The first 6 questions of the 2006 annual exam.

1. (a) The region R in the xy-plane is given by

$$R = \{(x, y) : x^2 \leq y \leq 4\}$$

Sketch the region R and evaluate

$$\int\int_R \exp y^{3/2} dx\, dy$$

(b) The region R in the xy-plane is given by

$$R = \{(x, y) : x^2 + y^2 \leq 4,\ x \geq 1,\ y \geq 0\}$$

Convert the integral

$$\int\int_R \frac{1}{\sqrt{x^2 + y^2 - 1}} dx\, dy$$

into polar coördinates and evaluate it.

Hint: Sketch the region R, describe the left boundary in polar coördinates - find r as a function of θ along this part of the boundary. In evaluating the resulting double integral the following standard integral may be useful

$$\int \tan\theta d\theta = -\log|\cos\theta| + C$$

2. Define a conservative vector field and show a vector field is conservative if and only if its line integral is path independent. Evaluate the integral

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3x^2 dx + 2yzdy + y^2 dz$$

from $(0, 1, 2)$ to $(1, -1, 7)$ by showing that $\mathbf{F}$ has a potential.

3. Show that for all integers m and n

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0$$

Outline how the formulae for the Fourier coefficients are derived. Calculate the Fourier series of the function

$$f(x) = x$$

for $|x| \leq \pi$ and $f(x + 2\pi) = f(x)$.

4. (a) Find the Fourier transform of the function

$$f(x) = \begin{cases} b & |x| < a \\ 0 & |x| \geq a \end{cases}$$

where a is a positive constant and b is a constant.

(b) Find the Fourier transform of the function

$$f(x) = e^{-a|x|}$$

where a is again a positive constant.

5. (a) Find a particular solution to

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x} \cos 2x$$

(b) Verify that $y(x) = 1$ is a particular solution to

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dt} + 4y = 4$$

and find solutions satifying the boundary condition $y(0) = y'(0) = 0$ where the dash denotes differenciation with respect to x.

6. Use the method of Frobenius to obtain a series solution to

$$8x^2y'' + 10xy' + (x - 1)y = 0$$

which is valid near $x = 0$. If the solution is non-singular at $x = 0$ and equal to one when $x = 1$ write down a solution which is accurate to within one percent when $x < 1$. Hint: the indicial equation leads to a solution which is non-singular at $x = 0$, write down the first two terms of this solution and evaluate it at $x = 1$.