Note III.3[1,2] 19 February 2008

Green's functions

Returning to the general second order linear inhomogeneous equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x) \tag{1}$$

If two solutions, y_1 and y_2, of the homogeneous equation are known then a particular solution of the full equation, y_p, can be found by the Green's function method; this is outlined here. Consider the equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = \delta(x - x'). \tag{2}$$

A solution of this equation is called a **Green's function** and denoted $G(x|x')$. A solution of the inhomogeneous equation is then

$$y_p(x) = \int_{-\infty}^{\infty} dx'\ f(x')\ G(x|x'). \tag{3}$$

The Green's function is not unique, but one formula for $G(x|x')$ is

$$G(x|x') = \theta(x - x')\frac{y_1(x)y_2(x') - y_2(x)y_1(x')}{y_1(x')y_2'(x') - y_2(x')y_1'(x')}. \tag{4}$$

The object in the denominator is called the **Wronksian** of y_1 and y_2. Of course, this still leaves the problem of solving the homogeneous equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0 \tag{5}$$

The Euler equation

For $a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0$ no general solution when the coefficients aren't constants. One important case that can be solved is Euler's equation.

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0 \tag{6}$$

where α, β, γ constants. This equation arises when studying Laplace's equation, the most important partial differential equation. Euler's equation is solved by transforming it into the constant coefficient case using a change of variable:

$$x = e^z \tag{7}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Based on notes I got from Chris Ford

Using

$$1 = \frac{dx}{dx} = \frac{d}{dx}e^z = \frac{dz}{dx}e^z \tag{8}$$

so $dz/dx = e^{-z}$ this gives

$$x\frac{dy}{dx} = e^z \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz} \tag{9}$$

and

$$\begin{aligned} x^2\frac{d^2y}{dx^2} &= x^2\frac{d}{dx}\frac{dy}{dx} = x^2\frac{d}{dx}\frac{1}{x}\frac{dy}{dz} \\ &= -\frac{dy}{dz} + x\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} \end{aligned} \tag{10}$$

so the Euler's equation becomes

$$\alpha\frac{d^2y}{dz^2} + (\beta - \alpha)\frac{dy}{dz} + \gamma y = 0 \tag{11}$$

which has constant coefficients. The auxiliary equation is

$$\alpha\lambda^2 + (\beta - \alpha)\lambda + \gamma = 0. \tag{12}$$

with general solution is

$$y_c = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} = C_1 x^{\lambda_1} + C_2 x^{\lambda_2} \tag{13}$$

where λ_1 and λ_2 are roots of the auxiliary equation. If $\lambda_1 = \lambda_2$ then

$$y_c = C_1 e^{\lambda_1 z} + C_2 z e^{\lambda_1 z} = C_1 x^{\lambda_1} + C_2 \log x x^{\lambda_1} \tag{14}$$

for $x \geq 0$.