Note IV.1[1,2] 13 May 2007

Part IV: Partial Differential Equations

A partial differential equation is a differential equation involving derivatives of more than one independent variable.

Some linear PDEs involving a scalar field ϕ

These equations are often known as the **equations of mathematical physics** since they are all important in physics and where all studied first because of their physical importance. However, they are now equally important in mathematics.

- **Laplace's equation**:
$$\nabla^2\phi = 0 \tag{1}$$
- **Poisson's equation**:
$$\nabla^2\phi = \rho \tag{2}$$
where ρ is some scalar field usually called a source term
- The **Helmholtz equation**:
$$(\nabla^2 + k^2)\phi = 0 \tag{3}$$
and the 'wrong sign' Helmholtz equation:
$$(\nabla^2 - k^2)\phi = 0 \tag{4}$$
where k is a real constant
- The **Heat or diffusion equation**:
$$\nabla^2\phi = D\frac{\partial\phi}{\partial t} \tag{5}$$
where D a constant and $\phi = \phi(\mathbf{r}, t)$ is time-dependent.
- The **wave equation**:
$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{6}$$
where c is the speed of wave propagation.

All of these equation involve the **Laplacian**:

$$\triangle = \nabla^2 = \nabla.\nabla = \text{ div grad } = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \tag{7}$$

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[2]Substantially based on lecture notes taken by John Kearney

Heat/Diffusion Equation

Imagine some material where the temperature is not constant but with no sources, or sinks, of heat. Temperature is a scalar field $\phi(\mathbf{r}, t)$. Heat current $\mathbf{j}(\mathbf{r}, t)$ is a vector field such that energy flux across an oriented surface S (Picture IV.1.1) is the surface integral $\int_S \mathbf{j}(\mathbf{r}, t)\mathbf{dA}$

Let S be a closed, static, surface. Heat flux out, Φ, of S is given by

$$\begin{aligned}\Phi = \int_S \mathbf{j}(\mathbf{r}, t)\mathbf{dA} = -\text{ rate of change of energy in D} &= -\alpha \frac{\partial}{\partial t}\int_D \phi(\mathbf{r}, t)dV \\ &= -\alpha \int_D \frac{\partial \phi}{\partial t}(\mathbf{r}, t)dV \qquad (8)\end{aligned}$$

where α is a constant, the heat capacity per unit volume.

Now, apply Gauß' theorem

$$\int_S \mathbf{j}(\mathbf{r}, t).\mathbf{dA} = \int_D \text{ div } \mathbf{j} dV \qquad (9)$$

so that

$$\int_D (\text{ div } \mathbf{j} + \alpha \frac{\partial \phi}{\partial t})dV = 0 \qquad (10)$$

where D is any 3d region with a smooth boundary. Thus

$$\text{div } \mathbf{j} + \alpha \frac{\partial \phi}{\partial t} = 0 \qquad (11)$$

Assume $\mathbf{j} = -\beta$ grad ϕ where β is the thermal conductivity constant and the - sign indicates that the heat flows from hot to cold regions

$$\nabla^2 \phi = D \frac{\partial \phi}{\partial t} \text{ with } D = \frac{\alpha}{\beta} \qquad (12)$$

Boundary Value Problems

Typically, we wish to solve a PDE subject to some boundary conditions. Assume ϕ satisfies some PDE (e.g. Laplace's equation) in a 3d region D with boundary $S = \delta D$ (Picture IV.1.2). There are three common, basic, types of boundary conditions

1. ϕ is given on S: **Dirichlet boundary conditions**.
2. $\partial_{\mathbf{n}}\phi = \mathbf{n} \cdot \nabla \phi$, directional derivative in direction of unit normal $\mathbf{n}$, is given on S: **Neumann boundary conditions**.
3. ϕ and $\partial_n \phi$ are given on S: **Cauchy boundary conditions**.
4. Can also have mixed b.c.s. where on different parts of S different b.c.s are imposed
5. This is not exhaustive since there are other types of b.c.s. such as periodic boundary conditions.

We will concentrate on the first two cases, usually the third case, Cauchy conditions, is too strong and there will be no solution. Typically, Dirichlet boundary conditions lead to a unique solution and Neumann, to a solution which is unique up to an overall additive constant. A useful distinction is often made between the simpler, **elliptic** equations and **parabolic** and **hyperbolic** equations. These terms refer to the relative signs of the derivative terms, something which has implications for the existence and stability of solutions. This won't be discussed here.

Laplace's Equation

A solution of Laplace's equation is called a **harmonic** function. Some simple (singular) examples:

- 3d $\phi = \frac{1}{r}$ is harmonic but singular at the origin.
- 2d $\phi = \log r, r = \sqrt{x^2 + y^2}$ harmonic but singular at $r = 0$.
- 1d $\phi = x$ is harmonic but singular at $\pm\infty$.
- 2d Any holomorphic function (Complex Analysis) is harmonic!

Suppose we wish to find a non-singular (e.g. C^∞) harmonic function is some domain D subject to some boundary conditions on $S = \partial D$. There are important theorems relating to the uniqueness of the solutions and to the value of the solutions, these theorems are very useful and are also typical of similar theorems for other similar equations. There are also important existence theorems, but we do not deal with these here.

Theorem: Uniqueness of the Laplace equation with Dirichlet and Neumann boundary conditions.

The solution of Laplace's equation under Dirichlet's boundary conditions, if it exists, is unique. The solution of the problem under Neumann boundary conditions, if it exists, is unique up to an additive constant. This will be proved after considering the Green's identities.

Green's Identities

These should not be confused with Green's theorem in the plane. Let ϕ and ψ be smooth functions, not necessarily harmonic, then

- **Green's First Identity**

$$\int_D (\phi \nabla^2 \psi + \nabla\phi . \nabla\psi) dV = \int_{S=\partial D} \phi \nabla \psi . \mathbf{dA} \tag{13}$$

- **Green's second Identity**

$$\int_D (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_{S=\partial D} (\phi\nabla\psi - \psi\nabla\phi).\mathbf{dA} \tag{14}$$

  To prove these, well, for the first identity, pply Gauss' theorem to the vector field $\mathbf{F} = \phi\nabla\psi$

$$\text{div } \mathbf{F} = \phi\nabla.(\nabla\psi) + \nabla\phi\cdot\nabla\psi = \phi\nabla^2\psi + \nabla\phi.\nabla\psi \tag{15}$$

  So, for the second identity, interchange ϕ and ψ in first identity the subtract from the first identity.

Now, to prove the uniquenss theorem, let ϕ_1 and ϕ_2 be harmonic in D and subject to the same boundary conditions on $S = \partial D$ (either DBCs or NBCs). Consider $\phi = \phi_1 - \phi_2$. Apply Green's 1st identity taking $\psi = \phi$

$$\int_D (\phi\triangle\phi + \nabla\phi\cdot\nabla\phi)dV = \int \phi\nabla\phi\cdot\mathbf{dA} \tag{16}$$

Now, the first term on the left is zero since $\phi = \phi_1 - \phi_2$ is harmonic. The right hand side is also zero because either ϕ is zero for Dirichlet boundary conditions or $\partial_{\mathbf{n}}\phi$ is zero for Neumann. Therefore

$$\int_D \nabla\phi.\nabla\phi dV = 0 \tag{17}$$

and $\nabla\phi.\nabla\phi$ is non-negative! This requires $\nabla\phi = 0$ or $\phi =$ constant, that is, $\phi_1 - \phi_2 = c$ proving the theorem for Neumann conditions. For Dirichlet c must be zero since ϕ_1 and ϕ_2 agree on S by assumption.

0.0.1 Gauss' Mean Value Theorem

The Gauss mean value theorem for harmonic functions: suppose ϕ is harmonic in $D \subset \mathbf{R}^3$. The average value of ϕ over the surface of a sphere of radius R centred at the point $\mathbf{r}$ is $\phi(\mathbf{r})$. This is true for any point in the interior of D. The radius R is any number such that the sphere, and every point inside it, is in D. To prove it, we assume without loss of generality consider a sphere centred at the origin. Idea is to show that the average

$$\bar{\phi}_R = \frac{1}{4\pi R^2}\int_{x^2+y^2+z^2=R^2} \phi dA \tag{18}$$

is independent of the radius R. Apply Green's 2nd identity to ϕ and $\psi = \frac{1}{r}$ in the regions $R_1 < r < R_2$ (Picture 231.IV.1.3).

$$\int_{R_1<r<R_2} (\phi\nabla^2\frac{1}{r} - \frac{1}{r}\nabla^2\phi)dV \tag{19}$$

and

$$= \int_{r=R_2,out} (\phi\nabla\frac{1}{r} - \frac{1}{r}\nabla\phi).\mathbf{dA} - \int_{r=R_1,out} (\phi\nabla\frac{1}{r} - \frac{1}{r}\nabla\phi).\mathbf{dA} \tag{20}$$

Now,

$$\int_{r=R_1,out} \frac{1}{r}\nabla\phi.\mathbf{dA} = \frac{1}{R_1}\int_{r=R_1,out} \nabla\phi.\mathbf{dA} \tag{21}$$

because $\frac{1}{r}$ is constant of sphere $r = R_1$, hence

$$\int_{r=R_1,out} \frac{1}{r}\nabla\phi.\mathbf{dA} = \frac{1}{R_1}\int_{r<=R_1} \text{div}\nabla\phi dV = \frac{1}{R_1}\int_{r<=R_1} \nabla^2\phi dV = 0 \tag{22}$$

Similarily

$$\int_{r=R_2,out} \frac{1}{r}\nabla\phi.\mathbf{dA} = 0 \tag{23}$$

Thus,

$$0 = \int_{r=R_2,out} \phi\nabla\frac{1}{r}.\mathbf{dA} - \int_{r=R_1,out} \phi\nabla\frac{1}{r}.\mathbf{dA} \tag{24}$$

so

$$\nabla\frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2} \text{ in both integrals} \mathbf{n} = \hat{\mathbf{r}} \tag{25}$$

and

$$0 = -\frac{1}{R_2^2}\int_{r=R_2} \phi dA + \frac{1}{R_1^2}\int_{r=R_1} \phi dA \tag{26}$$

or

$$\bar{\phi_{R_2}} = \bar{\phi_{R_2}} \tag{27}$$

Letting $R \to 0$, $\bar{\phi_R} = \phi(0)$, R being the radius of the outer sphere, completing the proof.

This leads to the **Maximum (minimum) Principle** for Harmonic Functions. Let ϕ be harmonic in a 3d (or 2d) domain D. Then ϕ never assumes its maximum (or minimum) value at an interior point of D unless ϕ is constant. To prove this, assume ϕ has a maximum at some point P in the interior of D. For R sufficiently small the sphere of radius R centred at P is inside D. For every point on the sphere $\phi < \phi(P)$ so $\bar{\phi_R} < \phi(P)$ contradicting the MVT. A similar argument holds if P is a minimum. If ϕ is harmonic in D it assumes its maximum and minimum values at the boundary $S = \partial D$

This has a physical interpretation. Heat satisfies the heat equation $\nabla^2 = D\frac{\partial\phi}{\partial t}$. If ϕ reaches a steady state $\frac{\partial\phi}{\partial t} = 0$, then $\nabla^2\phi = 0$, that is, the temperature is harmonic. Suppose we have a finite lump of matter and the boundary temperature (not necessarily constant) is fixed, for example, consider a square slab with three sides fixed to be at 0 degrees and the other at 100 degrees (Picture IV.1.4). The steady state temperature inside the slab is harmonic. Steady state temperature can never exceed 100 degrees, or fall below 0 degrees; heat would immediately flow out of, or enter, such a hot or cold spot.

A similar result for the whole of space is known as **Liouville's Theorem**: if ϕ is harmonic and bounded throughout $\mathbb{R}^3$ (or $\mathbb{R}^2$) then it is constant. The proof isn't given here. $\phi = \frac{1}{r}$ in three dimensions and $\phi = \log r$ in two dimensions are unbounded.

Some solutions

Uniqueness theorem very powerful; any solution with DBCs, however simple is the only solution.

- Let ϕ be a harmonic function which is constant, say $\phi = a$, on the boundary of D. $\phi(\mathbf{r}) = a$ is trivially a solution of Laplace's equation with the correct b.c.s.. It must be the unique solution to this boundary value problem.

- Let ϕ be harmonic in a 2d annulus with $\phi = 1$ on the other boundary $(r = R_2)$ $\phi = b$ on the inner boundary $(r = R_1)$. Now $\phi = C \log r + D$ $(r = \sqrt{x^2 + y^2})$ harmonic but singular at origin, with C and D constants.

$$\begin{aligned} a &= \phi(r = R_2) = C \log R_2 + D \\ b &= \phi(r = R_1) = C \log R_1 + D \\ a - b &= C \log \frac{R_2}{R_1} \\ D &= a - C \log R_2 \end{aligned} \tag{28}$$

  and

$$\phi = \frac{a - b}{\log \frac{R_2}{R_1}} \log r R_2 + a \tag{29}$$

In these two examples we have guessed a solution (which we know to be unique). This is clearly insufficient for most problems.