Note I.1[1] 2 October 2006

PART I Vector Analysis

Scalar fields

Standard calculus concerns functions whose domain is either the real line $\mathbf{R}$ or a subset of the real line.

- **Definition**: A function is a mapping
$$\phi : D \to \mathbf{R} \tag{1}$$
where D is a subset of $\mathbf{R}$.

Here we consider functions over more than one dimension, defined on

$$\mathbf{R}^3 = \{(x,y,z)|x,\, y,\, z \in \mathbf{R}\}.$$

A function over such a domain is called a scalar field:

- **Definition**: A **scalar field** on D is a mapping
$$\phi : D \to \mathbf{R} \tag{2}$$

In the examples we will look at D will be $\mathbf{R}^3$ or $\mathbf{R}^2$ or a subset of one of these. In the definition the *scalar* part refers to the fact that the target space is $\mathbf{R}$.

$$\phi(x,y,z) = xy + z \tag{3}$$

is an example of a scalar field. Physical examples would include the pressure or temperature field of a fluid: at every point in the fluid there is a pressure and so pressure is a scalar field.

Integration

We would like to define the integral of a scalar field. For functions

$$\int_a^b dx f(x) \tag{4}$$

may be defined as a limit of a Riemann sum $\mathcal{R}$. (Picture I.1.1). We want to generalize this to an integral of a scalar field, written

$$\int_D dV\phi \tag{5}$$

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or

$$\int_D d^3x\phi \tag{6}$$

Because they are easier to draw, we will first consider two-dimensional integrals. The region is divided up into rectangular cells (Picture I.1.2) and the **lower Riemann sum** is

$$\mathcal{R} = \sum_{r\in R} A_r\phi_r \tag{7}$$

where R is the set of rectangles, A_r is the area of the rectangle r and ϕ_r is the infimum of ϕ over r, to remind you of what this means, ϕ_r is the largest number such that $\phi_r \leq \phi(x,y)$ for all $(x,y) \in r$, for nice functions the infimum is the same as the minimum. One technical matter is that a prescription is needed to deal with incomplete rectangles at the boundary, one that works is to define A_r as the area of the full rectangle, even if it is at the boundary and define

$$\phi_r = \begin{cases} \inf\phi & r \text{ a complete rectangle} \\ \min(\inf\phi, 0) & r \text{ on the boundary} \end{cases} \tag{8}$$

Now, we define

$$\int_D dA\phi = \int_D dxdy\phi = \sup\mathcal{R} \tag{9}$$

where the supremum is taken over all possible grids. In two-dimension it is common to use dA instead of dV as the infinitesimal element.

In practice the integral is calculated as an **iterated integral**. The basic idea of an iterated integral is that you integrate in the x direction first and then in the y direction, or visa versa. In the underlying Riemann sum picture this corresponds to first summing the rectangles in horizontal slices and then summing the slices, or, in the other order, first summing the rectangles in vertical slices and then summing the slices (Picture I.1.3) and, as sketched in the picture, each of the slices should approach a one-dimensional Riemann integral. Hence, if $y = d(x)$ is the upper boundary of the curve and $y = c(x)$ the lower and a and b are the lowest and highest x values

$$\int_D dxdy\phi = \int_a^b dx \int_{c(x)}^{d(x)} dy\phi(x,y) \tag{10}$$

- **Example**: Consider the half-disk $x^2 + y^2 \leq 1$ and $x > 0$ (Picture I.1.5). The upper boundary is $d(x) = \sqrt{1-x^2}$ and the lower boundary is $c(x) = 0$. The area of the region is the integral of one over the region, this is clear from the definition given above in terms of the Riemann sum. Now
$$\text{Area} = \int_D dA1 = \int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} dy1 = \int_{-1}^1 dx\sqrt{1-x^2} = \frac{\pi}{2} \tag{11}$$
The **centers of mass** are defined as
$$\bar{x} = \frac{\int_D dA\rho(x,y)x}{\int_D \rho(x,y)dA}$$

$$\bar{y} = \frac{\int_D dA\rho(x,y)y}{\int_D \rho(x,y)dA} \tag{12}$$

where $\rho(x,y)$ is the density. For uniform density, $\rho(x,y) = 1$ it is easy to see that $\bar{x} = 0$ by symmetry, the $x < 0$ portion cancels the $x > 0$. Now,

$$\begin{aligned} \bar{y} &= \frac{2}{\pi}\int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy\, y \\ &= \frac{2}{\pi}\int_{-1}^{1} dx\, \frac{y^2}{2}\bigg|^{\sqrt{1-x^2}} \\ &= \frac{2}{\pi}\int_{-1}^{1} dx \frac{1-x^2}{2} = \frac{4}{3\pi} \end{aligned} \tag{13}$$

Now suppose that the density is $\rho = \sqrt{x^2+y^2}$ then the mass $\int_D dA\rho$ is

$$\int_D dA\rho = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \sqrt{x^2+y^2} \tag{14}$$

which is a messy integral, probably possible with a trigonometric substitution, but who would want to get into it? What is clear is that the integrated integral is not exploiting the symmetry of the problem. This motivates us to look at changes of variable; as in one-dimension, a change in variables can be used to bring an integral into an easier form.

Changes of variable

Let us first recall the situation with integration in one-dimension when making a change in variable. Say we have $f(x)$ some function of x and $u(x)$ is a new variable and, inverting, we know x in terms of u as $x(u)$, then

$$\int_a^b dx f(x) = \int_{u(a)}^{u(b)} du f(x(u)) \frac{dx}{du} \tag{15}$$

- **Example**: So $f(x) = x^4$ and we are interested in

$$\int_1^2 dx x^4 = \frac{x^5}{5}\bigg|_1^2 = \frac{1}{5} - \frac{32}{5} = -\frac{31}{5} \tag{16}$$

and a new variable is given by $u = x^2$. In the interval $[1,2]$ this relationship is invertible with $x = \sqrt{u}$ so $f(u) = (\sqrt{u})^4 = u^2$ and the extra factor in the integral is

$$\frac{dx}{du} = \frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}} \tag{17}$$

Finally $u(1) = 1$ and $u(2) = 4$ so

$$\int_1^2 dx x^4 = \int_1^4 du \frac{u^2}{2\sqrt{u}} = \frac{1}{2}\int_1^4 u^{3/2} = \frac{1}{5}\, u^{5/2}\big|_1^4 = -\frac{31}{5} \tag{18}$$

as before.

Now, for a two-dimensional integral, consider the change of variables from (x,y) to (u,v) related by

$$\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned} \tag{19}$$

then

$$\int_D dxdy\phi(x,y) = \int_D dudv\phi(x(u,v), y(u,v))J \tag{20}$$

where the **Jacobian** J is the absolute value of the determinant

$$J = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\| \tag{21}$$

Another notation for the Jacobian that is used is

$$J = \frac{\partial(x,y)}{\partial(u,v)} \tag{22}$$

An explanation will be given for the Jacobian, but only after we have considered the example of polar coördinates.

- **Example**: Polar coördinates are given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{23}$$

so r corresponds to the distance from the origin and θ is the angle distended with the x-axis at the origin (Picture I.1.6).

The Jacobian for this coördinate change is

$$J = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\| = \left\| \begin{array}{cc} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{array} \right\| = r(\cos^2\theta + \sin^2\theta) = r \tag{24}$$

Hence

$$\int_D dxdy\phi(x,y) = \int_D drd\theta r\phi(r,\theta) \tag{25}$$

It is easy to see in a picture (Picture I.1.7) why $dV = dxdy = rdrd\theta$. It is also clear that for simple domains containing the origin this can be written as an iterated integral

$$\int_D dxdy\phi(x,y) = \int_D drd\theta r\phi(r,\theta) \int_0^{2\pi} d\theta \int_0^{h(\theta)} drr\phi(r,\theta) \tag{26}$$

where $r = h(\theta)$ is the boundary and by simple we mean that the boundary can be written in this way (Picture I.1.8).

Now, to deal with the example from earlier 14, in polar coordinates we have

$$\int_D dA\rho = \int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \sqrt{x^2+y^2} = \int_0^{\pi} \int_0^{1} drr^2 = \frac{\pi}{3} \tag{27}$$

To work out $\bar{y}$ we would also need $\int_D dA\rho y$ which is

$$\int_D dA\rho y = \int_0^{\pi} d\theta \int_0^1 drr^3 \sin\theta = \frac{1}{4}\int_0^{\pi} \theta \sin\theta = \frac{1}{2} \tag{28}$$

giving

$$\bar{y} = \frac{3}{2\pi} \tag{29}$$

So, back to the interpretation of the Jacobian. Consider what happens if you vary u, using the Taylor expansion

$$\begin{aligned} x(u+\delta u, v) &\approx x(u,v) + \frac{\partial x}{\partial u}\delta u \\ y(u+\delta u, v) &\approx y(u,v) + \frac{\partial y}{\partial u}\delta u \end{aligned} \tag{30}$$

and if you vary y

$$\begin{aligned} x(u, v+\delta v) &\approx x(u,v) + \frac{\partial x}{\partial v}\delta v \\ y(u, v+\delta v) &\approx y(u,v) + \frac{\partial y}{\partial v}\delta v \end{aligned} \tag{31}$$

so the area element corresponding to δu and δv is a small parallelogram with edges

$$\begin{aligned} \mathbf{e}_1 &= \delta u \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\right) \\ \mathbf{e}_2 &= \delta v \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}\right) \end{aligned} \tag{32}$$

and there is a formula for the area of a parallelogram, it is

$$J = \left\| \begin{array}{cc} \frac{\partial x}{\partial u}\delta u & \frac{\partial x}{\partial v}\delta v \\ \frac{\partial y}{\partial u}\delta u & \frac{\partial y}{\partial v}\delta v \end{array} \right\| = J\delta u \delta v \tag{33}$$

Hence the small element of area corresponding to small variations in u and v is $J\delta u\delta v$ and, roughly speaking, the change of variable formula is the infinitesimal limit of this.