Vector Potentials

Recall that if curl $\mathbf{F} = 0$ in a simply-connected region then $\mathbf{F}$ is conservative meaning there exists a scalar potential ϕ such that $\mathbf{F} = \text{grad}\ \phi$.

There is a similar result for solenoidal vector fields:

Theorem If div $\mathbf{F} = 0$ in a region without inner boundaries there exists a vector field $\mathbf{A}$ such that $\mathbf{F} = \text{curl}\ \mathbf{A}$. $\mathbf{A}$ is called a *vector potential* for $\mathbf{F}$.

Example $\mathbf{F} = \mathbf{B}$ a constant vector (obviously solenoidal). A vector potential is $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$.

The above theorem will not be proved in full here. Rather a (constructive) proof is given for the special case of a *star-shaped region.*

[A region D is called star-shaped if it has a point O such that the line-segment joining O and any other point in D lies within D.]

If D is star-shaped there is a formula for the vector potential for a solenoidal vector field

$$\mathbf{A}(\mathbf{r}) = \int_0^1 dt\ \mathbf{F}(t\mathbf{r}) \times t\mathbf{r}.$$

where the point O in D is taken to be the origin ($\mathbf{r} = 0$).

Proof To prove this we have to show that taking the curl of the right hand side reproduces the original vector field $\mathbf{F}$. We have

$$\text{curl}\ \mathbf{A}(\mathbf{r}) = \int_0^1 dt\ \text{curl}\left(\mathbf{F}(t\mathbf{r}) \times t\mathbf{r}\right).$$

Using the vector identity

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

$$\text{curl}\ (\mathbf{F}(t\mathbf{r}) \times t\mathbf{r}) = 3t\mathbf{F}(t\mathbf{r}) + t(\mathbf{r} \cdot \nabla)\mathbf{F}(t\mathbf{r}) - 0 - t(\mathbf{F}(t\mathbf{r}) \cdot \nabla)\mathbf{r},$$

using div $\mathbf{r} = 3$ and div $\mathbf{F} = 0$.

A straightforward calculation gives

$$\left(\mathbf{F}(t\mathbf{r}) \cdot \nabla\right)\mathbf{r} = \mathbf{F}(t\mathbf{r})$$

We also require

$$(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r}) = t\frac{d}{dt}\mathbf{F}(t\mathbf{r}),$$

or

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\mathbf{F}(tx,ty,tz) = t\frac{d}{dt}\mathbf{F}(tx,ty,tz),$$

which is a generalisation of

$$a\frac{d}{da}g(at) = t\frac{d}{dt}g(at).$$

Inserting these two formulae into the expression for curl $(\mathbf{F}(t\mathbf{r})\times t\mathbf{r})$ gives

$$\text{curl }\mathbf{A} = \int_0^1 \,dt\,\left[2t\mathbf{F}(t\mathbf{r})+t^2\frac{d}{dt}\mathbf{F}(t\mathbf{r})\right] = \int_0^1 \,dt\,\frac{d}{dt}\left(t^2\mathbf{F}(t\mathbf{r})\right) = \mathbf{F}(\mathbf{r}),$$

using the FToC.

Note that the vector potential for a solenoidal vector field is not unique; if $\mathbf{A}$ is a vector potential for $\mathbf{F}$ then so is

$$\mathbf{A}' = \mathbf{A} + \text{grad }\phi, \tag{1}$$

where ϕ is *any* scalar field since

$$\text{curl grad }\phi = 0.$$

In electromagnetic theory this ambiguity is called *gauge freedom* and (1) is called a *gauge transformation.*

We have seen that in a simply-connected region an irrotational vector field can be written as a scalar and in a region without inner boundaries (though the proof was only given for star-shaped regions) a solenoidal vector field can be expressed as a curl. Now if $\mathbf{F}$ is neither solenoidal nor irrotational it can be *decomposed* into a gradient and a curl. There are a number of versions of this statement and they go under various names (Fundamental theorem of vector analysis, Helmholtz' theorem, Hodge decomposition).

Here we give a simple version of the decomposition theorem:

Any vector field $\mathbf{F}$ defined in a region D without inner boundaries can be written

$$\mathbf{F} = \text{grad } \phi + \text{curl } \mathbf{A}$$

Approach to Proof: Assume that $\mathbf{F}$ is not solenoidal and consider $\mathbf{F} - \nabla\phi$ where ϕ is (for now) any scalar field. We have

$$\text{div } (\mathbf{F} - \nabla\phi) = \text{div } \mathbf{F} - \nabla^2\phi.$$

Now if there exists a scalar field satisfying the equation

$$\nabla^2\phi = \text{div } \mathbf{F} \qquad (*)$$

then $\mathbf{F} - \nabla\phi$ is solenoidal in D and so we can write

$$\mathbf{F} - \text{grad } \phi = \text{curl } \mathbf{A}.$$

To complete the proof we need to show that (*) always has a smooth solution ϕ. (*) is actually a form of Poisson's equation. *More later.......*