Note III.4[1,2] 31 March 2008

Series Solutions

The idea behind series solutions is to write y as a power series about $x = 0$ or some other point

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{1}$$

and determine a *recursion relation* for the a_n coefficients.

Here is a simple example where we know what the answer should be

$$y' - y = 0 \tag{2}$$

Now, assuming convergence,

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \tag{3}$$

and so substituting into the equation gives

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{4}$$

The way series solutions works is we try to express this in the form

$$\sum_n \text{stuff}_n x^n = 0 \tag{5}$$

and the stuff_n gives the recursion relation. The problem here is the different powers of x in the sums, we fix this by doing a change of index $m = n - 1$ so

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{m=-1}^{\infty} (m+1) a_{m+1} x^m \tag{6}$$

and now the sum limits are different to the limits of the other term in the equation, we fix this by doing the first term seperately, in this case the $m = -1$ term is zero, so

$$\sum_{m=-1}^{\infty} (m+1) a_{m+1} x^m = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m \tag{7}$$

Now, we m is just a dummy index, we can name it back to n and the equation becomes

$$y' - y \;=\; \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231
[2]Based on notes I got from Chris Ford

$$= \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0 \tag{8}$$

and, since the x^n are independent, for this to be true for all x every coefficient of every power of x must vanish, giving

$$a_{n+1} = \frac{a_n}{n+1} \tag{9}$$

This is the recursion relation, it allows us to calculate higher a_n's from lower ones, for $n = 0$ it gives $a_1 = a_0$, for $n = 1$ it gives $a_2 = a_1/2 = a_0/2$ and so on. The a_0 is arbitrary, it is the arbitrary constant you would expect for a first order differential equation. Of course, in this case it is easy to see that $a_n = a_0/n!$, as you would expect: we know this equation is solved by $y = A\exp x$ where A is an arbitrary constant.

Here is another example, consider

$$y'' - xy = 0 \tag{10}$$

so assuming we can find a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{11}$$

we have

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}, \tag{12}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}, \tag{13}$$

Substituting into the equation we have

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \tag{14}$$

and so we take the first term and shift its index up three so that the exponent of x has the same form as in the second term

$$\begin{aligned} \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} &= \sum_{m=-3}^{\infty} a_{m+3}(m+3)(m+2)x^{m+1} \\ &= 2a_2 + \sum_{m=0}^{\infty} a_{m+3}(m+3)(m+2)x^{m+1} \end{aligned} \tag{15}$$

where we have removed the $m = -3$, -2 and -1 terms from the sum, but only the -1 term is non-zero. Renaming m to n we have

$$2a_2 + \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \tag{16}$$

and, again, substituting every coefficient of every power of x to zero we get recursion relations

$$a_2 = 0 \tag{17}$$

and

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)} \tag{18}$$

Now, we have a three-step recursion relation, but with $a_2 = 0$; so there are two arbitrary constants

$$\begin{aligned} a_0 &= y(0) \\ a_1 &= y'(0) \end{aligned} \tag{19}$$

where the constants are related to the initial conditions by substituting $x = 0$ into the series for y and y'.

Now, Consider Hermite's equation

$$y'' - 2xy' + 2\alpha y = 0 \tag{20}$$

So, we let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{21}$$

so

$$xy'(x) = \sum_{n=0}^{\infty} n a_n x^n, \tag{22}$$

and

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m \tag{23}$$

where we have changed index to $m = n + 2$ and used that the first two terms in the sum are zero. This gives

$$y'' - 2xy' + 2\alpha y = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2(\alpha - n)a_n\right] x^n = 0. \tag{24}$$

This implies that the content of the square bracket is zero for all n leading to the recursion relation

$$a_{n+2} = \frac{2(n-\alpha)a_n}{(n+1)(n+2)}. \tag{25}$$

From this two independent solutions can be obtained.

- An even solution: Set $a_1 = 0$. Recursion relation $\rightarrow$ a_3, a_5, a_7 etc. all zero. Fix $a_0 = 1$ and apply recursion relation

$$a_2 = -\frac{2\alpha a_0}{1 \cdot 2} = -\frac{2\alpha}{1 \cdot 2}$$

$$
\begin{aligned}
a_4 &= \frac{2(2-\alpha)a_2}{3\cdot 4} = 2^2\frac{(\alpha-2)\alpha}{1\cdot 2\cdot 3\cdot 4} \\
a_6 &= \frac{2(4-\alpha)a_4}{5\cdot 6} = -\frac{2^3(\alpha-4)(\alpha-2)\alpha}{6!}
\end{aligned}
\tag{26}
$$

so the pattern clear

$$
a_{2m} = \frac{(-2)^m}{(2m)!}(\alpha-2m+2)(\alpha-2m+4)...\alpha \tag{27}
$$

where we define $0! = 1$. An even solution of Hermite's equation reads

$$
y_e(x)\sum_{m=0}^{\infty}\frac{(-2)^m}{(2m)!}(\alpha-2m+2)(\alpha-2m+4)...\alpha\ x^{2m}. \tag{28}
$$

This series is convergent with radius of convergence=∞; the ratio test can be used to prove this, we don't persue issues of convergence here. For special values of α, even and positive, the series terminates and the solution is a polynomial of degree α. For example, when $\alpha = 2$ a_4, a_6, a_8 and so on, are all zero and

$$
y_{even}(x) = 1 - 2x^2 \tag{29}
$$

You can check that this satisfies Hermite's equation.

- An odd solution: Set $a_0 = 1$, $a_1 = 1$. Recursion relation $\rightarrow$ a_2, a_4, a_6 et. all zero. Odd coefficients can be worked out via the recursion formula. If α is an odd integer the series will terminate producing a polynomial of degree α.

The general solution of Hermite's equation is therefore

$$
y = C_1 y_e(x) + C_2 y_o(x) \tag{30}
$$

where y_e and y_o are the even and odd solutions.

Generating function

If α is a positive integer one of the solutions to Hermite's equation is polynomial. Such functions are called Hermite polynomials. Remarkably, all the polynomials can be combined into a single *generating function.* Consider $\Phi(x,h) = e^{2xh-h^2}$. Expanding this in powers of h:

$$
\Phi(x,h) = \sum_{n=0}^{\infty}\frac{h^n}{n!}H_n(x)
$$

$H_n(x)$ are polynomial solutions of Hermite's equation.