Note II.1[1] 30 January 2007

PART II Fourier Analysis

The Fourier series

First some terminology: a function $f(x)$ is **periodic** if $f(x+l)=f(x)$ for all x for some l, if l is the smallest such number, it is called the **period** of $f(x)$. It is even if $f(-x)=f(x)$, for all x and odd if $f(-x)=-f(x)$, again, for all x. $\sin x$, $\cos x$, $\sin 2x$, $\sin 3x$ and so on are examples of periodic functions: $\sin nx$ has period $2\pi/n$

Now, consider $\sin^3 x$, this is clearly periodic with periodic with period 2π: $\sin^3(x+2\pi)=\sin^3 x$. Using the usual trigonometric identities, or otherwise, it can be shown that

$$\sin^3 x = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x \tag{1}$$

In short, $\sin^3 x$ can be re-expressed in terms of sines. In fact, this is a much more common property than you might expect, the theory of Fourier series tells us that if $f(x)$ is odd and periodic with period 2π then there are b_ns such that

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \tag{2}$$

If it is even it has a cosine series instead

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos nx \tag{3}$$

where the half before a_0 is a standard convention, we will see soon why it is convenient. Anyway, these are **Fourier series**.

More generally, a periodic function $f(x)$ with period l has Fourier series

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos\frac{2\pi nx}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{2\pi nx}{l} \tag{4}$$

Leaving aside, for now, issues of convergence, it is easy to calculate what values the a_n and b_n must have. First, integrating both sides gives

$$\int_{-l/2}^{l/2} dx f(x) = \frac{l}{2}a_0 + \sum_{n=1}^{\infty} a_n \int_{-l/2}^{l/2} dx \cos\frac{2\pi nx}{l} + \sum_{n=1}^{\infty} b_n \int_{-l/2}^{l/2} dx \sin\frac{2\pi nx}{l} = \frac{l}{2}a_0 \tag{5}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

where I have assumed I can bring the integrals into the sum signs, the sines and cosines both integrate to zero: sine and cosine integrate to zero if integrated over a whole number of periods and $\cos 2n\pi/l$ and $\sin 2n\pi/l$ have period l/n. This means that

$$a_0=\frac{2}{l}\int_{-l/2}^{l/2} f(x)dx \tag{6}$$

In fact, the method for calculating the other coefficients is not too different; we multiply across by a sine or cosine and then integrate using the formulae

$$\begin{aligned}
\int_{-l/2}^{l/2} dx \sin\frac{2\pi mx}{l}\sin\frac{2\pi nx}{l} &= \frac{l}{2}\delta_{mn}\\
\int_{-l/2}^{l/2} dx \cos\frac{2\pi mx}{l}\cos\frac{2\pi nx}{l} &= \frac{l}{2}\delta_{mn}\\
\int_{-l/2}^{l/2} dx \sin\frac{2\pi mx}{l}\cos\frac{2\pi nx}{l} &= 0
\end{aligned} \tag{7}$$

which can be proved, for example, by writing the trigonometric functions in terms of complex exponentials. Hence, multiplying across by $\cos 2\pi mx/l$ and integrating, we get

$$\begin{aligned}
\int_{-l/2}^{l/2} dx f(x)\cos\frac{2\pi mx}{l} &= \frac{1}{2}\int_{-l/2}^{l/2} dx a_0 \cos\frac{2\pi mx}{l}\\
&\quad+\sum_{n=1}^{\infty} a_n \int_{-l/2}^{l/2} dx \cos\frac{2\pi nx}{l}\cos\frac{2\pi mx}{l}\\
&\quad+\sum_{n=1}^{\infty} b_n \int_{-l/2}^{l/2} dx \sin\frac{2\pi nx}{l}\cos\frac{2\pi mx}{l}\\
&= \frac{l}{2}a_m
\end{aligned} \tag{8}$$

so, using this and a similar calculation for sine, we get

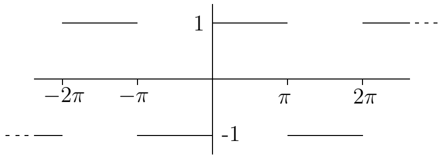
$$\begin{aligned}
a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x)\cos\frac{2\pi mx}{l}\\
b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x)\sin\frac{2\pi mx}{l}
\end{aligned} \tag{9}$$

where the first equation holds for $n\geq 0$ and the second for $n>0$. It is to have all the a_n obey the same general expression that there is the convention to put the half is put in front of the a_0. As a point of terminology, the a_n and b_n are called **Fourier coefficients** and the sines and cosines, or sometimes the sines and cosine along with their coefficient, are called **Fourier modes**.

- **Example:** Consider the block wave with period $l = 2\pi$ (Picture II.1.1)

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} \tag{10}$$

with $f(x + 2\pi) = f(x)$.



So

$$a_n = \frac{1}{\pi}\int_{-l/2}^{l/2} dx f(x)\cos nx = 0 \tag{11}$$

because the integrand is odd, and

$$b_n = \frac{1}{\pi}\int_{-l/2}^{l/2} dx f(x)\sin nx = \frac{2}{\pi}\int_0^\pi dx \sin nx = -\left.\frac{2\cos nx}{n\pi}\right|_0^\pi = \frac{2}{\pi n}[1-(-1)^n] \tag{12}$$

where we have used $\cos n\pi = (-1)^n$. Hence

$$f(x) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nx \tag{13}$$

This series is not obviously convergent; the point of Fourier series is that there are theorems to tell us it is. However, there are particular values of x where we can see that the answer is correct, for example, at $x = \pi/2$, we have $\sin{(2m+1)\pi/2} = (-1)^m$ where m is an integer so $2m - 1$ is odd. Putting this back into the series gives

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\dots\right) \tag{14}$$

and the right hand side can be derived by Taylor expanding $\tan^{-1} x$. It is interesting to note that the series as written, up to 1/9 gives $1 \approx 1.06$; the Fourier series gives workable but not efficient approximations and its importance is not in its ability to approximate functions with high numerical accuracy, rather, it quickly captures features of the function, preserving its periodicity and encoding its behaviour at lengths scales bigger than l/n, where n is where the series is truncated. Another interesting thing to look at is the behaviour at $x = 0$ where the function is discontinuous. Since all the sines are zero, the Fourier series gives zero at $x = 0$. This interpolates the discontinuity. This is a feature of the Fourier series, the series does not see what happens at individual points and interpolates over any finite discontinuities. A graph of the Fourier series is given in Note 3.

There are lots of versions of the theorem which tells us the Fourier series exists, different versions impose different conditions on the function and have convergence properties for the series; the version we quote is actually quite vague about the convergence and pretty restrictive on the function and we will call it **Dirichlet's Theorem**: If f is periodic and has, in any period, a finite number of maxima and minima and a finite number of discontinuities and $\int_{-l/2}^{l/2}|f(x)|^2dx$ is finite then the Fourier series converges and converges to $f(x)$ at all points where $f(x)$ is continuous. At a point a where $f(x)$ is discontinuous it converges to

$$\frac{1}{2}\left[\lim_{x\to a+} f(x) + \lim_{x\to a-} f(x)\right] \tag{15}$$

One annoying thing about Dirichlet's theorem, as quoted, is that it appears to exclude the block wave used in the example, the block wave doesn't have a finite number of maxima and minima, obviously this isn't the sort of function the statement is trying to exclude, it is aimed at functions that oscillate infinitely fast. To fix it you could extend Dirichlet's theorem to functions $f(x)$ such that there is a function $g(x)$, satisfying the properties described by the theorem, such that $g(f(x))$ has the properties required by the theorem.

Complex Fourier series

As often happens, apart from the slight inconvenience of being complex, complex Fourier series are more straightforward than real ones, there is only once type of Fourier coefficient, c_n, instead of three, a_0, a_n and b_n for the real series. It is easy to see the existence of a complex exponential series follows from the existence of the sine and cosine series, just replace

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{e^{-ix} - e^{-ix}}{2i} \end{aligned} \tag{16}$$

to get a series of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/l}. \tag{17}$$

Rather than try to work out the formula for the c_n from the formulas for the a_n and b_n, we can just take this as a series for $f(x)$ and calculate the c_n by a similar trick to the one we used before, we multiply across by $\exp(-2\pi i m x/l)$ and integrate

$$\int_{-l/2}^{l/2} dx e^{-2\pi i m x/l} f(x) = \sum_{n=-\infty}^{\infty} c_n \int_{-l/2}^{l/2} e^{2\pi i(n-m)x/l} \tag{18}$$

and use

$$\int_{-l/2}^{l/2} dx e^{2\pi i(n-m)x/l} = l\delta_{nm} \tag{19}$$

which is clear if you note the integrand is one for $n = m$ and otherwise, it is easy to see from

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{20}$$

that it integrates to zero. This means that

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) e^{-2\pi i n x/l} \tag{21}$$

It is interesting to ask what the consequence of $f(x)$ being real is on the c_n, using a star to mean the complex conjugate lets take the complex conjugate of this equation, using $f*(x) = f(x)$:

$$c_n^* = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) e^{2\pi i n x/l} = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) e^{-2\pi i(-n)x/l} = c_{-n} \tag{22}$$

- **Example:** It is easy to redo the last by integrating; since we have already done the integrations when working out the b_n's, we will use the previous real series to work out the Fourier coefficients for the complex series, so,

$$f(x) = \frac{4}{\pi}\sum_{n>0 \text{ and odd}} \frac{1}{n}\sin nx = \frac{2}{\pi}\sum_{n>0\text{ and odd}} \frac{1}{n}\left(e^{inx} - e^{-inx}\right) = \frac{2}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{inx} = \tag{23}$$

so

$$c_n = \begin{cases} 2/(\pi i n) & n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

- **Example:** Consider $f(x) = e^x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$. So,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx e^{-inx} e^x = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx e^{(1-in)x} \\ &= \frac{e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}}{2\pi(1-in)} = (-1)^n \frac{e^{\pi} - e^{-\pi}}{2\pi(1-in)} \\ &= \frac{\sinh\pi}{\pi}\frac{(-1)^n}{1-in} \end{aligned} \tag{25}$$

and so

$$f(x) = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1-in} e^{inx} \tag{26}$$

At $x = 0$ this gives the amusing formula

$$1 = \frac{\sinh\pi}{\pi}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{1-in} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+in}\right) = \frac{\sinh\pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} \tag{27}$$

where the $n = 1$ terms cancel the one.

Parseval's Theorem

Parseval's theorem is a relation between the L^2 size of $f(x)$ and the Fourier coefficients:

$$\frac{1}{l}\int_{-l/2}^{l/2} |f(x)|^2 dx = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{28}$$

or for the complex series

$$\frac{1}{l}\int_{-l/2}^{l/2} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{29}$$

This theorem is very impressive, it relates a natural measure for the size of the function on the space of periodic functions to the natural measure for the size of an infinite vector on the space of coefficients. It is easy to prove and convenient too for the complex series

$$\int_{-l/2}^{l/2} dx f(x) f^*(x) = \sum_{m,n} c_n c_m^* \int_{-l/2}^{l/2} dx e^{2\pi i(n-m)x/l} = \sum_{m,n} c_n c_m^* \delta_{nm} = l\sum_n |c_n|^2. \tag{30}$$

- **Example:** So, going back to the block wave example, it is easy to check that

$$\frac{1}{2\pi}\int_{-l/2}^{l/2} dx |f(x)|^2 = 1 \tag{31}$$

so

$$1 = \frac{8}{\pi^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots\right). \tag{32}$$