Note III.5[1][2] 12 April 2008

The method of Fröbenius

For the general homogeneous ordinary differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1}$$

the series method works, as in the Hermite case, where both p and q are smooth. If p and q have singularities the series method sometimes fails, and example is Euler's equation $\alpha x^2 y'' + \beta x y' + \gamma y = 0$ or $p(x) = \beta/(\alpha x)$ and $q(x) = \gamma/(\alpha x^2)$. The explicit solution $y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$ picked up by power series method, unless both roots λ_1 and λ_2 are positive integers, because x^λ is not of the form of the ansatz

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

for any a_ns, unless λ is itself a natural number.

One way out is to expand about a point other than $x = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \tag{3}$$

where $p(c)$ and $q(c)$ are finite. However, a singular point can often be the 'most symmetric' point and, in many cases, exactly the point we are interested in.

Frobenius (or generalised series) method allows one to expand about a **regular singularity**, described later, of p and q. Without loss of generality consider an expansion about $x = 0$. Consider a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \, x^{n+s} \tag{4}$$

where s is some real number. Unlike in the standard power series method a_0 is always taken to be non-zero; the odd solution of Hermite's equation would emerge as an $s = 1$ Frobenius series with $a_0 \neq 0$. Starting with s arbitrary consistency will lead to a quadratic equation for s called the **indicial equation.**

The Bessel Equation

The Bessel equation is

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0 \tag{5}$$

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[2]Based on notes I got from Chris Ford

or, multiplying across by x^2,

$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0 \tag{6}$$

It is one of the important equation of applied mathematics and engineering mathematics because it is related to the Laplace operator in cylindrical coördinates. The Bessel equation is solved by series solution methods, in fact, to solve the Bessel equation you need to use the method of Fröbenius. It might be expected that Fröbenius is needed because of the singularities at $x = 0$, however, lets pretend we hadn't noticed and try to use the ordinary series solution method:

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{7}$$

Now, by calculating directly

$$x^2 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n \tag{8}$$

and

$$x^2 y' = \sum_{n=0}^{\infty} n a_n x^n \tag{9}$$

so the equation becomes

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^n = 0 \tag{10}$$

Hence, if we want to go up to the highest power we need to increase everything to the form x to the $n+2$. By letting $m + 2 = n$ we get

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^{m+2} \tag{11}$$

and

$$\sum_{n=0}^{\infty} n a_n x^n = \sum_{m=0}^{\infty} (m+2) a_{m+2} x^{m+2} \tag{12}$$

and, finally,

$$\sum_{n=0}^{\infty} n a_n x^n = a_0 + a_1 x + \sum_{m=0}^{\infty} a_{m+2} x^{m+2} \tag{13}$$

Putting this all back in to the equation, renaming m to n in the usual way, we get

$$a + 0 + a_1 x \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+2) a_{n+2} x^n - \nu^2 a_{n+2} + a_n \right] x^{n+2} = 0 \tag{14}$$

which gives recursion relation

$$a_{n+2} = -\frac{a_n}{(n+2)^2 - \nu^2} \tag{15}$$

along with $a_0 = a_1 = 0$. Thus, while we get a perfectly good two step recursion relation, the extra conditions, on a_0 and a_1 lead to the solution being trivial. Hence, the solution of the series form is trivial and, clearly, to find the actual solution, a more general series ansatz is needed.

Fröbenius means that you look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{16}$$

Now, in terms of this series we have

$$\begin{aligned} x^2 y'' &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \\ xy' &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \\ x^2 y &= \sum_{n=0}^{\infty} a_n x^{n+r+2} \\ \nu^2 y &= \sum_{n=0}^{\infty} \nu^2 a_n x^{n+r} \end{aligned} \tag{17}$$

As usual, we move to the highest power, in this case $n+r+2$, without going through the details, this gives

$$x^2 y'' = r(r-1)a_0 x^r + r(r+1)a_1 x^{r+1} + \sum_{n=0}^{\infty} a_{n+2}(n+r+2)(n+r+1)x^{n+r+2} \tag{18}$$

and

$$xy' = ra_0 t^r + r(r+1)a_1 x^{r+1} + \sum_{n=0}^{\infty} a_{n+2}(n+r+2)x^{n+r+2} \tag{19}$$

and finally

$$\nu^2 y = \nu^2 a_0 x^r + \nu^2 a_1 x^{r+1} + \sum_{n=0}^{\infty} a_{n+2} x^{n+r+2} \tag{20}$$

Now, if we put this all in one equation and set the x^r terms to zero, we have

$$[r(r-1) + r - v^2]a_0 = 0 \tag{21}$$

or, put another way, either $a_0 = 0$ or $r = \pm\nu$. The x^{r+1} term gives

$$[(r+1)^2 - \nu^2]a_1 = 0 \tag{22}$$

so, with $r = \pm nu$ $a_1 = 0$

Now, the recusion relation is

$$[(n+r+2)(n+r+1) + (n+r+2) - \nu^2]a_{n+2} = -a_n \tag{23}$$

so, with $r = \pm\nu$ we have

$$a_{n+2} = -\frac{a_n}{(n \pm \nu + 2)^2 - \nu^2} \tag{24}$$

and so there are two solutions to the Bessel equation, one corresponding to $r = \nu$ and the other with $r = -\nu$. If $\nu = 0$ the situation is more complicated, this example is dealt with in a problem sheet.

Here we consider the case $\nu = \frac{1}{2}$ so that $s = \pm\frac{1}{2}$. The recursion relation can be written

$$a_{m+2} = -\frac{a_m}{(m+2+\frac{1}{2})^2 - \frac{1}{4}} = -\frac{a_m}{(m+2)(m+3)}. \tag{25}$$

Since $a_1 = 0$ the recursion relation implies that a_3, a_5, a_7 etc. are all zero. Fixing $a_0 = 0$ and applying the recursion relation gives

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 3} = -\frac{1}{2 \cdot 3} \\ a_4 &= -\frac{a_2}{4 \cdot 5} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{5!} \\ a_6 &= -\frac{1}{7!} \end{aligned} \tag{26}$$

and so on. Thus, the solution is

$$\begin{aligned} y(x) &= x^{\frac{1}{2}} - \frac{x^{\frac{5}{2}}}{3!} + \frac{x^{\frac{9}{2}}}{5!} - ... \\ &= x^{-\frac{1}{2}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \right) \\ &= x^{-\frac{1}{2}} \sin x. \end{aligned} \tag{27}$$

where we have used $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$)

The other root $s = -\frac{1}{2}$ leads to

$$y(x) = x^{-\frac{1}{2}} \cos x, \tag{28}$$

and so the general solution of the $\nu = \frac{1}{2}$ problem is

$$y(x) = x^{-\frac{1}{2}} \left(C_1 \cos x + C_2 \sin x \right). \tag{29}$$

Fuch's theorem

The method of Frobenius gives a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \, (x-c)^{n+s}$$

where p or q are singular at $x = c$. Method does not always give the general solution, the $\nu = 0$ case of Bessel's equation is an example where it doesn't. There is a theorem dealing with the applicability of the Frobenius method in the case of **regular singularities.**

$x = c$ is a regular singular point if $(x-c)p(x)$ and $(x-c)^2 q(x)$ can be expanded as a power series about $x = c$. All the singular ODEs we have met have regular singularities, an example of an ODE with a non-regular singularity $x^3 y'' + y = 0$ since here $q(x) = 1/x^3$ so that $x^2 q(x) = 1/x$ cannot be expanded about $x = 0$.

If p and q are non-singular at $x = c$, $x = c$ is called an **ordinary point** of the ODE $y''(x) + p(x)y'(x) + q(x)y(x) = 0$.

Fuchs' Theorem states that if $x = c$ is a regular singular or ordinary point of the ODE

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{30}$$

if and only if the two solutions are Frobenius series or one solution is a Frobenius series, $S_1(x)$ and the other solution is of the form $y(x) = S_1(x)\log(x-c) + S_2(x)$ where $S_2(x)$ is another Frobenius series, it is not a solution on its own. The proof of this isn't given.

The second case occurs when the indicial equation has equal roots and sometimes when the roots differ by an integer, an example is the $\nu = 1$ case of Bessel's equation).

Finding a Second Solution

If one solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ can be found another one can be constructed. Let $u(x)$ be a solution then try $y(x) = u(x)v(x)$ then a short calculation gives

$$y'' + py' + qy = (u'' + pu' + qu)v + (2u' + pu)v' + uv'' = 0. \tag{31}$$

Now since u is, by assumption, a solution the first term on the right hand side is zero giving

$$(2u' + pu)v' + uv'' = 0. \tag{32}$$

This is a first order linear ODE for $v'(x)$.