Note III.1[1,2] 19 February 2007

Part III: ODEs

A **differential equation** is an equation involving derivatives. An **ordinary differential equation (ODE)** is a differential equation involving a function, or functions, of only one variable. If the ODE involves the nth (and lower) derivatives it is said to be an n**th order ODE.** Let y be a function of one variable x, for neatness, we will try to always use x as the dependent variable and prime for derivative. An equation of the form

$$h_1(x, y(x), y'(x)) = 0 \tag{1}$$

is a first order ODE.

$$h_2(x, y(x), y'(x), y''(x)) = 0 \tag{2}$$

is second order. A function satisfying the ODE is called **a solution** of the ODE.

Linear ODEs (2 types)

There are two types of linear ODEs

1. **Homogeneous**: If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where A and B are arbitrary constants.
2. **Inhomogeneous**: If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where $A + B = 1$.

where, obviously, the point is in a homogeneous equation, all the terms are y terms, whereas the inhomogeneous equation has an extra **forcing** term.

- **Homogeneous example:** The equation

  $$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

  is homogeneous, where $p(x)$ and $q(x)$ are some, given, functions of x. Now substituting $Ay_1 + By_2$ gives

  $$(Ay_1+By_2)''+p(Ay_1+By_2)'+q(Ay_1+By_2) = A(y_1''+py_1'+qy_1)+B(y_2''+py_2'+qy_2) = 0 \tag{4}$$

  when y_1 and y_2 are solutions.

- **Inhomogeneous example:** The equation

  $$y'' + p(x)y' + q(x)y = f(x) \tag{5}$$

  is homogeneous, where $p(x)$, $q(x)$ and $f(x)$ are some, given, functions of x. Now substituting $Ay_1 + By_2$ gives

  $$(Ay_1+By_2)''+p(Ay_1+By_2)'+q(Ay_1+By_2) = A(y_1''+py_1'+qy_1)+B(y_2''+py_2'+qy_2) = (A+B)f \tag{6}$$

  when y_1 and y_2 are solutions. Hence $Ay_1 + By_2$ is a solution is $A + B = 1$.

The general first order linear ODE, for a single function, can be written

$$a(x)y'(x) + b(x)y(x) = f(x) \tag{7}$$

where a, b and $f(x)$ are arbitrary functions. The equation is homogeneous if $f = 0$. A common standard form is write the equation as

$$y'(x) + p(x)y(x) = f(x) \tag{8}$$

where $p = b/a$ and f/a has been renamed back to f.

The general 2nd order linear ODE is

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x) \tag{9}$$

where a, b, c and f are arbitrary functions and the equation is homogeneous if $f = 0$. Again, another standard form is

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{10}$$

First order linear differential equations.

All solutions of

$$y'(x) + p(x)y(x) = f(x) \tag{11}$$

can be written

$$y(x) = Cy_1(x) + y_p(x) \tag{12}$$

where $y_1(x)$ is a solution of the **corresponding** homogeneous equation $y'(x)+p(x)y(x) = 0$ and $y_p(x)$ is one solution of the full equation. This can be demonstrated by explicit construction.

$$y'(x) + p(x)y(x) = f(x) \tag{13}$$

can be rewritten

$$\frac{d}{dx}e^{I(x)}y(x) = e^{I(x)}f(x) \tag{14}$$

where

$$I(x) = \int_a^x dz p(z). \tag{15}$$

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[2]Based partly on lecture notes taken by John Kearney

and, here, a is an arbitrary constant. Now, $I'(x) = p(x)$ and I is called an **integrating factor**. Integrate from a to x

$$e^{I(x)}y(x) - e^{I(a)}y(a) = \int_a^x dz e^{I(z)} f(z). \tag{16}$$

with $e^{I(a)} = 1$. This gives

$$y(x) = Cy_1(x) + y_p(x), \tag{17}$$

with $y_1(x) = e^{-I(x)}$, $y_p(x) = e^{-I(x)} \int_a^x dz e^{I(z)} f(z)$ and $C = y(a)$. In practise, this method will always find a solution, but, often, it is quicker just to stare at the equation and then guess a solution and check it works.

- **Example** Find all solutions of the ODE 1

  $$y'(x) + \frac{1}{x} y(x) = x^3. \tag{18}$$

  Here $p(x) = 1/x$ which has a non-integrable singularity at $x = 0$! Work with $x > 0$, or $x < 0$. First, the integrating factor $I(x) = \int dx p(x) = log x + c$. Set $c = 0$, or $a = 1$. $e^{I(x)} = x$ so that the ODE can be written

  $$\frac{d}{dx}(xy) = x^4. \tag{19}$$

  Integrating gives $xy = \frac{1}{5}x^5 + C$ or $y = \frac{1}{5}x^4 + C/x$, that is $y_1(x) = 1/x$, $y_p(x) = \frac{1}{5}x^4$.

Second order case

All solutions, or the **general solution** of

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{20}$$

are given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x) \tag{21}$$

where y_1, y_2 are linearly independent solutions of the **corresponding** homogeneous equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{22}$$

and $y_p(x)$ is a solution of the full equation. C_1 and C_2 are arbitrary constants. This isn't proved here, but it is easy to understand why it would be the case: this is a second order equation so it nears to arbitrary constant, in the initial value problem, one matches $y(0)$ and the other $y'(0)$. Now, if you have a solution, adding a solution of the corresponding homogeneous problem gives you another solution and the homogeneous problem also has a two-dimensional space of solutions, so it all mathes up. $y_p(x)$ is called a **particular integral**. The general solution is sometimes written

$$y(x) = y_c(x) + y_p(x) \tag{23}$$

where $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ is called the **complementary function**. It is the general solution of the homogeneous form of the ODE.

Constant Coeffcients

We now consider the special case where the coefficients a, b and c are constants

$$ay''(x) + by'(x) + cy(x) = f(x). \tag{24}$$

This type of equation has a nice interpretation as a **damped/driven oscillator** where we will use t instead of x as the variable, since it is time. y is the displacement from equilibrium. Recall the equation for a simple harmonic oscillator

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) \tag{25}$$

Now add in a damping force proportional to the velocity dy/dt and a driving force $f(t)$, which may be periodic or non-periodic,

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) - \gamma \frac{dy(t)}{dt} + d(t) \tag{26}$$

which is a linear ODE with constant coeffcents.

So, back to the general constant coefficient form with x as the variable, the first step in solving ODEs of this type is to find two solutions of the homogeneous equation

$$ay''(x) + by'(x) + cy(x) = 0. \tag{27}$$

This equation has simple exponential solutions of the form $y(x) = e^{\lambda x}$. Differentiating $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$ so that

$$ay''(x) + by' + cy = (a\lambda^2 + b\lambda + c)y \tag{28}$$

which is zero provided

$$a\lambda^2 + b\lambda + c = 0. \tag{29}$$

This is called an **auxiliary equation**. Thus $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ where λ_1 and λ_2 are roots of the quadratic auxiliary equation. The complementary function, if $\lambda_1 \neq \lambda_2$, is $y_c(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

If $\lambda_1 = \lambda_2$ we only have one exponential solution. In this case a second solution of the ODE is $y(x) = xe^{\lambda_1 x}$ and $y_c(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$. In the oscillator model this special case corresponds to critical damping. This trick is justified by the fact it works; there are ways to derived it, for example, by converting the equation into two first order equations using $y_1 = y$ and $y_2 = y'$ and then diagonalizing the corresponding matrix equation and solving using an integrating factor. In practise, the easiest thing is to keep adding powers of x until you have two solutions.

- **Example**: $y'' + 3y' + 2y = 0$ has auxiliary equation $\lambda^2 + 3\lambda + 2 = 0$ with roots $\lambda_1 = 1$, $\lambda_2 = 2$ so the general solution is

  $$y(x) = C_1 e^x + C_2 e^{2x} \tag{30}$$

  This corresponds to over damping.

- **Example**: $y'' + 2y' + y = 0$ has auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$ with two equal roots $\lambda = 1$ and so the general solution is

$$y(x) = (C_1 + C_2 x)e^x \tag{31}$$

- **Example**: If the auxiliary equation $\lambda^2 + \lambda + 1 = 0$ with complex roots $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$ the general complex solution is

$$y(x) = C_1 e^{-\frac{1}{2}x + i\frac{1}{2}\sqrt{3}x} + C_2 e^{-\frac{1}{2}x - i\frac{1}{2}\sqrt{3}x} \tag{32}$$

where C_1 and C_2 are complex constants. The general real solution can be obtained by imposing the constraint $C_2 = \bar{C}_1$:

$$y(x) = e^{-\frac{1}{2}x}\left[C_1\left(\cos\frac{1}{2}\sqrt{3}x + i\sin\cos\frac{1}{2}\sqrt{3}x\right) + \text{ c.c.}\right] \tag{33}$$

Writing $C_1 = \frac{1}{2}(A - iB)$ where A and B are real constants gives

$$y(x) = e^{-\frac{1}{2}x}\left(A\cos\frac{1}{2}\sqrt{3}x + B\sin\frac{1}{2}\sqrt{3}x\right) \tag{34}$$

this is the underdamped case, it still oscillates.