Note I.4[1] 4 October

Line and surface integrals

For a vector field there are *natural* ways of integrating over one and two-dimensional subspaces of $\mathbf{R}^3$ to get a number, rather than a vector. These are line and surface integrals.

Line integrals

Consider two points P_1 and P_2 joined by a smooth or piecewise smooth curve C (Picture I.4.1). A small segment of C can be represented by a vector $\delta\mathbf{l}$, meaning that for two proximate points on the curve at $\mathbf{x}_1$ and $\mathbf{x}_2$ with $\delta\mathbf{l} = \mathbf{x}_2 - \mathbf{x}_1$ then all the points on the curve between $\mathbf{x}_1$ and $\mathbf{x}_2$ are close to the straight line $\mathbf{x}_1 + t\delta\mathbf{l}$ where $0 \leq t \leq 1$. Anyway, the idea of the line integral is that it is the limit of the sum

$$\mathcal{L} = \sum_{k=0\cdots N-1} \mathbf{F}(\mathbf{x}_k)\cdot\delta\mathbf{l}_k \tag{1}$$

where $\mathbf{x}_0 = P_1$, $\mathbf{x}_N = P_2$, the other x_k are intermediate points on the curve and $\nabla\mathbf{l}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$ where the limit is the infinitesimal limit where N becomes infinite and *all* the lengths of the $\nabla\mathbf{l}$ go to zero. With a bit of effort and a lot of fiddling, this can be made into a rigorous definition, but the important idea is that the line integral

$$\int_C \mathbf{F}\cdot\mathbf{dl} \tag{2}$$

is the integral along the curve of the projection of $\mathbf{F}$ onto the tangent. Note that this definition *orients* C, reversing the orientation reverses the sign of the integral.

The obvious physical example is work against a force: the work done moving a particle from P_1 to P_2 along the curve C against a position dependent force $\mathbf{F}(x,y,z)$ is the line integral $\int_c \mathbf{F}\cdot\mathbf{dl}$.

In practise the line integral is usually calculated using a parametric form of the formula. Suppose the points on C are given by $\mathbf{x}(u)$ where u is a parameter, a real number, and it runs from a to b so $\mathbf{x}(a) = P_1$ and $\mathbf{x}(b) = P_2$. In other words there is a map

$$\begin{array}{rcl} [a,b] & \hookrightarrow & \mathbf{R}^3 \\ u & \to & \mathbf{x}(u) \end{array} \tag{3}$$

Now, by Taylor,

$$\mathbf{x}(u+\delta u) \approx \mathbf{x}(u) + \frac{d\mathbf{x}}{du}\delta u \tag{4}$$

so we can identify

$$\delta\mathbf{l} \leftrightarrow \frac{d\mathbf{x}}{du}\delta u \tag{5}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

and can conclude that

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \int_a^b du \mathbf{F}(\mathbf{x}(u))\cdot\frac{d\mathbf{x}(u)}{du} \tag{6}$$

- **Example** Integrate the vector field

$$\mathbf{F} = \frac{1}{2}y\mathbf{i} - \frac{1}{2}x\mathbf{j} \tag{7}$$

over the semi-circular arc of unit radius in the $z = 0$ plane. (Picture I.4.2). So, to get a parameterization of the curve take

$$\begin{aligned} x(u) &= \cos u \\ y(u) &= \sin u \\ z(u) &= 0 \end{aligned} \tag{8}$$

with $0 \le u \le \pi$. Now,

$$\frac{d\mathbf{x}(u)}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j} \tag{9}$$

and substituting for x and y in the formula for $\mathbf{F}$ we get

$$\mathbf{F} = \frac{1}{2}\sin u\mathbf{i} - \frac{1}{2}\cos u\mathbf{j} \tag{10}$$

so that

$$\mathbf{F}\cdot\frac{d\mathbf{x}(u)}{du} = -\frac{1}{2}\sin^2 u - \frac{1}{2}\cos^2 u = -\frac{1}{2} \tag{11}$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{l} = -\frac{1}{2}\int_0^\pi du = -\frac{\pi}{2} \tag{12}$$

Conservative vector fields and path independence

- **Definition**: A vector field is called **conservative** if it is the gradient of a scalar field, so $\mathbf{F}$ is conservative if $\mathbf{F} = \nabla\phi$ for some ϕ.

If curl$\mathbf{F} \neq 0$ then $\mathbf{F}$ cannot be conservative, however, the converse need not hold.

- **Definition**: A vector field is called **path independent** if the line integral between any two points is the same for any path.

Any conservative field is path-independent: choose any smooth curve joining points P_1 and P_2 parameterized by $u \in [a, b]$, then

$$\mathbf{F}\cdot\frac{d\mathbf{x}}{du} = \frac{\partial\phi}{\partial x}\frac{dx}{du} + \frac{\partial\phi}{\partial y}\frac{dy}{du} + \frac{\partial\phi}{\partial z}\frac{dz}{du} = \frac{d\phi(\mathbf{x}(u))}{du} \tag{13}$$

so by the Fundamental Theorem of Calculus

$$\int_C \mathbf{F}\cdot d\mathbf{l} = \phi(\mathbf{x}(b)) - \phi(\mathbf{x}(a)) \tag{14}$$

and this answer does not depend on the path.

Now, for a conservative field, let C_a and C_b be two curves with the same endpoints P_1 and P_2 (Picture I.4.3). Since a conservative field is path independent,

$$\int_{C_a} \mathbf{F}\cdot d\mathbf{l} = \int_{C_b} \mathbf{F}\cdot d\mathbf{l} \tag{15}$$

Now consider the closed curve $C = C_a - C_b$ where the minus in C_b means we have reversed the orientation,

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \int_{C_a} \mathbf{F}\cdot d\mathbf{l} - \int_{C_b} \mathbf{F}\cdot d\mathbf{l} = 0 \tag{16}$$

and for any closed curve C

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = 0 \tag{17}$$

- **Example**: Back to the previous example of the semicircle. It is easy to extend the calculation to the full closed circle to show

  $$\oint_C \mathbf{F}\cdot d\mathbf{l} = -\frac{1}{2}\int_0^{2\pi} du = -\pi \tag{18}$$

  so

  $$\mathbf{F} = \frac{1}{2}y\mathbf{i} - \frac{1}{2}x\mathbf{j} \tag{19}$$

  cannot be conservative. This is consistent with $\text{curl}\mathbf{F} = -\mathbf{k} \neq 0$.

In fact, for a continuous vector field $\mathbf{F}$ in an open and connected domain D, the following are equivalent

1. $\mathbf{F}$ is conservative.
2. $\oint_C \mathbf{F}\cdot d\mathbf{l} = 0$ for all closed paths in D
3. $\mathbf{F}$ is path independent.

We have already seen that (2) and (3) are equivalent and that (1) implies (3), to finish, then, we need only prove (3) implies (1). Let P be any point in D and let

$$\phi(\mathbf{x}) = \int_{C(P,\mathbf{x})} \mathbf{F}\cdot d\mathbf{l} \tag{20}$$

where $C(P, \mathbf{x})$ is any curve joining P and $\mathbf{x}$. Since the line integral is path independent, ϕ is uniquely defined. Now, we want to show that $\mathbf{F} = \nabla\phi$. Again, the result is path independent, so, to prove

$$F_1 = \partial_x \phi \tag{21}$$

we use a path that goes from P to $P' = (x', y, z)$ where P' is chosen so that the straight line segment from P' to (x, y, z) is in D (Picture I.4.4). Now

$$\phi(\mathbf{x}) = \int_{C(P,P')} \mathbf{F} \cdot \mathbf{dl} + \int_{x_1}^{x} \mathbf{F} \cdot \mathbf{dl} \tag{22}$$

so

$$F_1 = \partial_x \phi \tag{23}$$

The other components follow by a similar trick.

If D is **simply connected** all loops are contractile (Picture I.4.5). In this case curl$\mathbf{F} = 0$ is sufficient for $\mathbf{F}$ to be conservative, that is, on simply connected domains, irrotational implies conservative. This will be proved later using the Stokes theorem.