A note on the uniqueness of solutions to the Laplace Eqn[1]

9 January 2007

Proving uniqueness of solutions to the Laplace equation is typical application of the Gauss theorem. Of course, before proving uniqueness of solutions you need to prove a solution exists, this is actually trickier and not something we will deal with here. The Laplace equation is

$$\nabla^2\phi = 0 \tag{1}$$

where ϕ is defined on some region D with boundary δD and here we will be dealing with the common case where D has compact support, there is a R such that $D \subset \{(x,y) \in \mathbf{R}^2 | x^2 + y^2 < R\}$ and where there are Dirichlet conditions, so ϕ is given on δD. Now, say there are two solutions to this problem ϕ_1 and ϕ_2, then, by linearity

$$\tilde{\phi} = \phi_1 - \phi_2 \tag{2}$$

also solves the Laplace equation, but with zero boundary conditions on D:

$$\tilde{\phi}|_{\delta D} = 0 \tag{3}$$

Now, consider the energy like integral

$$E = \int_D dV |\nabla\tilde{\phi}| \tag{4}$$

The usual vector identities give the integration by parts like expression

$$E = \int_D dV \nabla\tilde{\phi}\cdot\nabla\tilde{\phi} = \int_D dV \nabla(\tilde{\phi}\nabla\tilde{\phi}) - \int_D \tilde{\phi}\nabla^2\tilde{\phi} \tag{5}$$

Now, the second term is zero because $\tilde{\phi}$ satisfies the Laplace equation and, by Gauss, the first term is a surface integral

$$E = \int_{\delta D} \tilde{\phi}\nabla\tilde{\phi}\cdot\mathbf{dS} = 0 \tag{6}$$

because $\tilde{\phi}$ vanishes on the boundary. Now, $|\nabla\tilde{\phi}|^2$ is non-negative so is integral can only be zero if it vanishes everywhere. Hence

$$\nabla\tilde{\phi} = 0 \tag{7}$$

hence $\tilde{\phi}$ is a constant and, by its boundary condition, this constant must be zero, hence $\phi_1 = \phi_2$, showing the solution, if it exists, is unique. It should be clear that this arguement can also be used to show uniqueness for **van Neumann** boundary conditions where $\nabla\phi\cdot\mathbf{n}$ is specified on the boundary.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`