Note IV.1[1,2] 13 May 2007

Separation of variables

The seperation of variables is the main, systematic, way to solve PDEs. Idea is to reduce PDEs involving two or more variables to ODEs in each variable by assuming, for example, that $\phi(x, y) = X(x)Y(y)$ where X depends on x only and Y depends on y only. While this is a strong restiction, not many functions in x and y are in this form, we will find that, in practice, we will get a countably infinite number of solutions and these can be added, by linearity.

Separation of variables and Laplace's equation

Consider the two-dimensional Laplace equation $\triangle\phi(x, y) = 0$ on the square region $0 \le x \le \pi$ and $0 \le y \le \pi$ with zero Dirichlet boundary conditions on three sides: $\phi(0, y) = \phi(\pi, y) = \phi(x, \pi)$ and, on the third side, a constant Dirichlet condition $\phi(x, 0) = 1$ (Picture 231.IV.2.1).

Now, under the assumption

$$\phi(x, y) = X(x)Y(y) \tag{1}$$

Laplace's equation becomes

$$\nabla^2\phi = X''(x)Y(y) + X(x)Y''(y) = 0 \tag{2}$$

where prime is used to denote differenciation with respect to the arguement, so $X' = dX/dx$ and $Y' = dY/dy$. Now divide through by $X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \tag{3}$$

with the left hand side independent of y and the right hand side independent of x. Since x and y are independent variables, this equation can only hold if each side is equal to the same constant

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{ constant independent of x and y } = E \tag{4}$$

Now, we have two seperate equations

$$\begin{aligned} X'' &= EX \\ Y'' &= -EY \end{aligned} \tag{5}$$

There are three possibilities: E can be zero, strictly positive or strictly negative.

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[2] Substantially based on lecture notes taken by John Kearney

- Zero: $E = 0$ so $X'' = Y'' = 0$ and

$$\begin{aligned} X(x) &= Ax + B \\ Y(y) &= Cy + D \end{aligned} \tag{6}$$

- Positive: $E = k^2$, k real and non-zero. Hence

$$\begin{aligned} X'' &= k^2 X \\ Y'' &= -k^2 Y \end{aligned} \tag{7}$$

  with solutions

$$\begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= C \sin ky + D \cos ky \end{aligned} \tag{8}$$

- Negative: $E = -k^2$, k real and non-zero. Hence

$$\begin{aligned} X'' &= -k^2 X \\ Y'' &= k^2 Y \end{aligned} \tag{9}$$

  with solutions

$$\begin{aligned} X(x) &= A \sin kx + B \cos kx \\ Y(y) &= Ce^{ky} + De^{-ky} \end{aligned} \tag{10}$$

So, there are lots of solutions, lots of arbitrary constants and, in addition, this new sort of constant, E or k that arrived as part of the seperation of variables assumption. However, we haven't implimented the boundary conditions yet. In fact, none of these solutions satisfies all the boundary conditions on its own.

Lets start with the left and right conditions

$$\phi(0, y) = \phi(\pi, y) = 0 \tag{11}$$

Since $\phi = XY$ this means that $X(0) = X(\pi) = 0$. For the $E = 0$ solution $X = Ax + B$, $X(0) = 0$ gives $B = 0$ and, the $X(\pi) = 0$ gives $A = 0$, so these solutions are trivial. Now, for $E = k^2$

$$X(x) = Ae^{kx} + Be^{-kx} \tag{12}$$

$X(0) = 0$ if $A = -B$ but, then

$$X(\pi) = A\left(e^{k\pi} - e^{-k\pi}\right) \tag{13}$$

and this can never be zero for non-zero k. Finally, for $E = -k^2$

$$X(x) = A \sin kx + B \cos kx \tag{14}$$

and, so, $X(0)=0$ if $B=0$, then

$$X(\pi)=A\sin\pi k \tag{15}$$

and this is zero if $k=n\in\mathbf{N}$, a natural number. Thus, there is an infinite series of solutions satisfying the differential equation and the boundary conditions at $x=0$ and π.

This is a linear problem and so we can add solutions to give general solution

$$\phi(x,y)=\sum_{n=1}^{\infty}(A_n e^{ny}+B_n e^{-ny})\sin nx \tag{16}$$

Now, we need only match the two remaining boundary conditions. First, the zero condition implies

$$A_n e^{n\pi}+B_n e^{-n\pi}=0 \tag{17}$$

and so we are left with

$$\phi(x,0)=\sum_{n=1}^{\infty}C_n\sin nx=1 \tag{18}$$

where $C_n=A_n+B_n$. This is obviously in Fourier series form and so we might hope to choose C_ns so that we have a Fourier series for $f(x)=1$, the boundary condition. This is more or less what we do do, the only problem is that $f(x)=1$ is even while the sines are odd. We need to make $f(x)$ into an odd periodic function, so that there is a sine series for it, such that $f(x)=1$ for $0<x<\pi$, since that is boundary condition. Recall

$$f(x)=1 \quad 0<x<\pi-1 \quad -\pi<x<0 \tag{19}$$

has the Fourier Series expansion

$$f(x)=\frac{4}{\pi}\sum_{n\text{ odd }>0}\frac{1}{n}\sin nx \tag{20}$$

Restricting to $0<x<\pi$ we have

$$1=\frac{4}{\pi}\sum_{n\text{ odd }>0}\frac{1}{n}\sin nx \tag{21}$$

Now, solving for the A_n and B_n we get

$$\phi(x,y)=\frac{4}{\pi}\sum_{n\text{ odd }>0}\frac{\sin nx}{n}\frac{e^{ny}}{e^{2\pi n}-1} \tag{22}$$

solving the problem.

If the bottom boundary condition has been replaced by different function the method would not change: extend the function to an odd periodic function and then Fourier analysis it. For general Dirichlet boundary conditions, the easiest thing is to express ϕ as the sum of four ϕ, each with one non-zero boundary condition, one for each side (Picture 231.IV.2.2). Solving the individual problems, each with one non-zero side is easy, they follow the same as for the one above, in fact, one quick way to solve them would be to do a change of variables rotating the above solution. Neumann conditions are equally straight-forward, more complicated boundaries are not; for this whole thing to work, the boundary conditions in individual parts of the boundary need to expressed as conditions on one of the seperated functions, X and Y here. To do this for more complicated boundaries usually requires a change of variables, to, for example, polar coördinates for annular or wedge shaped regions. This is not done here.

Periodic Strip

Periodic in x direction $\pi(x + 2\pi, y) = \phi(x, y)$ and y in some finite range, say $0 \leq y \leq 1$ with Dirichlet boundary conditions at $y = 0$ and $y = 1$.

$$\phi(x, 1) = g(x) \tag{23}$$

where $g(x + 2\pi) = g(x)$ and

$$\phi(x, y = 0) = f(x) \tag{24}$$

where $f(x + 2\pi) = f(x)$.

So, separation of variables:

$$\phi(x, y) = X(x)Y(y) \tag{25}$$

Solutions must be periodic in x,

$$\begin{aligned} X(x) &= A\cos nx + B\sin nx \\ Y(y) &= Ce^{ny} + De^{-ny} \end{aligned} \tag{26}$$

and $X(x) = A$ with $Y(y) = C + Dy$. Now, redefining the constants

$$\phi(x, y) = k_1 + k_2 y + \sum_{n=1}^{\infty} \left(a_n e^{ny} + \tilde{a_n} e^{-ny}\right) \cos nx + \sum_{n=1}^{\infty} \left(b_n e^{ny} + \tilde{b_n} e^{-ny}\right) \sin nx \tag{27}$$

Obtain coefficients k_1, k_2, a_n, $\tilde{a_n}$, b_n, $\tilde{b_n}$ through boundary conditions at $y = 0$, and $y = 1$.

The heat equation

For convenience we will take the 1+1-dimensional heat equation to be

$$\triangle\phi(x, t) = \frac{\partial\phi(x, t)}{\partial t} \tag{28}$$

with zero Dirichlet boundary conditions at $x = 0$ and π and an initial condition

$$\phi(x, 0) = f(x) \tag{29}$$

for some known $f(x)$.

We begin, as usual, by seperating

$$\phi(x,t) = X(x)T(t) \tag{30}$$

giving

$$\frac{X''}{X} = \frac{T'}{T} \tag{31}$$

As before, this must mean that each side is equal to the same constant, which we will call E. There are, again, three cases

- Zero: $E = 0$ so $X'' = T' = 0$ and

$$\begin{aligned} X(x) &= Ax + B \\ T(t) &= C \end{aligned} \tag{32}$$

- Positive: $E = k^2$, k real and non-zero. Hence

$$\begin{aligned} X'' &= k^2 X \\ T' &= k^2 T \end{aligned} \tag{33}$$

$$\begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ T(t) &= Ce^{k^2 t} \end{aligned} \tag{34}$$

- Negative: $E = -k^2$, k real and non-zero. Hence

$$\begin{aligned} X'' &= -k^2 X \\ T' &= -k^2 T \end{aligned} \tag{35}$$

  with solutions

$$\begin{aligned} X(x) &= A\sin kx + B\cos kx \\ T(t) &= Ce^{-k^2 t} \end{aligned} \tag{36}$$

As before, most of these solutions will not satisfy the x boundary conditons, applying the boundary conditions and adding all solutions gives

$$\phi(x,t) = \sum_{n=1}^{\infty} A_n \sin nx e^{-n^2 t} \tag{37}$$

and, so for $t = 0$

$$\phi(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \tag{38}$$

and so this is used to match the initial condition. If there are Neumann boundary conditions the sine will be replaced by a cosine and the constant solution would have to be included

$$\phi(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx e^{-n^2 t} \tag{39}$$

Obviously in this case the even periodic extension of $f(x)$ is required.