Part III: ODEs

A differential equation is an equation involving a function (or functions) and its derivatives.

An ordinary differential equation (ODE) is a differential equation involving a function (or functions) of one variable.

If the ODE involves the nth (and lower) derivatives it is said to be an nth order ODE.

Let y be a function of one variable (which we will always call x)

An equation of the form

$$h_1(x, y(x), y'(x)) = 0$$

is a rst order ODE.

$$h_2(x, y(x), y'(x), y''(x)) = 0$$

is second order.

A function satisfying $y(x)$ the ODE is called a solution of the ODE.

0.1 Linear ODEs (2 types)

i) homogeneous. If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where A and B are arbitrary constants.

ii) inhomogeneous. If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where $A + B = 1$.

The general 1st order linear ODE (for a single function) can be written

$$a(x)y'(x) + b(x)y(x) = f(x)$$

a, b and f are arbitrary functions. The equation is homogeneous if $f = 0$.

It is sometimes written in the form

$$y'(x) + p(x)y(x) = f(x).$$

($p = b/a$ and f/a has been renamed as f)

The general 2nd order linear ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x) \qquad (*)$$

1 a, b, c and f are arbitrary functions (homogeneous if $f = 0$).

We will sometimes write (*) in the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x). \qquad (**)$$

(**) looks more economical, but we will use (*) as well.

0.2 1st Order Case

All solutions of

$$y'(x) + p(x)y(x) = f(x)$$

can be written

$$y(x) = Cy_1(x) + y_p(x).$$

where $y_1(x)$ is a solution of the homogeneous equation $y'(x) + p(x)y(x) = 0$ and $y_p(x)$ is one solution of the full equation.

Proof

by explicit construction.

$$y'(x) + p(x)y(x) = f(x)$$

can be rewritten

$$\frac{d}{dx} e^{I(x)} y(x) = e^{I(x)} f(x)$$

where

$$I(x) = \int_a^x dz p(z).$$

(a is an arbitrary constant) which has the property $I'(x) = p(x)$. I is called an integrating factor. Now integrate from a to x

$$e^{I(x)} y(x) e^{I(a)} y(a) = \int_a^x dz e^{I(z)} f(z).$$

Note that $e^{I(a)} = 1$. This gives

$$y(x) = Cy_1(x) + y_p(x),$$

with $y_1(x) = e^{I(x)}$, $y_p(x) = e^{I(x)} \int_a^x dz e^{I(z)} f(z)$ and $C = y(a)$.

Example

Find all solutions of the ODE 1

$$y'(x) + \frac{1}{x} y(x) = x^3.$$

Here $p(x) = 1/x$ which has a non-integrable singularity at $x = 0$! Work with $x > 0$ (or $x < 0$).

$I(x) = \int dx p(x) = logx + c$. Set $c = 0$ (or $a = 1$). $e^{I(x)} = x$ so that the ODE can be written

$$\frac{d}{dx}(xy) = x.x^3 = x^4.$$

Integrating gives $xy = \frac{1}{5}x^5 + C$ or $y = \frac{1}{5}x^4 + C/x$, i.e. $y_1(x) = 1/x$, $y_p(x) = \frac{1}{5}x^4$.

0.3 2nd Order Case

All solutions (or the general solution) of (*) or (**) can be written

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x).$$

y_1, y_2 are linearly independent solutions of the homogeneous equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0 \qquad \text{or} \qquad y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

or and $y_p(x)$ is a solution of the full equation. C_1 and C_2 are arbitrary constants.

Proof

Not given.

$y_p(x)$ is called a particular integral. The general solution is sometimes written

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ is called the complementary function. It is the general solution of the homogeneous form of the ODE.

0.4 Constant Coeffcients

We now consider (*) in the special case that a, b and c are constants

$$ay''(x) + by'(x) + cy(x) = f(x).$$

This type of equation has a nice interpretation as a damped/driven oscillator (where x is time and y is the displacement from equilibrium). Recall the equation for a simple harmonic oscillator

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t)$$

Now add in a damping force proportional to the velocity dy/dt and a driving force $f(t)$ (which may be periodic or non-periodic)

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) - \gamma\frac{dy(t)}{dt} + d(t)$$

which is a linear ODE with constant coeffcents.

The rst step in solving ODEs of this type is to nd two solutions of the homogeneous equation

$$ay''(x) + by'(x) + cy(x) = 0.$$

This equation has simple exponential solutions of the form $y(x) = e^{\lambda x}$. Differentiating $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$ so that

$$ay''(x) + by' + cy = (a\lambda^2 + b\lambda + c)y$$

which is zero provided

$$a\lambda^2 + b\lambda + c = 0.$$

This is called an auxiliary equation. Thus $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ where λ_1 and λ_2 are roots of the (quadratic) auxiliary equation. The complementary function (if $\lambda_1 = \lambda_2$) is $y_c(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

If $\lambda_1 = \lambda_2$ we only have one exponential solution. In this case a second solution of the ODE is $y(x) = xe^{\lambda_1 x}$ and $y_c(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$ (in the oscillator model this special case corresponds to critical damping).

Examples

i) $y'' + 3y' + 2y = 0$. Auxiliary equation $\lambda^2 + 3\lambda + 2 = 0$ roots $\lambda_1 = 1$, $\lambda_2 = 2$. General solution $y(x) = C_1 e^x + C_2 e^{2x}$ (over-damping).

ii) $y'' + 2y' + y = 0$. Auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$ with two equal roots $\lambda = 1$. General solution $y(x) = (C_1 + C_2 x)e^x$

iii) Auxiliary equation $\lambda^2 + \lambda + 1 = 0$ with complex roots $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$.

The general complex solution is

$$y(x) = C_1 e^{-\frac{1}{2}x + i\frac{1}{2}\sqrt{3}x} + C_2 e^{-\frac{1}{2}x - i\frac{1}{2}\sqrt{3}x}$$

where C_1 and C_2 are complex constants. The general real solution can be obtained by imposing the constraint $C_2 = \bar{C}_1$:

$$y(x) = e^{\frac{1}{2}x}\left[C_1 e^{\frac{1}{2}x}\left(\cos\frac{1}{2}\sqrt{3}x + i\sin\cos\frac{1}{2}\sqrt{3}x\right) + \text{ c.c.}\right]$$

Writing $C_1 = \frac{1}{2}(AiB)$ where A and B are real constants gives

$$y(x) = e^{\frac{1}{2}}x\left(A\cos\frac{1}{2}\sqrt{3}x + B\sin\frac{1}{2}\sqrt{3}x\right)$$

(underdamped- still oscillates).