Note I.2[1] 4 October 2006

Three dimensions

The three-dimensional case is a straight-forward extension of the two-dimensional analysis. The Cartesian iterated integral has the form

$$\int_D dV\phi = \int_a^b dx \int_{c(x)}^{d(x)} dy \int_{e(x,y)}^{f(x,y)} dz\phi(x,y,z) \tag{1}$$

where $z = e(x,y)$ and $z = f(x,y)$ describe the upper and lower surfaces bounding the domain D (Picture I.2.1). The general coördinate transform

$$\begin{aligned} x &= x(u,v,w) \\ y &= y(u,v,w) \\ z &= z(u,v,w) \end{aligned} \tag{2}$$

has

$$dxdydz = Jdudvdw \tag{3}$$

where the Jacobian is

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \left\| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array} \right\| \tag{4}$$

Of course this formula is more general than this, it applies for any transformation between coördinate systems.

Two commonly used coördinate systems are **spherical polar coördinates** and **cylindrical polar coördinates**. The spherical polar coördinates are r, θ and ϕ where r is the distance from the origin, θ, called the **polar angle**, is the angle distended with the z-axis and ϕ, called the **azimuthal angle**, is the angle the projection onto the xy-plane makes with the x-axis (Picture I.2.2). The spherical polars are related to Cartesians by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{5}$$

and $J = r^2\sin\theta$. The polar coördinates are z, ρ and ϕ, z is the distance along the z-axis, as usual, ρ is the length of the projection on the xy-plane and ϕ is the angle the projection distends with the x-axis (Picture I.2.3), they are related to the Cartesians by

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned} \tag{6}$$

and the Jacobian is $J = \rho$.

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/231

- **Example**: A ball of radius a has a cylindrical hole of radius $b < a$ drilled though its center, what is its volume? Well try cylindrical polars with the z-axis corresponding to the axis of the hole (Picture I.2.4). Now to integrate over the remaining material we need to work out the ranges for the various coördinates. Obviously $0 \le \phi \le 2\pi$, by trigonometry, at z $\rho = \sqrt{a^2 - z^2}$ and the sphere and the cylinder touch when $\rho = b$, which happens when $z = \pm\sqrt{a^2 - b^2}$, hence $b \le \rho \le \sqrt{a^2 - z^2}$ and $-\sqrt{a^2 - b^2} \le z \le \sqrt{a^2 - b^2}$ and the iterated integral is

$$V = \int_V dV = \int_0^{2\pi} d\phi \int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} dz \int_b^{\sqrt{a^2-z^2}} d\rho\rho \tag{7}$$

where the final ρ is the Jacobian. Now, we can do this integral

$$\begin{aligned} \int_0^{2\pi} d\phi \int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} dz \int_b^{\sqrt{a^2-z^2}} d\rho\rho &= 2\pi \int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} dz \frac{a^2 - z^2 - b^2}{2} \\ &= \pi \left[(a^2 - b^2)z - \frac{z^3}{3} \right] \Bigg|_{z=-\sqrt{a^2-b^2}}^{z=\sqrt{a^2-b^2}} \\ &= \frac{4\pi}{3}(a^2 - b^2)^{3/2} \end{aligned} \tag{8}$$

Vector fields

A scalar field, defined already, maps points in $\mathbf{R}^3$ to real numbers; now we define

- **Definition**: A **vector field** is a mapping

$$\mathbf{F} : D \to \mathbf{R}^3 \tag{9}$$

where D is a subset of $\mathbf{R}^3$

so a vector field maps points in $\mathbf{R}^3$ to three-dimensional vectors.

- **Example**:

$$\mathbf{F} = (xy, y^2, z) \tag{10}$$

also written

$$\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j} + z\mathbf{k} \tag{11}$$

is a vector field, where we have used the usual basis

$$\begin{aligned} \mathbf{i} &= (1,0,0) \\ \mathbf{j} &= (0,1,0) \\ \mathbf{k} &= (0,0,1) \end{aligned} \tag{12}$$

Physical examples include the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$ in electromagnetism and the fluid velocity $\mathbf{u}(x,y,z)$ in a fluid.

Vector calculus

Now, the issue is how to define the derivatives of scalar and vector fields. In practice, there are three differential operators used; these will be defined and, hopefully, by looking at examples it will become clearer as to why these particular operators are the ones that are important for physically and mathematically.

- **Definition**: The **gradient** of a scalar field ϕ is the vector field
$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \tag{13}$$
so gradient is a map
$$\begin{aligned} \operatorname{grad} : \text{scalar fields} &\mapsto \text{vector fields} \\ \phi &\to \operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \end{aligned} \tag{14}$$
It is common and useful to also use the symbolic notation
$$\operatorname{grad}\phi = \nabla\phi \tag{15}$$
where ∇, called **nabla** is the *vector operator*
$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{16}$$

Hence, for example,

- **Example**: The gradient of the scalar field $\phi(x, y, z) = xy + y\cos z$ is
$$\operatorname{grad}\phi = y\mathbf{i} + (x + \cos z)\mathbf{j} - y\sin z\mathbf{k} \tag{17}$$
and physical examples include the force on a particle
$$\mathbf{F} = -\nabla V \tag{18}$$
in a potential energy field $V(x, y, z)$.

Probably the easiest way to understand the gradient is to relate it to the directional derivative; it is easy to see that a sensible definition of the derivative of ϕ is the direction given by a unit vector $\hat{\mathbf{e}} = (e_1, e_2, e_3)$ is
$$D_{\hat{\mathbf{e}}}\phi := \lim_{h\to 0}\frac{\phi(\mathbf{x} + h\hat{\mathbf{e}}) - \phi(\mathbf{x})}{h} \tag{19}$$
but, by expanding $\phi(\mathbf{x} + h\hat{\mathbf{e}}) = \phi(x + he_1, y + he_2, z + he_3)$ using the Taylor expansion
$$D_{\hat{\mathbf{e}}}\phi = \hat{\mathbf{e}}\cdot\nabla\phi \tag{20}$$

Obviously this is maximum for $\hat{\mathbf{e}}$ in the same direction as $\nabla\phi$ so the direction of gradient gives the direction that ϕ has its greatest variation in and the length of the gradient is the directional derivative in that direction. Similarly, the gradient of ϕ is perpendicular to the level surfaces of ϕ, so $\operatorname{grad}\phi$ is perpendicular to the surface ϕ =constant. Finally, we define

- **Definition**: The **stationary points** of a scalar field are points where the gradient of the field is zero.

The remaining two differential operators act on vector fields, the divergence, sends a vector field to a scalar field and, we will see, the curl sends a vector field to another vector field.

- **Definition**: The **divergence** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is
$$\operatorname{div}\mathbf{F} := \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \tag{21}$$
or in the symbolic notation
$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} \tag{22}$$
Hence
$$\begin{aligned} \operatorname{div} : \text{vector fields} &\mapsto \text{scalar fields} \\ \mathbf{F} &\to \operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} \end{aligned} \tag{23}$$

So for example

- **Example**: The divergence of the vector field $\mathbf{F} = (xy, \sin z, z)$ is $\operatorname{div}\mathbf{F} = 1 + y$.

We will see that it is significant when a vector field has no divergence and

- **Definition**: A vector field is called **solenoidal** if it has a zero divergence.

In electromagnetism the magnetic field $\mathbf{B}$ is solenoidal by the Maxwell equations and in fluid flow the **continuity equation** for an incompressible liquid has a solenoidal velocity field. In fact, the continuity equation is a good way of getting a handle on how the divergence works, consider a compressible fluid with density field $\rho(x, y, z; t)$ and velocity field $\mathbf{u}(x, y, z; t)$, at a given time t and at a given point (x, y, z) ρ gives the density of the fluid and $\mathbf{u}$ gives its velocity. The field $\rho\mathbf{u}$ is the mass transport and the continuity equation is
$$\frac{\partial\rho}{\partial t} = -\operatorname{div}(\rho\mathbf{u}) \tag{24}$$
so the amount of fluid at a point changes according to the divergence of the mass transport field, hence, roughly speaking we can think of the divergence as giving the net accumulation of the vector field at the point.