(\frac{27}{2} - 9\cos\phi\right) = 27\pi \end{aligned} \tag{5}$$