**Note III.3**[1,2] 19 February 2008

## Green's functions

Returning to the general second order linear inhomogeneous equation

$$a(x)y''(x)+b(x)y'(x)+c(x)y(x)=f(x) \tag{1}$$

If two solutions, $y_1$ and $y_2$, of the homogeneous equation are known then a particular solution of the full equation, $y_p$, can be found by the Green's function method; this is outlined here. Consider the equation

$$a(x)y''(x)+b(x)y'(x)+c(x)y(x)=\delta(x-x'). \tag{2}$$

A solution of this equation is called a **Green's function** and denoted $G(x|x')$. A solution of the inhomogeneous equation is then

$$y_p(x)=\int_{-\infty}^{\infty} dx'\ f(x')\ G(x|x'). \tag{3}$$

The Green's function is not unique, but one formula for $G(x|x')$ is

$$G(x|x')=\theta(x-x')\frac{y_1(x)y_2(x')-y_2(x)y_1(x')}{y_1(x')y_2'(x')-y_2(x')y_1'(x')}. \tag{4}$$

The object in the denominator is called the **Wronksian** of $y_1$ and $y_2$. Of course, this still leaves the problem of solving the homogeneous equation

$$a(x)y''(x)+b(x)y'(x)+c(x)y(x)=0 \tag{5}$$

## The Euler equation

For $a(x)y''(x)+b(x)y'(x)+c(x)y(x)=0$ no general solution when the coefficients aren't constants. One important case that can be solved is Euler's equation.

$$\alpha x^2y''+\beta xy'+\gamma y=0 \tag{6}$$

where $\alpha$, $\beta$, $\gamma$ constants. This equation arises when studying Laplace's equation, the most important partial differential equation. Euler's equation is solved by transforming it into the constant coefficient case using a change of variable:

$$x=e^z \tag{7}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

[2]Based on notes I got from Chris Ford



Using

$$1=\frac{dx}{dx}=\frac{d}{dx}e^z=\frac{dz}{dx}e^z \tag{8}$$

so $dz/dx=e^{-z}$ this gives

$$x\frac{dy}{dx}=e^z\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz} \tag{9}$$

and

$$\begin{aligned} x^2\frac{d^2y}{dx^2} &= x^2\frac{d}{dx}\frac{dy}{dx}=x^2\frac{d}{dx}\frac{1}{x}\frac{dy}{dz} \\ &= -\frac{dy}{dz}+x\frac{d^2y}{dz^2}\frac{dz}{dx}=-\frac{dy}{dz}+\frac{d^2y}{dz^2} \end{aligned} \tag{10}$$

so the Euler's equation becomes

$$\alpha\frac{d^2y}{dz^2}+(\beta-\alpha)\frac{dy}{dz}+\gamma y=0 \tag{11}$$

which has constant coefficients. The auxiliary equation is

$$\alpha\lambda^2+(\beta-\alpha)\lambda+\gamma=0. \tag{12}$$

with general solution is

$$y_c=C_1e^{\lambda_1 z}+C_2e^{\lambda_2 z}=C_1x^{\lambda_1}+C_2x^{\lambda_2} \tag{13}$$

where $\lambda_1$ and $\lambda_2$ are roots of the auxiliary equation. If $\lambda_1=\lambda_2$ then

$$y_c=C_1e^{\lambda_1 z}+C_2ze^{\lambda_1 z}=C_1x^{\lambda_1}+C_2\log x x^{\lambda_1} \tag{14}$$

for $x\geq 0$.