

A note on the uniqueness of solutions to the Laplace Eqn¹

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Proving uniqueness of solutions to the Laplace equation is typical application of the Gauss theorem. Of course, before proving uniqueness of solutions you need to prove a solution exists, this is actually trickier and not something we will deal with here. The Laplace equation is

$$\nabla^2 \phi = 0 \quad (1)$$

where ϕ is defined on some region D with boundary δD and here we will be dealing with the common case where D has compact support, there is a R such that $D \subset \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 < R\}$ and where there are Dirichlet conditions, so ϕ is given on δD . Now, say there are two solutions to this problem ϕ_1 and ϕ_2 , then, by linearity

$$\tilde{\phi} = \phi_1 - \phi_2 \quad (2)$$

also solves the Laplace equation, but with zero boundary conditions on D :

$$\tilde{\phi}|_{\delta D} = 0 \quad (3)$$

Now, consider the energy like integral

$$E = \int_D dV |\nabla \tilde{\phi}|^2 \quad (4)$$

The usual vector identities give the integration by parts like expression

$$E = \int_D dV \nabla \tilde{\phi} \cdot \nabla \tilde{\phi} = \int_D dV \nabla \cdot (\tilde{\phi} \nabla \tilde{\phi}) - \int_D \tilde{\phi} \nabla^2 \tilde{\phi} \quad (5)$$

Now, the second term is zero because $\tilde{\phi}$ satisfies the Laplace equation and, by Gauss, the first term is a surface integral

$$E = \int_{\delta D} \tilde{\phi} \nabla \tilde{\phi} \cdot \mathbf{dS} = 0 \quad (6)$$

because $\tilde{\phi}$ vanishes on the boundary. Now, $|\nabla \tilde{\phi}|^2$ is non-negative so is integral can only be zero if it vanishes everywhere. Hence

$$\nabla \tilde{\phi} = 0 \quad (7)$$

hence $\tilde{\phi}$ is a constant and, by its boundary condition, this constant must be zero, hence $\phi_1 = \phi_2$, showing the solution, if it exists, is unique. It should be clear that this argument can also be used to show uniqueness for **van Neumann** boundary conditions where $\nabla \phi \cdot \mathbf{n}$ is specified on the boundary.

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