Note I.3[1] 5 October

- **Definition**: The **curl** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is the vector field

$$\begin{aligned} \text{curl : vector fields} &\mapsto \text{vector fields} \\ \mathbf{F} &\to \text{curl}\,\mathbf{F} \end{aligned} \tag{1}$$

  with

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \tag{2}$$

  and, in the symbolic notation this is

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} \tag{3}$$

  This is the easiest way to remember the formula, using the determinant formula for the cross product

$$\text{curl} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \tag{4}$$

- **Example**: If $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ then applying the formula above gives

$$\text{curl}\,\mathbf{F} = -x\mathbf{k} \tag{5}$$

Again, it is not easy at first to get a picture of what the curl does. One rough idea is that it measures the rotation at a point of the vectors in a vector field. Certainly, this is what happens when you take the curl of the rotational field. Consider the velocity field

$$\mathbf{u} = \mathbf{w} \times \mathbf{r} \tag{6}$$

where $\mathbf{r} = (x, y, z)$ is the position vector and $\mathbf{w} = (w_!, w_2, w_3)$ is some constant vector. Now, $\mathbf{u}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{w}$ and the length of $\mathbf{u}$ is constant on circles around $\mathbf{w}$, hence the velocity field corresponds to rotation around $\mathbf{w}$. We will take its curl, first

$$\begin{aligned} \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix} \\ &= (w_2 z - w_3 x)\mathbf{i} + (w_3 x - w_1 z)\mathbf{j} + (w_1 y - w_2 x)\mathbf{k} \end{aligned} \tag{7}$$

Now, substituting this into the curl formula we get

$$\nabla \times \mathbf{u} = 2\mathbf{w} \tag{8}$$

- **Definition**: A vector field is called **irrotational** if it has zero curl.

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The gradient of a scalar field is irrotational:

$$\text{curl}\,\text{grad}\,\phi = 0 \tag{9}$$

or, using the symbolic notation $\nabla \times \nabla\phi = 0$. This is proved by calculation, using the subscript to denote component, so $\mathbf{v} = (v_1, v_2, v3)$ for any vector, we have

$$\begin{aligned}(\nabla \times \nabla\phi)_1 &= \partial_y(\nabla\phi)_3 - \partial_z(\nabla\phi)_2 \\ &= \partial_y\partial_z\phi - \partial_z\partial_y\phi\end{aligned} \tag{10}$$

and the other components follow in the same way. We have used the useful notation where

$$\partial_x = \frac{\partial}{\partial x} \tag{11}$$

and so on.

Vector identities

There are a number of usefull identities involving grad, div and curl. These are usually proved by direct calculation, expand out the various terms.

Let ϕ and ψ be scalar fields and $\mathbf{F}$ and $\mathbf{G}$ be vector fields, then

1.
$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{12}$$
This is a direct consequence of the product rule.

2.
$$\nabla(\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi\nabla \cdot \mathbf{F} \tag{13}$$
and again this follows by just expanding it out
$$\begin{aligned}\nabla(\phi\mathbf{F}) &= \partial_x(\phi F_1) + \partial_y(\phi F_2) + \partial_z(\phi F_3) \\ &= F_1\partial_x\phi + F_2\partial_y\phi + F_3\partial_z\phi + \phi(\partial_x F_1 + \partial_y F_2 + \partial_z F_3) \\ &= \nabla\phi \cdot \mathbf{F} + \phi\nabla \cdot \mathbf{F}\end{aligned} \tag{14}$$

3.
$$\nabla \times (\phi\mathbf{F}) = \nabla\phi \times \mathbf{F} + \phi\nabla \times \mathbf{F} \tag{15}$$
This can easily be proved too, just check, say, the x-component by direct calculation.

4.
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot \nabla \times \mathbf{G} \tag{16}$$
The proof of this is handwritten as (Picture I.3.1)

5. $$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G} \tag{17}$$

   This is one of the harder ones to prove since it involves the unusual operator

   $$(\mathbf{G} \cdot \nabla) = G_1\partial_x + G_2\partial_y + G_3\partial_z \tag{18}$$

   and the proof is given as an exercise on the problem sheets.

6. $$\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G}) + (\mathbf{G} \cdot \nabla)\mathbf{F}) \tag{19}$$

   This is also given as an exercise.

7. $$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{20}$$

   or, the curl of a vector field is solenoidal. This is one of the important vector identities which hints at some of the beautiful constructions in differential geometry. It is easy enough to prove by direct calculation.

8. $$\nabla \times \nabla\phi = 0 \tag{21}$$

   was proved above.

9. $$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \triangle\mathbf{F} \tag{22}$$

   where

   $$\triangle\phi = \nabla \cdot \nabla\phi = (\partial_x^2 + \partial_y^2 + \partial_z^2)\phi \tag{23}$$

   is the **Laplacian**, an operator which occurs frequently in physically significant equations. Obviously

   $$\triangle\mathbf{F} = (\triangle F_3, \triangle F_2, \triangle F_3) \tag{24}$$