A note on conservative fields[1]

3 November 2005

Conservative fields

A smooth vector field $\mathbf{F}$ is *conservative* iff there exists a smooth scalar field ϕ such that

$$\mathbf{F} = \operatorname{grad}\phi \tag{1}$$

ϕ is often called a *potential* for $\mathbf{F}$.

Since $\operatorname{curl}\operatorname{grad}\phi = 0$ for any scalar field ϕ, $\operatorname{curl}\mathbf{F} = 0$ is a neccessary condition for $\mathbf{F}$ to be conservative. It isn't suffiecient, however, this is something we will return to, but, for now, we notice that it makes it easy to spot fields that aren't conservative, for example, if $\mathbf{F} = (-y, x, 0)$ then

$$\nabla \times \mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \tag{2}$$

On the other hand, it is easy to see that $\mathbf{F} = \mathbf{r}$ is conservative because

$$\nabla \frac{r^2}{2} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3}$$

and therefore $\phi = r^2/2$ is a potential for $\mathbf{F}$. Of course, $\operatorname{curl}\mathbf{r} = 0$.

Path independent fields

A field $\mathbf{F}$ is *path independent* if, for any two points p_1 and p_2 the line integral along any path between those points has the same value.

In other words, a field is path independent if the line integral doesn't depend on the path it is taken along. Obviously, the integral around a closed path should be zero for a path independent field and, in fact we can state a lemma:

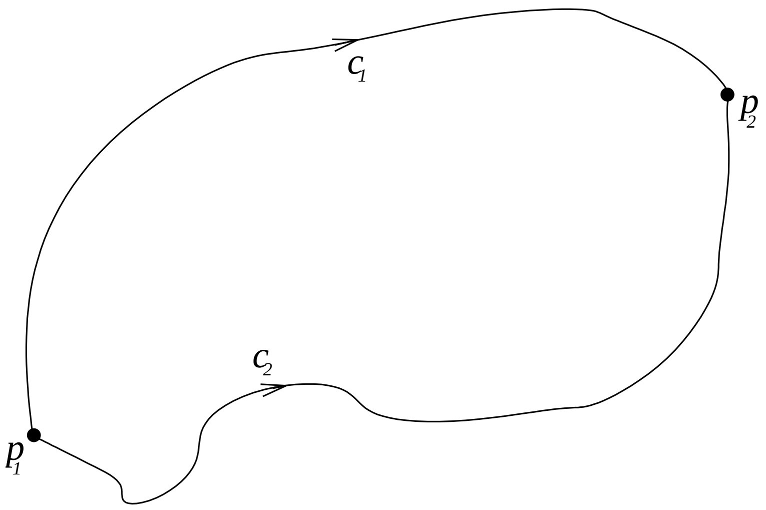
Lemma: A smooth vector field $\mathbf{F}$ is path independent iff

$$\oint_c \mathbf{F} \cdot \mathbf{dl} = 0 \tag{4}$$

for any closed curve c.[2]

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/231`

[2] The integral sign with a circle on it is a standard notation for an integral around a closed path.



Proof: For a path independent $\mathbf{F}$ choose any two points on a closed curve c and label the two curves from p_1 to p_2 as c_1 and c_2 so that $c = c_1 - c_2$. Now, by path independence,

$$\int_{c_1} \mathbf{F} \cdot \mathbf{dl} = \int_{c_2} \mathbf{F} \cdot \mathbf{dl} \tag{5}$$

and so

$$\oint_{c_1 - c_2} \mathbf{F} \cdot \mathbf{dl} = \oint_c \mathbf{F} \cdot \mathbf{dl} = 0 \tag{6}$$

Conversely, two difference paths between two points p_1 and p_2 can be subtacted from each other to give a closed path $c = c_1 - c_2$ so

$$0 = \oint_c \mathbf{F} \cdot \mathbf{dl} = \oint_{c_1 - c_2} \mathbf{F} \cdot \mathbf{dl} \tag{7}$$

implies

$$\int_{c_1} \mathbf{F} \cdot \mathbf{dl} = \int_{c_2} \mathbf{F} \cdot \mathbf{dl} \tag{8}$$

$\square$

Conservative fields are path independent

Theorem: On a connected domain, a smooth vector field $\mathbf{F}$ is path-independent iff it is conservative.
Proof: First, given a conservative field $\mathbf{F} = \nabla \phi$ for some smooth ϕ consider the line integral along some curve c from p_1 to p_2. Let $\mathbf{r}(t)$ be a parameterization of the curve with $p_1 = \mathbf{r}(t_1)$ and $p_2 = \mathbf{r}(t_2)$. Now,

$$\int_c \mathbf{F} \cdot \mathbf{dl} \quad = \quad \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} \nabla\phi \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial\phi}{dx}\frac{dx}{dt} + \frac{\partial\phi}{dy}\frac{dy}{dt} + \frac{\partial\phi}{dy}\frac{dy}{dt} \right) dt
\end{aligned} \tag{9}
$$

where we have used the fact that $\mathbf{F}$ is conservative and we have written the dot product out explicitely. Now, we use the chain rule to rewrite the dot product

$$
\frac{\partial\phi}{dx}\frac{dx}{dt} + \frac{\partial\phi}{dy}\frac{dy}{dt} + \frac{\partial\phi}{dy}\frac{dy}{dt} = \frac{d\phi}{dt} \tag{10}
$$

where $\phi(x, y, z)$ is a function of t along the curve through the t dependence of x, y and z: $\phi(t) = \phi(x(t), y(t), z(t))$. This gives

$$
\int_c \mathbf{F} \cdot \mathbf{dl} = \int_{t_1}^{t_2} \frac{d\phi}{dt} dt = \phi(t)]_{t_1}^{t_2} = \phi(t_2) - \phi(t_1) \tag{11}
$$

which depends only on the value of the ϕ at the begining and end of the curve and is therefore path independent.

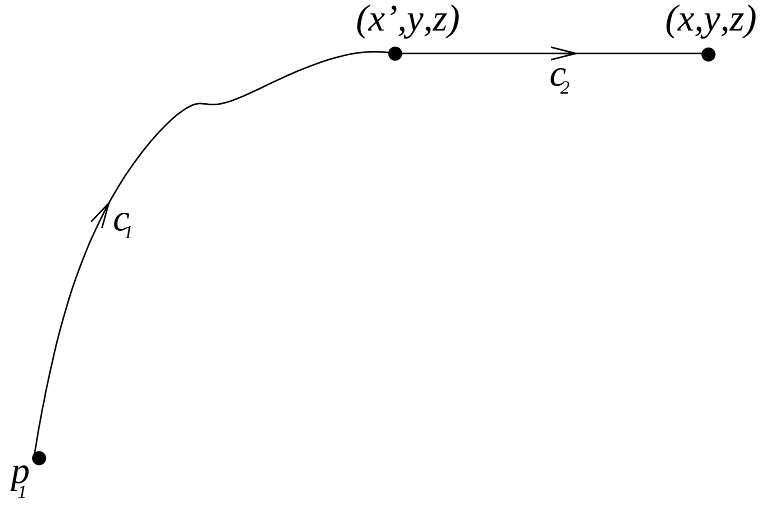
To go the other way, for give path independent $\mathbf{F}$ and some fixed point p let

$$
\phi(\mathbf{r}) = \int_c \mathbf{F} \cdot \mathbf{dl} \tag{12}
$$

where c is a path from p to $\mathbf{r}$. Since the field is path independent ϕ is well defined. It clearly depends on the choice of the reference point p and so ϕ isn't unique, however, two choices only differ by an overall constant. We now want to prove that $\mathbf{F} = \nabla\phi$. We begin by proving

$$
F_1 = \frac{\partial\phi}{\partial x} \tag{13}
$$

where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. If this hold it should hold for all paths, so lets choose a path $c = c_1 + c_2$ where c_1 goes from p to a point (x', y, z) and c_2 runs from (x', y, z) to (x, y, z).



Now,

$$\begin{aligned}
\phi(\mathbf{r}) &= \int_c \mathbf{F}\cdot\mathbf{dl} = \int_{c_1} \mathbf{F}\cdot\mathbf{dl} + \int_{c_2} \mathbf{F}\cdot\mathbf{dl} \\
&= \int_{c_1} \mathbf{F}\cdot\mathbf{dl} + \int_{x'}^{x} F_1 dx \qquad (14)
\end{aligned}$$

where in the last line we have used that $dy = dz = 0$ along the curve c_2. Now, noting that the first integral doesn't depend on x, we differenciate

$$\frac{\partial}{\partial x}\phi = \frac{d}{dx}\int_{x'}^{x} F_1 dx = F_1 \qquad (15)$$

using the fundamental theorem of calculus. Now, a similar arguement could be used for the other component and so the theorem is proved. $\square$

Conservative fields and irrotational fields

All conservative fields are irrotational because

$$\nabla\times\mathbf{F} = \nabla\times\nabla\phi = 0 \qquad (16)$$

but, the converse isn't true in general, an irrotational field is not neccessarily conservative. However, Stoke's theorem can be used to prove that it is true locally: if a field is irrotation then for any point there is a neighbourhood of the point for which it is conservative.

A more general theorem applies to simply connected domain. A domain is *simply connected* if any two paths between the same two points can be deformed into each other smoothly. The plane is simply connected but the plane minus a point, say $\mathbf{R}^2\setminus(0,0)$[3] is not because the missing point gets in the way. So, in the $\mathbf{R}^2\setminus(0,0)$ example the path from $(-1,0)$ to $(1,0)$ along the upper half circle can't be deformed into the path along the lower half circle because the missing point gets in the way. Anyway, without proof, we have a theorem:
Theorem: On a simply connected domain a smooth vector field is irrotational iff it is conservative.

[3]The backslash is a set minus, that is

$$\mathbf{R}^2\setminus(0,0) = \{(x,y)\in\mathbf{R}^2 | (x,y)\neq(0,0)\} \qquad (17)$$