Part I

231 Part IV Partial Differential Equations

1 Some linear PDEs involving a scalar field ϕ

(aka the Equations of Mathematical Physics)

i)

$$\nabla^2\phi = 0 \qquad \text{Laplace's equation (elliptic)}$$

ii)

$$\nabla^2\phi = \rho \qquad \text{Poisson's equation (elliptic)}$$

ϕ is some scalar field usually called a source term

iii)

$$(\nabla^2 + k^2)\phi = 0 \qquad \text{Helmholtz equation (elliptic)}$$

$$(\nabla^2 - k^2)\phi = 0 \qquad \text{'wrong sign' Helmholtz}$$

k is a real constant

iv)

$$\nabla^2\phi = D\frac{\partial\phi}{\partial t} \qquad \text{Heat/diffusion equation (parabolic)}$$

D a constant, $\phi = \phi(\vec{r}, t)$ time-dependent

v)

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \qquad \text{Wave equation (hyperbolic)}$$

c = speed of sound / light

$$\text{Laplacian } \nabla^2 = \nabla.\nabla = \text{ div grad } = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

2 Heat/Diffusion Equation

Image some material where the temperature is not constant but with no sources (or sinks) of heat.
Temperature is a scalar field $\phi(\vec{r},t)$. Heat current $\vec{j}(\vec{r},t)$ is a vector field s.t. energy flux across an oriented surface S.

INSERT IMAGE OF A VECTOR $\vec{n}$ OUT OF A SURFACE S

is the surface integral $\int_S \vec{j}(\vec{r},t)d\vec{A}$
Let S be a closed (but static) surface. Heat flux out of S

$$= \int_S \vec{j}(\vec{r},t)d\vec{A}$$

$$= - \text{ rate of change of energy in D}$$

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$$= -\alpha \frac{\partial}{\partial t} \int_D \phi(\vec{r},t)dV$$

$$= -\alpha \int_D \frac{\partial \phi}{\partial t}(\vec{r},t)dV$$

α constant (heat capacity per unit volume)

Apply Gauss' theorem

$$\int_S \vec{j}(\vec{r},t).d\vec{A} = \int_D \text{ div } \vec{j}dV$$

so that

$$\int_D (\text{ div } \vec{j} + \alpha \frac{\partial \phi}{\partial t})dV = 0$$

where D is any 3d region with a smooth boundary. Thus

$$\text{div } \vec{j} + \alpha \frac{\partial \phi}{\partial t} = 0$$

Assume $\vec{j} = -\beta \text{ grad } \phi$
where β is the thermal conductivity constant

and the - sign indicates that the heat flows from hot to cold regions

$$\nabla^2\phi = D\frac{\partial\phi}{\partial t} \text{ with } D = \frac{\alpha}{\beta}$$

Laplace's Equation is a special case of Poisson's equation and the heat equation

$$\text{Poisson } \rho = 0 \Rightarrow \text{ no source } \Rightarrow \nabla^2\phi = 0$$

$$\text{Heat } \frac{\partial\phi}{\partial t} = 0 \Rightarrow \text{ steady state } \Rightarrow \nabla^2\phi = 0$$

3 Boundary Value Problems

Often wish to solve a PDE subject to some boundary conditions.

Assume ϕ satisfies some PDE (e.g. Laplace's equation) in a 3d region D with boundary $S = \partial D$

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3.1 3 basic kinds of b.c.s.

i) ϕ is given on S $\rightarrow$ Dirichlet boundary conditions

ii) $\partial_n\phi = \vec{n}.\nabla\phi$ (directional derivative in direction of unit normal $\vec{n}$) is given on S $\rightarrow$ Neumann boundary conditions

iii) ϕ and $\partial_n\phi$ are given on S $\rightarrow$ Cauchy boundary conditions

Can also have mixed b.c.s. where on different parts of S different b.c.s are imposed

This is not exhaustive since there are other types of b.c.s. such as periodic boundary conditions.

3.2 Elliptic Case

Usually the Cauchy b.c.s. are too strong, i.e. no solutions.
Dirichlet b.c.s. more or less lead to a unique solution.
Neumann b.c.s. lead to a unique solution (up to an arbitrary constant in Laplace/Poisson cases).

3.3 Other cases (parabolic and hyperbolic)

more complicated.

4 Laplace's Equation

A solution of Laplace's equation is called a harmonic function.
Some simple (singular) examples:

3d $\phi = \frac{1}{r}$ is harmonic but singular at the origin.

2d $\phi = \log r, r = \sqrt{x^2 + y^2}$ harmonic but singular at $r = 0$.

1d $\phi = x$ is harmonic but singular at $\pm\infty$.

2d Any holomorphic function (Complex Analysis) is harmonic!

Suppose we wish to find a non-singular (e.g. C^∞) harmonic function is some domain D subject to some boundary conditions on $S = \partial D$.

4.1 Theorem: Uniqueness of solns to DBCs and NBCs

The solution of Laplace's equation under Dirichlet's boundary conditions (DBCs), if it exists, is unique. The solution of the problem under Neumann boundary conditions, if it exists, is unique up to an additive constant.

4.1.1 Proof

postponed; digression on vector analysis required.

4.1.2 Green's Identities

(not to be confused with Green's theorem in the plane) Let ϕ and ψ be smooth functions, (not necessarily harmonic)

Green's 1st Identity

$$\int_D (\phi \nabla^2 \psi + \nabla \phi . \nabla \psi) dV = \int_{S=\partial D} \phi \nabla \psi . d\vec{A}$$

Green's 2nd Identity

$$\int_D (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \int_{S=\partial D} (\phi\nabla\psi - \psi\nabla\phi).d\vec{A}$$

Proofs

1st identity. Apply Gauss' theorem to the vector field $\vec{F} = \phi\nabla\psi$

$$\text{div } \vec{F} = \phi\nabla.(\nabla\psi) + \nabla\phi.\nabla\psi$$

vector identity

$$= \phi\nabla^2\psi + \nabla\phi.\nabla\psi$$

2nd identity. Interchange ϕ and ψ in first identity the subtract from the first identity.

4.1.3 Proof of uniqueness Theorem

Let ϕ_1 and ϕ_2 be harmonic in D and subject to the same boundary conditions on $S = \partial D$ (either DBCs or NBCs). Consider $\phi = \phi_1 - \phi_2$. Apply Green's 1st identity taking $\psi = \phi$

$$\int_D (\phi\nabla^2\psi + \nabla\phi\nabla\psi)dV = \int \phi\nabla\phi.d\vec{A}$$

$$= 0 \text{ (since } \phi = \phi_1 - \phi_2 \text{ is harmonic)} \;= \int \partial_n\phi dA$$

$$= 0 \text{ for DBCs (} \phi = 0 \text{)}$$

$$= 0 \text{ for NBCs (} \partial_n\phi = 0 \text{)}$$

Therefore

$$\int_D \nabla\phi.\nabla\phi dV = 0$$

and $\nabla\phi.\nabla\phi$ is non-negative!
which requires $\nabla\phi = 0$ or $\phi =$ constant i.e. $\phi_1 - \phi_2 = c$ proving the theorem for NBCs. DBCs: c must be zero since ϕ_1 and ϕ_2 agree on S by assumption.

END FIRST THREE LECTURES

START SECOND THREE LECTURES

5 Gauss' Mean Value Theorem

for Harmonic Functions

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Suppose ϕ is harmonic in $D \subset \mathbb{R}^3$. The average value of ϕ over the surface of a sphere of radius R centred at the point $\vec{r}$ is $\phi(\vec{r})$.

5.0.4 Note.

This is true for any point in the interior of D. The radius R is any number s.t. the sphere (and every point inside it) is in D.

5.0.5 Proof

Without loss of generality consider a sphere centred at the origin. Idea is to show that the average

$$\bar{\phi}_R = \frac{1}{4\pi R^2}\int_{x^2+y^2+z^2=R^2} \phi dA$$

is independent of the radius R.
Apply Green's 2nd identity to ϕ and $\psi = \frac{1}{r}$ in the regions $R_1 < r < R_2$

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$$\int_{R_1<r<R_2} (\phi\nabla^2\frac{1}{r} - \frac{1}{r}\nabla^2\phi)dV$$

$$= \int_{r=R_2,out} (\phi\nabla\frac{1}{r} - \frac{1}{r}\nabla\phi).d\vec{A} - \int_{r=R_1,out} (\phi\nabla\frac{1}{r} - \frac{1}{r}\nabla\phi).d\vec{A}$$

Now,

$$\int_{r=R_1,out} \frac{1}{r}\nabla\phi.d\vec{A} = \frac{1}{R_1}\int_{r=R_1,out} \nabla\phi.d\vec{A}$$

because $\frac{1}{r}$ is constant of sphere $r = R_1$

$$= \frac{1}{R_1}\int_{r<=R_1} \text{div}\nabla\phi dV$$

$$= \frac{1}{R_1}\int_{r<=R_1} \nabla^2\phi dV = 0$$

$$\text{Similarly } \int_{r=R_2,out} \frac{1}{r}\nabla\phi.\vec{dA} = 0$$

Thus,

$$0 = \int_{r=R_2,out} \phi\nabla\frac{1}{r}.\vec{dA} - \int_{r=R_1,out} \phi\nabla\frac{1}{r}.\vec{dA}$$

$$\nabla\frac{1}{r} = -\frac{\hat{\vec{r}}}{r^2} \text{ in both integrals} \vec{n} = \hat{\vec{r}}$$

$$0 = -\frac{1}{R_2^2}\int_{r=R_2} \phi dA + \frac{1}{R_1^2}\int_{r=R_1} \phi dA$$

$$\text{or } \bar{\phi}_{R_2} = \bar{\phi}_{R_2}$$

Letting $R \to 0$, $\bar{\phi_R} = \phi(0)$, R being the radius of the outer sphere.

6 Maximum (minimum) Principle

for Harmonic Functions

Let ϕ be harmonic in a 3d (or 2d) domain D. Then ϕ never assumes its maximum (or minimum) value at an interior point of D unless ϕ is constant.

6.0.6 Proof

Assume ϕ has a maximum at some point P in the interior of D. For R sufficiently small the sphere of radius R centred at P is inside D. For every point on the sphere $\phi < \phi(P)$ so $\bar{\phi_R} < \phi(P)$ contradicting the MVT. A similar argument holds if P is a minimum.

If ϕ is harmonic in D it assumes its maximum and minimum values at the boundary $S = \partial D$

6.1 Physical Interpretation

Heat Equation $\nabla^2 = D\frac{\partial\phi}{\partial t}$. If ϕ reaches a steady state $\frac{\partial\phi}{\partial t} = 0$, then $\nabla^2\phi = 0$, i.e. the temperature is harmonic. Suppose we have a finite lump of matter and the boundary temperature (not necessarily constant) is fixed, e.g. consider a square slab with three sides fixed to be at 0 degrees and the other at 100 degrees. The steady state temperature inside the slab is harmonic. Steady state temperature can never exceed 100 degrees (or fall below 0 degrees); heat would immediately flow out of (or enter) such a hot spot (or cold spot).

SQUARE HEAT IMAGE

7 Liouville's Theorem

If ϕ is harmonic and bounded throughout $\mathbb{R}^3$ (or $\mathbb{R}^2$) then it is constant.

7.0.1 Proof

Not given.

7.0.2 Note

$\phi = \frac{1}{r}$ 3d $\phi = \log r$ (2d) are unbounded.

7.1 Liouville's Theorem - Complex Analysis Version

If f is holomorphic throughout $\mathbb{C}$ (sometimes called an entire function) and bounded $|f| < C$ then f is constant.

7.2 Solutions

Uniqueness theorem very powerful; any solution with DBCs, however simple is the only solution.

7.3 Examples

7.3.1 Example i

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Let ϕ be a harmonic function which is constant, say $\phi = a$, on the boundary of D. $\phi(\vec{r}) = a$ is trivially a solution of Laplace's equation with the correct b.c.s.. It must be the unique solution to this boundary value problem.

7.3.2 Example ii

Let ϕ be harmonic in a 2d annulus with $\phi = 1$ on the other boundary $(r = R_2)$ $\phi = b$ on the inner boundary $(r = R_1)$

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$\phi = C \log r + D$ $(r = \sqrt{x^2 + y^2})$ harmonic but singular at origin,
C, D constants.

$$a = \phi(r = R_2) = C \log R_2 + D$$

$$b = \phi(r = R_1) = C \log R_1 + D$$
$$a - b = C \log \frac{R_2}{R_1}$$
$$D = a - C \log R_2$$

$$\phi = \frac{a-b}{\log \frac{R_2}{R_1}} \log r R_2 + a$$

In these 2 examples we have guessed a solution (which we know to be unique).

This is clearly insufficient for most problems, e.g. the square where ϕ is zero on three sides, and another value, say 1, on the remaining side.

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8 Separation of Variables

(a more systematic approach to solving linear PDEs)
Idea is to reduce PDEs involving 2 or more variables to ODEs in each variable.

$$\text{Try } \phi(x, y) = X(x)Y(y)$$

where X depends on x only and Y depends on y only. Laplace's equation becomes

$$\nabla^2 \phi = X''(x)Y(y) + X(x)Y''(y) = 0$$

divide through by $X(x)Y(y)$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

with the first term independent of y and the second independent of x.

$$\text{Therefore, } \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{ constant independent of x and y}$$

3 possibilities constant i) positive, ii) zero, iii) negative.

i)

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k^2 \text{ 2 ODEs, k constant}$$

$X''(x) = k^2 X(x)$	$Y''(y) = -k^2 Y(y)$
with solutions	with solutions
$X(x) = e^{kx}, e^{-kx}$	$Y(y) = \sin ky, \cos ky$

For each k 4 independent solutions of Laplace's equation
$\phi = e^{kx} \sin ky, e^{kx} \cos kye^{-kx} \sin ky, e^{-kx} \cos ky$
ii) constant= 0

$$\begin{array}{ll} X''(x) = 0; & Y''(y) = 0; \\ X(x) = Ax + B & Y(y) = Cy + D \end{array}$$

$\Rightarrow$ 4 independent solutions of Laplace's equation

iii) constant negative

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k^2$$

$$X(x) = \sin kx, \cos kx \qquad Y(y) = e^{ky}, e^{-ky}$$

For each k 4 solutions

$$\phi = \sin kx, e^{ky}, \sin kx, e^{-ky}, \cos kx, e^{ky}, \cos kx, e^{-ky}$$

Lots of solutions! None of these satisfy the b.c.s. for one square problem, however we can implement left and right b.c.s.

$$\phi(0, y) = \phi(\pi, y) = 0$$

solutions (from iii)) $\sin kxe^{ky}$ and $\sin kxe^{-ky}$
satisfy left and right b.c.s. if k is an integer.
Consider a linear combination of these solutions

$$\phi(x, y) = \sum_{n=1}^{\infty} \sin nx(b_n e^{ny} + \tilde{b_n} e^{-ny})$$

Now try to find b_n and $\tilde{b_n}$ such that upper and lower b.c.s. are satisfied.

8.0.3 Upper boundary condition

$$0 = \phi(x, \pi) = \sum_{n=1}^{\infty} \sin nx(b_n e^{n\pi} + b_n e^{-n\pi})$$

This is satisfied if

$$b_n e^{n\pi} + \tilde{b_n} e^{-n\pi} = 0 \text{ for all n.}$$

8.0.4 Lower boundary condition

$$\phi(x,0) = \sum_{n=1}^{\infty} \sin nx(b_n + \tilde{b_n}) =^? 1$$

1 is even while the sines are odd. Recall

$$f(x) = 1 \quad 0 < x < \pi - 1 \quad -\pi < x < 0$$

has the Fourier Series expansion

$$f(x) = \frac{4}{\pi} \sum_{n \text{ odd } > 0} \frac{1}{n} \sin nx$$

Restricting to $0 < x < \pi$ we have

$$1 = \frac{4}{\pi} \sum_{n \text{ odd } > 0} \frac{1}{n} \sin nx$$

half-range sine series

$$\text{Set } b_n + \tilde{b_n} = \frac{4}{\pi n} \text{ n odd 0 n even}$$

$$b_n = \frac{4}{\pi n(1 - e^{2\pi n}} \text{ n odd 0 n even}$$

$$\phi(x,y) = \frac{4}{\pi} \sum_{n \text{ odd } > 0} \frac{\sin nx}{n} \frac{e^{ny}}{e^{2\pi n} - 1}$$

9 Half-range Sine Series

An even function f can be expanded in sine waves over a half-period

Suppose $f(x + 2\pi) = f(x)$, $f(-k) = f(k)$

The function f odd defined as

$$f_{odd}(x) = \begin{matrix} f(x), & 0 < x < \pi \\ -f(x), & -\pi < x < 0 \end{matrix}$$

is odd and agrees with $f(x)$ for $o < x < \pi$. The usual Fourier expansion of f_{odd} contains only sines

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad 0 < x < \pi$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} dx \sin nx f(x)$$

(true if f odd, even or neither)

This is known as a half-range Fourier sine series. If f is odd it coincides with the usual Fourier series.

In Q2 of problem sheet 19 the cosine must be expanded as a sine series.

Back to our square boundary value problem.

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Lines of constant ϕ must converge at the two lower corners.

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Closed loops of constant ϕ do not occur – uniqueness theorem would force ϕ to be constant inside any such loop.

Other constant boundary conditions:

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a, b, c, d constant

can be dealt with through linear combinations of the basic $a = 1$, $b = c = d = 0$ square and similar solutions

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interchange x, and y in $a = 1$, $b = c = d = 0$ solution

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$x \to 1 - x$ in previous solution

$\phi =$ constant is also a solution.

9.1 Periodic Strip

Periodic in x direction $\pi(x+2\pi, y) = \phi(x, y)$ and y in some finite range, say $0 \leq y \leq 1$ with DBCs at $y = 0$ and $y = 1$.

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$\phi(x, y = 1) = g(x) \qquad g(x + 2\pi) = g(x)$
$\phi(x, y = 0) = f(x) \qquad f(x + 2\pi) = f(x)$
separation of variables:
$\phi(x, y) = X(x)Y(y)$
Solutions (must be periodic in x)
$X(x) = \cos nx$ or $\sin nx \qquad Y(y) = e^{ny}$ or e^{-ny}
and $X(x) = A + Bx \qquad Y(y) = C + Dy$
$B = 0$ periodicity
(redefine $k_1 = AC$, $k_2 = AD$)

$$\phi(x, y) = k_1 + k_2 y + \sum_{n=1}^{\infty} \left(a_n e^{ny} + \tilde{a_n} e^{-ny}\right) \cos nx + \sum_{n=1}^{\infty} \left(b_n e^{ny} + \tilde{b_n} e^{-ny}\right) \sin nx$$

Obtain coefficients k_1, k_2, a_n, $\tilde{a_n}$, b_n, $\tilde{b_n}$ through boundary conditions at $y = 0$, and $y = 1$.

Note that separation of variables in Cartesian coordinates is not always possible and even when it is, it is not always useful (if the boundary conditions are not suited).

10 Seperation of Variables in other Coordinate Systems

In situations with circular symmetry polar coordinates are advantageous. Separation of variables in polar coordinates

$$\phi(r, \theta) = R(r)\Theta(\theta)$$

But to solve, for example, Laplace's equation require Laplacian in these coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

For more compliacated coordinate systems, e.g. spherical polars, this becomes messy

10.1 Curvilinear Coordinates

Consider a general change of coordinates

$$\begin{aligned} x &= x(u_1, u_2, u_3) \\ y &= y(u_1, u_2, u_3) \text{ or } \underline{r} = \underline{r}(u_1, u_2, u_3) \\ z &= z(u_1, u_2, u_3) \end{aligned}$$

e.g. $(u_1, u_2, u_3) = (r, \theta, \phi)$ spherical polars.
Can rewrite grad, div and curl as derivatives w.r.t. the new coordinates

$$\begin{aligned} grad\phi &= \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j} + \frac{\partial \phi}{\partial z}\underline{k} \\ &= \left(\frac{\partial \phi}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial \phi}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial \phi}{\partial u_3}\frac{\partial u_3}{\partial x}\right)\underline{i} + \text{ cyclic perms} \\ &= \frac{\partial \phi}{\partial u_1}\nabla u_1 + \frac{\partial \phi}{\partial u_2}\nabla u_2 + \frac{\partial \phi}{\partial u_3}\nabla u_3 \end{aligned}$$

Working with other coordinate systems convenient to replace basis vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ with unit vectors $\underline{e_{u_1}}$, $\underline{e_{u_2}}$, $\underline{e_{u_3}}$ pointing in the direction of increasing u_1, u_2, u_3.
Define

$$\begin{aligned} \underline{e_{u_1}} &= \frac{1}{h_{u_1}}\frac{\partial \underline{r}}{\partial u_1} \qquad h_{u_1} = \left|\frac{\partial \underline{r}}{\partial u_1}\right| \\ \underline{e_{u_2}} &= \frac{1}{h_{u_2}}\frac{\partial \underline{r}}{\partial u_2} \qquad h_{u_2} = \left|\frac{\partial \underline{r}}{\partial u_2}\right| \\ \underline{e_{u_3}} &= \frac{1}{h_{u_3}}\frac{\partial \underline{r}}{\partial u_3} \qquad h_{u_3} = \left|\frac{\partial \underline{r}}{\partial u_3}\right| \end{aligned}$$

(can abbreviate $\underline{e}_1 = \underline{e}_{u_1}$, $\underline{e}_2 = \underline{e}_{u_2}$ etc and $h_1 = h_{u_1}$, $h_2 = h_{u_2}$ etc.)

10.2 Spherical Polar Coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

$$\frac{\partial \underline{r}}{\partial r} = \sin\theta\cos\phi\underline{i} + \sin\theta\cos\phi\underline{j} + \cos\theta\underline{k}$$

$$h_r = \left|\frac{\partial \underline{r}}{\partial r}\right| = \sqrt{(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta} = 1$$

$$\begin{aligned} \underline{e_r} &= \sin\theta\cos\phi\underline{i} + \sin\theta\sin\phi\underline{j} + \cos\theta\underline{k} \\ &= \frac{\underline{r}}{r} \text{ sometimes written } \underline{\hat{r}} \end{aligned}$$

$$\frac{\partial \underline{r}}{\partial \theta} = r\cos\theta\cos\phi\underline{i} + r\cos\theta\sin\phi\underline{j} - r\sin\theta\underline{k}$$

$$h_\theta = \left|\frac{\partial \underline{r}}{\partial \theta}\right| = r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta} = r$$

$$\underline{e_\theta} = \frac{1}{h_\theta}\frac{\partial \underline{r}}{\partial \theta} = \cos\theta\cos\phi\underline{i} + \cos\theta\sin\phi\underline{j} - \sin\theta\underline{k}$$

$$\frac{\partial \underline{r}}{\partial \phi} = -r\sin\theta\sin\phi\underline{i} + r\sin\theta\cos\phi\underline{j}$$

$$h_\phi = \left|\frac{\partial \underline{r}}{\partial \phi}\right| = r\sqrt{sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi} = r\sin\theta$$

$$\underline{e_\phi} = \frac{1}{h_\phi}\frac{\partial \underline{r}}{\partial \phi} = -\sin\phi\underline{i} + \cos\phi\underline{j}$$

Note that the new basis vectors $\underline{e_r}$, $\underline{e_\theta}$ and $\underline{e_\phi}$ are orthogonal $\underline{e_r}.\underline{e_\theta} = \underline{e_\theta}.\underline{e_\phi} = \underline{e_\phi}.\underline{e_r} = 0$
<u>Claim</u> The three (un-normalised) basis vectors

$$\frac{\partial \underline{r}}{\partial u_1}, \frac{\partial \underline{r}}{\partial u_2}, \frac{\partial \underline{r}}{\partial u_3}$$

are dual to the three gradients

$$\nabla u_1, \nabla u_2, \nabla u_3$$

i.e.

$$\nabla u_i.\frac{\partial \underline{r}}{\partial u_j} = \delta_{ij} \qquad \text{, where } \delta \text{ is the Kronecker delta}$$

<u>Proof</u>

$$\nabla u_1.\frac{\partial \underline{r}}{\partial u_1} = \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial u_1} + \frac{\partial u_1}{\partial y}\frac{\partial y}{\partial u_1} + \frac{\partial u_1}{\partial z}\frac{\partial z}{\partial u_1} = \frac{\partial u_1}{\partial u_1} = 1$$

11 Orthogonal Curvilinear Coordinates

Assume basis vectors $\underline{e_i}$ are orthogonal

$$\underline{e_i}\underline{e_j} = \delta_{ij}$$

(e.g. spherical polars)

Write

$$\nabla u_i . \frac{\partial \underline{r}}{\partial u_j} = \delta_{ij}$$

as

$$\nabla u_i . h_j \underline{e_j} = \delta_{ij}$$

or

$$h_j \nabla u_i . \underline{e_j} = \delta_{ij}$$

This implies that $\underline{e_i} = h_i \nabla u_i$ or

$$\nabla u_i = \frac{1}{h_i} \underline{e_i} \text{ (orthogonal coord. system)}$$

Inserting this into gradient formula

$$\begin{aligned} \text{grad}\phi &= \frac{\partial \phi}{\partial u_1} \nabla u_1 + \frac{\partial \phi}{\partial u_2} \nabla u_2 + \frac{\partial \phi}{\partial u_3} \nabla u_3 \\ &= \frac{1}{h_1} \frac{\partial \phi}{\partial u_1} \underline{e_1} + \frac{1}{h_2} \frac{\partial \phi}{\partial u_2} \underline{e_2} + \frac{1}{h_3} \frac{\partial \phi}{\partial u_3} \underline{e_3} \end{aligned}$$

For example in spherical polars (r, θ, ϕ) $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin\theta$

$$\text{grad}\Phi = \frac{\partial \Phi}{\partial r} \underline{e_r} + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \underline{e_\theta} + \frac{1}{r \sin\theta} \frac{\partial \Phi}{\partial \phi} \underline{e_\phi}$$

(large Φ not to be confused with angle ϕ!)

A vector field $\underline{F} = F_x \underline{i} + F_y \underline{y} + F_z \underline{k}$ can be written in terms of the 'new' basis vectors $\underline{e_1}$, $\underline{e_2}$, $\underline{e_3}$.

$$\underline{F} = F_{u_1} \underline{e_{u_1}} + F_{u_2} \underline{e_{u_2}} + F_{u_3} \underline{e_{u_3}}$$

abbreviated to

$$\underline{F} = F_1 \underline{e_1} + F_2 \underline{e_2} + F_3 \underline{e_3}$$

For example

$$\begin{aligned} \underline{F} &= \frac{1}{r^3}(x\underline{i} + y\underline{j} + z\underline{k}) \\ &= \frac{1}{r^2} \underline{e_r} \end{aligned}$$

or $F_r = \frac{1}{r^2}$, $F_\theta = F_\phi = 0$

11.0.1 Divergence and curl

Divergence of a vector field $\underline{F} = F_1\underline{e_1} + F_2\underline{e_2} + F_3\underline{e_3}$

$$\div\underline{F} = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\partial}{\partial u_2}(F_2h_3h_1) + \frac{\partial}{\partial u_3}(F_3h_1h_2)]$$

(coord system orthogonal)
Proof Based on identity

$$\div\frac{\underline{e_1}}{h_2h_3} = \div\frac{\underline{e_2}}{h_3h_1} = \div\frac{\underline{e_3}}{h_1h_2} = 0$$

Write $\underline{F} = (F_1h_2h_3)\frac{\underline{e_1}}{h_2h_3} + (F_2h_3h_1)\frac{\underline{e_2}}{h_3h_1} + (F_3h_1h_2)\frac{\underline{e_3}}{h_1h_2}$
Use $\div\phi\underline{G} = \phi\div\underline{G} + \nabla\phi\cdot\underline{G}$

$$\div\underline{F} = \nabla(F_1h_2h_3)\cdot\frac{\underline{e_1}}{h_2h_3} + \nabla(F_2h_3h_1)\cdot\frac{\underline{e_2}}{h_3h_1} + \nabla(F_3h_1h_2)\cdot\frac{\underline{e_3}}{h_1h_2}$$

$$\nabla(F_1h_2h_3) = \frac{\underline{e_1}}{h_1}\frac{\partial}{\partial u_1}(F_1h_2h_3) + \frac{\underline{e_2}}{h_2}\frac{\partial}{\partial u_2}(F_2h_3h_1) + \frac{\underline{e_3}}{h_3}\frac{\partial}{\partial u_3}(F_3h_1h_2)$$

and similarly for $\nabla(F_2h_3h_1)$ and $\nabla(F_3h_1h_2)$.
Using $\underline{e_i}\cdot\underline{e_j} = \delta_{ij}$ gives result.
To prove that $\div\frac{\underline{e_1}}{h_2h_3} = 0 +$ cyclic perms.
write $\underline{e_1} = \underline{e_2}\times\underline{e_3}$ (orthonormality, can be $\underline{e_1} = \underline{e_2}\times\underline{e_3}$)

$$\frac{\underline{e_1}}{h_2h_3} = \frac{\underline{e_2}}{h_2}\times\frac{\underline{e_3}}{h_3} = \nabla u_2\times\nabla u_3$$

Using identity

$$\dot{\nabla(}\underline{F}\times\underline{G}) = (\nabla\times\underline{F})\cdot\underline{G} - (\nabla\times\underline{G})\cdot\underline{F}$$

$$\nabla\cdot\frac{\underline{e_1}}{h_2h_3} = (\nabla\times\nabla u_2).\nabla u_3 - (\nabla\times\nabla u_3)\cdot\nabla u_2 = 0$$

$$\text{curl grad} = 0$$

Curl in orthogonal curv. coords.

$$\text{curl }\underline{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underline{e_1} & h_2\underline{e_2} & h_3\underline{e_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

Proof: not given

Laplacian

$$\begin{aligned} \nabla^2\phi &= \text{div grad } \phi \\ &= \div () \\ &= [] \end{aligned}$$

$$\Rightarrow \nabla^2 = \frac{1}{}[]$$