Note I.5[1] 4 October

Surface integrals

Consider a two-dimensional surface S embedded in a three-dimensional space (Picture I.5.1) with $\mathbf{F}$ a vector field defined in the domain which contains S. Now, consider approximating the surface with small flat pieces, for each piece we construct a vector $\delta\mathbf{A}$ which is normal to the surface and whose magnitude is the area of the piece. Now a scalar can be formed by adding

$$\sum \mathbf{F}\cdot\delta\mathbf{A} \tag{1}$$

where the sum is taken over all the small pieces $\delta\mathbf{A}$ and in the sum the field is evaluated at the center of the piece. Roughly speaking, the surface integral

$$\int_S \mathbf{F}\cdot\mathbf{dA} \tag{2}$$

is the infinitesimal limit of this sum, it is the integral over the surface of the projection of $\mathbf{F}$ onto the normal. As with the line integral, with a bit of care, this rough description can be turned into a definition, but we don't do that here. Physically surface integrals measure net flow of a fluid or the net electric or magnetic flux through a surface.

It is important to note that we have **oriented** the surface by choosing a direction for the normal; at any point in a surface there are two possible normals and a surface is **orientable** if it is possible to smoothly choose one of these two possible normals over all the surface. It is not easy here to define what we mean by smoothly choose, but it is easy to to explain, we mean that the normal doesn't hop from one side to the next going from one point to a nearby point. The usual example of an unorientable surface is the Moebius strip and this is illustrated in Picture I.5.2.

To compute surface integrals convert they are usually converted into a standard two-dimensional integral using a parametric representation for the surface

$$\begin{aligned} x &= x(u,v)\\ y &= y(u,v)\\ z &= z(u,v) \end{aligned} \tag{3}$$

or $\mathbf{x}=\mathbf{x}(u,v)$ where (u,v) belongs to some domain D in $\mathbf{R}^2$. Using the same approach as for the line integrals we will compute the element of area $\delta\mathbf{A}$ corresponding to small variations in u and v. By expanding $\mathbf{x}(u+\delta u,v)$ and $\mathbf{x}(u,v+\delta v)$ using the Taylor expansion we find that to leading order the area element is a parallelogram with sides

$$\begin{aligned} \mathbf{a} &= \frac{\partial\mathbf{x}}{\partial u}\delta u\\ \mathbf{b} &= \frac{\partial\mathbf{x}}{\partial v}\delta v \end{aligned} \tag{4}$$

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Now the area of the parallelogram is $|\mathbf{a}\times\mathbf{b}|$ and the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ and hence to the area element so

$$\delta\mathbf{A} = \mathbf{a}\times\mathbf{b} = \frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}\delta u\delta v \tag{5}$$

From this we conclude that the surface integral has parametric form

$$\int_S \mathbf{F}\cdot dA = \int_D dudv\mathbf{F}(\mathbf{x})\cdot\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v} \tag{6}$$

where the integrand depends on U and v through the parameterization $\mathbf{x} = \mathbf{x}(u, v)$.

- **Example**: Consider the flux of $\mathbf{F} = xy\mathbf{k} + z\mathbf{i}$ through the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 2, 0)$ (Picture I.5.4) and with orientation upwards, in the z-direction. The surface is parameterized by $x(u, v) = u$ with $0 \leq u \leq 1$, $y(u, v) = v$ with $0 \leq v \leq 2(1-u)$ and $z = 0$. Hence

$$\begin{aligned}\mathbf{x} &= u\mathbf{i} + v\mathbf{j}\\ \frac{\partial\mathbf{x}}{\partial u} &= \mathbf{i}\\ \frac{\partial\mathbf{x}}{\partial v} &= \mathbf{j}\end{aligned} \tag{7}$$

and so

$$\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v} = \mathbf{k} \tag{8}$$

and

$$\mathbf{F}(\mathbf{x})\cdot\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v} = xy = uv \tag{9}$$

so

$$\int_S \mathbf{F}\cdot\mathbf{dA} = \int_0^1\int_0^{2(1-u)} dvuv = \frac{v}{1}2\int_0^1 duu[2(1-u)]^2 = \frac{1}{6} \tag{10}$$

A vector field can also be integrated over a closed surface. On a closed surface the choice of orientation is a choice between an inwards or outward pointing normal vector.

- **Example**: Compute the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of the sphere $x^2 + y^2 + z^2 = a^2$. So, we can parameterize the sphere with the polar and azimuthal angles (θ, ϕ). However, rather than calculating the normal to the parameterized surface using the cross product rule above, it is easier just to note that the normal to a sphere is a radial line and so $\mathbf{F}\cdot\hat{\mathbf{n}} = a$ and $\mathbf{dA} = \hat{\mathbf{n}}dA$, hence

$$\int_S \mathbf{F}\cdot\mathbf{dA} = a\times\text{surface area} = 4\pi a^3 \tag{11}$$

The integral theorems

There are a number of important theorems relating multi-dimensional integration and the operators of vector calculus. These are all basically consequences of the Fundamental Theorem of Calculus and can be thought of as higher-dimensional prescriptions for integration by parts. Their proofs, though not difficult, tend to quite involved and they will only be sketched here.

The Stokes Theorem

Let S be a piecewise smooth orientable surface bounded by a piecewise smooth curve C (Picture I.5.5). The orientations of S and C are chosen such that at the edge of the surface $\mathbf{n}\times\delta\mathbf{l}$ points into the surface. (Picture I.5.6). Let $\mathbf{F}$ be a continuously differentiable vector field defined in some domain containing S then

$$\int_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dA} = \oint_C \mathbf{F}\cdot\mathbf{dl} \tag{12}$$

Sometimes the notation ∂S is used for the correctly oriented boundary of S. Before discussing the proof of the Stokes Theorem, we will first look at Green's Theorem. Green's Theorem can be viewed as Stokes' Theorem for flat surfaces and Green's Theorem is used to prove Stokes' Theorem.

Green's Theorem in the Plane

Let D be a region in the xy-plane bounded by a piecewise smooth curve C. If $f(x,y)$ and $g(x,y)$ have continuous first derivatives

$$\int_D dA\left(\frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}f(x,y)\right) = \int_C \left(f(x,y)dx + g(x,y)dy\right) \tag{13}$$

with C oriented anti-clockwise. It is easy to check that Stokes' Theorem for a flat surface, taken without loss of generality to lie in the xy-plane and oriented upwards, reduces to Green's Theorem with $F_1(x,y,0)$ identified with $f(x,y)$ and $F_2(x,y,0)$ identified with $g(x,y)$, F_3 does not enter since $\mathbf{dl}$ is perpendicular to $\mathbf{k}$, the normal.

To prove Green's Theorem we first consider a **simple** region, D where the integral over D can be written as an iterated Cartesian integral in any order, (Picture I.5.8), so the integral of a scalar field ϕ can be written as

$$\int_D dA\phi = \int_a^b dx\int_{c(x)}^{d(x)} dy\phi = \int_c^d dy\int_{a(y)}^{b(y)} dx\phi \tag{14}$$

So, now compute

$$\int_D \frac{\partial f}{\partial y}dA \quad = \quad \int_a^b dx\int_{c(x)}^{d(x)} dy\frac{\partial f}{\partial y}$$

$$
\begin{aligned}
&= \int_a^b dx[f(x,d(x)) - f(x,c(x))] \\
&= -\oint_C dx f(x,y) \qquad (15)
\end{aligned}
$$

where we have used to the Fundamental Theorem of Calculus to get the second equals and in the last line we have put the upper, $f(x,d(x))$, and lower, $f(x,c(x))$ together into a anti-clockwise closed contour integral. Using the opposite order of integration we get

$$
\int_D dA \frac{\partial g}{\partial x} = \oint dy g(x,y) \qquad (16)
$$

and the theorem follows as the difference of these two.

A **regular region** is a non-simple region that can be split into simple parts. As illustrated in Picture I.5.9 the boundary contribution from shared boundaries cancels and so the formula for a regular region is the sum of the formula for simple regions. It is also clear that the integral for an internal closed boundary curve needs to be taken *clockwise*.

Proving the Stokes Theorem

Like Green's Theorem, Stokes' Theorem is proved by building the general case out of a particular special case where the theorem reduces to something already known, in Green's Theorem this was the Fundamental Theorem of Calculus and here, it will be Green's Theorem. Consider a vector field of the form $\mathbf{F} = F_3(x,y,z)\mathbf{k}$, so $F_1 = F_2 = 0$. Assume S is of the form $z = h(x,y)$ with (x,y) in some domain D in the xy-plane (Picture I.5.10). Now compute $\int_S \text{curl}\,\mathbf{F}\cdot\mathbf{dA}$ with the upwards orientation.

$$
\text{curl}\,\mathbf{F} = \partial_y F_3 \mathbf{i} - \partial_x F_3 \mathbf{j} \qquad (17)
$$

The surface is parameterized by

$$
\mathbf{x} = x\mathbf{i} + y\mathbf{j} + h(x,y)\mathbf{k} \qquad (18)
$$

and so

$$
\begin{aligned}
\frac{\partial \mathbf{x}}{\partial x} &= \mathbf{i} + \frac{\partial h}{\partial x}\mathbf{k} \\
\frac{\partial \mathbf{x}}{\partial y} &= \mathbf{j} + \frac{\partial h}{\partial y}\mathbf{k} \qquad (19)
\end{aligned}
$$

and hence

$$
\frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y} = \mathbf{k} - \frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j} \qquad (20)
$$

giving

$$
\text{curl}\,\mathbf{F}\cdot\frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y} = -(\partial_y F_3)\frac{\partial h}{\partial x} + (\partial_x F_3)\frac{\partial h}{\partial y} = -\frac{\partial}{\partial y}\left(F_3\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial x}\left(F_3\frac{\partial h}{\partial y}\right) \qquad (21)
$$

where, for the last line you need the cross terms to cancel, taking care to account for the two ways F depends on x: explicitly and through the dependence of h, so that

$$\frac{\partial}{\partial x}\left(F_3\frac{\partial h}{\partial y}\right) = \partial_x F_3\frac{\partial h}{\partial y} + \partial_z F_3\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} + F_3\frac{\partial^2 h}{\partial x\partial y} \tag{22}$$

Now

$$\int_S \mathrm{curl}\mathbf{F}\cdot\mathbf{dA} = \int_D dA\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \tag{23}$$

where

$$\begin{aligned} f(x,y) &= F_3(x,y,h(x,y))\frac{\partial h(x,y)}{\partial x} \\ g(x,y) &= F_3(x,y,h(x,y))\frac{\partial h(x,y)}{\partial y} \end{aligned} \tag{24}$$

Thus, we can apply Green's Theorem

$$\begin{aligned} \int_S \mathrm{curl}\mathbf{F}\cdot\mathbf{dA} &= \int_{C'=\partial D}\left(F_3(x,y,h(x,y))\frac{\partial h(x,y)}{\partial x}dx + F_3(x,y,h(x,y))\frac{\partial h(x,y)}{\partial y}dy\right) \\ &= \int_{C=\partial S} F_3 dz \end{aligned} \tag{25}$$

since

$$dz = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy \tag{26}$$

Finally $F_3dz = \mathbf{F}\cdot\mathbf{dl}$ because $\mathbf{F} = (0,0,F_3)$. This shows the theorem holds for $\mathbf{F} = F_3\mathbf{k}$ and S of the form $z = h(x,y)$. For more general S, split S up into a finite number of surfaces $\{S_1, S_2, \ldots, S_n\}$ that have the simple form (Picture I.5.11). Now if $\mathbf{F} = F_1\mathbf{i}$ or $\mathbf{F} = F_2\mathbf{j}$ a similar proof works using y and z or, respectively, x and z to parameterize the surface. Adding up these three results gives the result for a general vector field.

Applications of Stokes' Theorem

1. **Scalar potential** In a simply connect region $\mathrm{curl}\,\mathbf{F} = 0$ implies that $\mathbf{F}$ is conservative. To see this take any close curve C in the region. Since the region D is simply connected their is a surface S in D whose boundary is C. Since $\mathrm{curl}\,\mathbf{F} = 0$ on $S \subset D$ Stokes' Theorem implies that

$$\oint_C \mathbf{F}\cdot\mathbf{dl} = 0 \tag{27}$$

   In a connected domain this is equivalent to $\mathbf{F}$ being conservative.

2. **Area of a plane region**. Let D be a region in the xy-plane with $C = \partial D$. Apply Green's Theorem to the functions $f = y$ and $g = 0$ to get

$$\mathrm{Area} = \int_D dA = -\oint_C y dx \tag{28}$$

In a similar way

$$\text{Area} = \oint_C x dy \tag{29}$$

Centroid integrals can also be written as line integrals, these will be given on a problem sheet.

3. **Cauchy's Theorem**, an important theorem in complex analysis is a consequence of Green's Theorem.