Note III.2[1,2] 16 February 2008

Inhomogeneous 2nd order

Returning to the inhomogeneous form $ay''(x) + by'(x) + cy(x) = f(x)$ we still require a particular integral y_p, that is *one* solution of the full equation. If f is exponential then it is straightforward: y_p is an exponential proportional to f.

- **For example**: consider
$$y'' + 3y' + 2y = e^x. \tag{1}$$
Trying $y_p = Ce^x$ where C is a constant gives
$$(1 + 3 + 2)Ce^x = e^x \tag{2}$$
so
$$C = \frac{1}{6} \tag{3}$$
and, therefore,
$$y_p(x) = \frac{1}{6}e^x \tag{4}$$
is a particular integral. To obtain the general solution we require the general solution of the homogeneous equation, see earlier example, $y_c = C_1e^{-x} + C_2e^{-2x}$. Thus, the general solution is
$$y(x) = y_c + y_p = C_1e^{-x} + C_2e^{-2x} + \frac{1}{6}e^x \tag{5}$$

If f is a solution of the homogeneous ODE this doesn't work.

- **For example**: consider $y''+3y'+2y = e^{-x}$. Trying $y_p = Ce^{-x}$ gives $C = \infty$. Much as in the equal-root case for the homogeneous equation try $y_p = Cxe^{-x}$. Differentiating
$$\begin{aligned} y_p' &= Ce^{-x} - Cxe^{-x} \\ y_p'' &= -2Ce^{-x} + Cxe^{-x} \end{aligned} \tag{6}$$
Inserting these into the ODE gives
$$Cxe^{-x}(1 - 3 + 2) + Ce^{-x}(-2 + 3) = e^{-x} \tag{7}$$
so that $C = 1$ or $y_p = xe^{-x}$. The general solution can be written
$$y = (C_1 + x)e^{-x} + C_2e^{-2x} \tag{8}$$

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[2]Based on notes I got from Chris Ford

Now, for more general cases, we exploit the linearity. If f is a sum of exponentials, as, for example, $\sin x = (e^{ix} - e^{-ix})/(2i)$, just add up the corresponding particular integrals corresponding to each exponential term; so, for example, if

$$y'' + 3y' + 2y = \sin x. \tag{9}$$

We solve

$$y'' + 3y' + 2y = \frac{1}{2i}e^{ix} \tag{10}$$

and

$$y'' + 3y' + 2y = -\frac{1}{2i}e^{-ix} \tag{11}$$

and add the two solutions.

If f is not a finite sum of exponentials decompose f into complex exponentials: we use Fourier analysis.

- **Example with Fourier analysis**: Obtain the general solution of

$$y''(x) + 3y'(x) + 2y(x) = f(x) \tag{12}$$

where f is the periodic square wave

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} \tag{13}$$

with $f(x + 2\pi) = f(x)$. The complementary function is $y_c = C_1 e^{-x} + C_2 e^{-2x}$, as in earlier example. For the particular integral we expand f as a Fourier series, this we have done before;

$$f(x) = \frac{4}{\pi} \sum_{n \text{ odd } > 0} \frac{1}{n} \sin nx = \frac{2}{\pi i} \sum_{n \text{ odd } \in Z} \frac{1}{n} e^{inx} \tag{14}$$

Now, we find the particular integral for

$$y'' + 3y' + 2y = e^{inx} \tag{15}$$

Trying $y_p = Ce^{inx}$ gives

$$C(-n^2 + 3in + 2)e^{inx} = e^{inx} \tag{16}$$

so that

$$C = \frac{1}{-n^2 + 3in + 2} \tag{17}$$

Now, adding the individual particular integrals together to get the particular integral of full problem

$$y_p(x) = \frac{2}{\pi i} \sum_{n \text{ odd } \in Z} \frac{e^{inx}}{n(-n^2 + 3in + 2)} \tag{18}$$

If f is not periodic write it as a Fourier integral

- **Fourier integral example**: Find the general solution of the ODE

$$y''(x) + 2y'(x) + 2y(x) = f(x) \tag{19}$$

where $f(x) = e^{-x^2}$. The auxiliary equation is $\lambda^2 + 2\lambda + 2 = 0$ with two complex roots $\lambda = -1 \pm i$ so that

$$y_c = e^{-x}(A\cos x + B\sin x) \tag{20}$$

Write f as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k) \tag{21}$$

where, using the standard formula for the Fourier integral of a Gaußian,

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{-ikx}\ e^{-x^2} = \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}k^2} \tag{22}$$

Now, we obtain PI for $y'' + 2y' + 2y = e^{ikx}$. Trying $y = Ce^{ikx}$ gives

$$C(-k^2 + 2ik + 2)e^{ikx} = e^{ikx} \tag{23}$$

giving

$$C = \frac{1}{-k^2 + 2ik + 2}. \tag{24}$$

The particular integral of the whole problem is obtained by integrating over the particular integrals for the individual problems labelled by k:

$$y_p(x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} dk\ \frac{e^{ikx}e^{-k^2/4}(-k^2 + 2ik + 2)}{.} \tag{25}$$