PART III ODEs

A differential equation is an equation involving a function (or functions) and its derivatives.

An ordinary differential equation (ODE) is a differential equation involving a function (or functions) of one variable.

If the ODE involves the nth (and lower) derivatives it is said to be an nth order ODE.

Let y be a function of one variable (which we will always call x)

An equation of the form

$$h_1(x, y(x), y'(x)) = 0$$

is a first order ODE.

$$h_2(x, y(x), y'(x), y''(x)) = 0$$

is second order.

A function satisfying $y(x)$ the ODE is called a *solution* of the ODE.

Linear ODEs (2 types)

i) homogeneous. If y_1 and y_2 are solutions so is $Ay_1 + By_2$ where A and B are arbitrary constants.

ii) inhomogeneous. If y_1 and y_2 are solutions so is Ay_1+By_2 where $A+B = 1$.

The general 1st order linear ODE (for a single function) can be written

$$a(x)y'(x) + b(x)y(x) = f(x)$$

a, b and f are arbitrary functions. The equation is homogeneous if $f = 0$.

It is sometimes written in the form

$$y'(x) + p(x)y(x) = f(x).$$

($p = b/a$ and f/a has been renamed as f)

The general 2nd order linear ODE

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x) \tag{$*$}$$

a, b, c and f are arbitrary functions (homogeneous if $f = 0$).
We will sometimes write (*) in the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x). \qquad (**)$$

(**) looks more economical, but we will use (*) as well.

1st Order Case
All solutions of

$$y'(x) + p(x)y(x) = f(x)$$

can be written

$$y(x) = Cy_1(x) + y_p(x).$$

where $y_1(x)$ is a solution of the homogeneous equation $y'(x) + p(x)y(x) = 0$ and $y_p(x)$ is one solution of the full equation.
Proof by explicit construction.

$$y'(x) + p(x)y(x) = f(x)$$

can be rewritten

$$\frac{d}{dx}e^{I(x)}y(x) = e^{I(x)}f(x)$$

where

$$I(x) = \int_a^x dz\ p(z),$$

(a is an arbitrary constant) which has the property $I'(x) = p(x)$. I is called an integrating factor. Now integrate from a to x

$$e^{I(x)}y(x) - e^{I(a)}y(a) = \int_a^x dz\ e^{I(z)}\ f(z).$$

Note that $e^{I(a)} = 1$. This gives

$$y(x) = Cy_1(x) + y_p(x),$$

with $y_1(x) = e^{-I(x)}$, $y_p(x) = e^{-I(x)} \int_a^x dz\ e^{I(z)} f(z)$ and $C = y(a)$.

Example Find all solutions of the ODE

$$y'(x) + \frac{1}{x}y(x) = x^3.$$

Here $p(x) = 1/x$ which has a non-integrable singularity at $x = 0$! Work with $x > 0$ (or $x < 0$).

$I(x) = \int dx\ p(x) = \log x + c$. Set $c = 0$ (or $a = 1$). $e^{I(x)} = x$ so that the ODE can be written

$$\frac{d}{dx}(xy) = x \cdot x^3 = x^4.$$

Integrating gives $xy = \frac{1}{5}x^5 + C$ or $y = \frac{1}{5}x^4 + C/x$, i.e. $y_1(x) = 1/x$, $y_p(x) = \frac{1}{5}x^4$.

2nd Order Case

All solutions (or the *general solution*) of (*) or (**) can be written

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x).$$

y_1, y_2 are linearly independent solutions of the homogeneous equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0 \qquad \text{or} \qquad y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

and $y_p(x)$ is a solution of the full equation. C_1 and C_2 are arbitrary constants.
Proof Not given.

$y_p(x)$ is called a particular integral. The general solution is sometimes written

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ is called the complementary function. It is the general solution of the homogeneous form of the ODE.

Constant Coefficients

We now consider (*) in the special case that a, b and c are constants

$$ay''(x) + by'(x) + cy(x) = f(x).$$

This type of equation has a nice interpretation as a damped/driven oscillator (where x is time and y is the displacement from equilibrium). Recall the equation for a simple harmonic oscillator

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t).$$

Now add in a damping force proportional to the velocity dy/dt and a driving force $f(t)$ (which may be periodic or non-periodic)

$$\frac{d^2y(t)}{dt^2} = -\omega^2 y(t) - \gamma\frac{dy(t)}{dt} + f(t),$$

which is a linear ODE with constant coefficents.

The first step in solving ODEs of this type is to find two solutions of the homogeneous equation

$$ay''(x) + by'(x) + cy(x) = 0.$$

This equation has simple exponential solutions of the form $y(x) = e^{\lambda x}$. Differentiating $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$ so that

$$ay'' + by' + cy = (a\lambda^2 + b\lambda + c)y$$

which is zero provided

$$a\lambda^2 + b\lambda + c = 0.$$

This is called an *auxiliary equation.* Thus $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ where λ_1 and λ_2 are roots of the (quadratic) auxiliary equation. The complementary function (if $\lambda_1 \neq \lambda_2$) is $y_c(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

If $\lambda_1 = \lambda_2$ we only have one exponential solution. In this case a second solution of the ODE is $y(x) = xe^{\lambda_1 x}$ and $y_c(x) = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$ (in the oscillator model this special case corresponds to critical damping).

Examples i) $y'' + 3y' + 2y = 0$. Auxiliary equation $\lambda^2 + 3\lambda + 2 = 0$ roots $\lambda_1 = -1$, $\lambda_2 = -2$. General solution $y(x) = C_1 e^{-x} + C_2 e^{-2x}$ (over-damping).

ii) $y'' + 2y' + y = 0$. Auxiliary equation $\lambda^2 + 2\lambda + 1 = 0$ with two equal roots $\lambda = -1$. General solution $y(x) = (C_1 + C_2 x)e^{-x}$.

iii) Auxiliary equation $\lambda^2 + \lambda + 1 = 0$ with complex roots $\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$. The general complex solution is

$$y(x) = C_1 e^{-\frac{1}{2}x + i\frac{1}{2}\sqrt{3}x} + C_2 e^{-\frac{1}{2}x - i\frac{1}{2}\sqrt{3}x},$$

where C_1 and C_2 are complex constants. The general real solution can be obtained by imposing the constraint $C_2 = \bar{C}_1$:

$$y(x) = e^{-\frac{1}{2}x}\left[C_1 e^{-\frac{1}{2}x}\left(\cos\frac{1}{2}\sqrt{3}x + i\sin\frac{1}{2}\sqrt{3}x\right) + \text{c.c.}\right].$$

Writing $C_1 = \frac{1}{2}(A - iB)$ where A and B are real constants gives

$$y(x) = e^{-\frac{1}{2}x}\left(A\cos\frac{1}{2}\sqrt{3}x + B\sin\frac{1}{2}\sqrt{3}x\right).$$

(underdamped- still oscillates).

Returning to the inhomogeneous form $ay''(x)+by'(x)+cy(x) = f(x)$ we still require a particular integral y_p, i.e. *one* solution of the full equation. If f is exponential then it is straightforward - y_p is an exponential proportional to f. For example consider

$$y'' + 3y' + 2y = e^x.$$

Trying $y_p = Ce^x$ where C is a constant gives $(1+3+2)Ce^x = e^x$ so $C = \frac{1}{6}$, therefore $y_p(x) = \frac{1}{6}e^x$ is a PI. To obtain the general solution we require the general solution of the homogeneous equation (see earlier example) $y_c = C_1e^{-x} + C_2e^{-2x}$. General solution $y(x) = y_c + y_p = C_1e^{-x} + C_2e^{-2x} + \frac{1}{6}e^x$.

If f is a solution of the homogeneous ODE this doesn't work, e.g. $y'' + 3y' + 2y = e^{-x}$. Trying $y_p = Ce^{-x}$ gives $C = \infty$. Much as in the equal-root case for the homogeneous equation try $y_p = Cxe^{-x}$. Differentiating $y_p' = Ce^{-x} - Cxe^{-x}$ and $y_p'' = -2Ce^{-x} + Cxe^{-x}$. Inserting these into the ODE gives $Cxe^{-x}(1-3+2) + Ce^{-x}(-2+3) = e^{-x}$ so that $C = 1$, i.e. $y_p = xe^{-x}$. The general solution can be written $y = (C_1 + x)e^{-x} + C_2e^{-2x}$.

If f is a sum of exponentials, e.g. $\sin x = (e^{ix} - e^{-ix})/(2i)$ just add up the PIs corresponding to each exponential term.

If f is not a finite sum of exponentials decompose f into complex exponentials (Fourier analysis).

Example Obtain the general solution of $y''(x)+3y'(x)+2y(x) = f(x)$ where f is the periodic square wave

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases} \qquad f(x+2\pi) = f(x).$$

CF: $y_c = C_1e^{-x} + C_2e^{-2x}$ (as in earlier example)

PI: Expand f as a Fourier series (done it before)

$$f(x) = \frac{4}{\pi}\sum_{n \text{ odd } >0}\frac{1}{n}\sin nx = \frac{2}{\pi i}\sum_{n \text{ odd } \in Z}\frac{1}{n}e^{inx}.$$

Find PI for $y'' + 3y' + 2y = e^{inx}$. Trying $y_p = Ce^{inx}$ gives $C(-n^2 + 3in + 2)e^{inx} = e^{inx}$ so that

$$C = \frac{1}{-n^2 + 3in + 2}.$$

PI of full problem

$$y_p(x) = \frac{2}{\pi i} \sum_{n \text{ odd } \in Z} \frac{e^{inx}}{n(-n^2 + 3in + 2)}.$$

If f is not periodic write it as a Fourier integral

Example Find the general solution of the ODE $y''(x) + 2y'(x) + 2y(x) = f(x)$ where $f(x) = e^{-x^2}$.

CF: Auxiliary equation $\lambda^2 + 2\lambda + 2 = 0$ with two complex roots $\lambda = -1 \pm i$ so that $y_c = e^{-x}(A \cos x + B \sin x)$.

PI: Write f as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk \; e^{ikx} \; \tilde{f}(k)$$

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \; e^{-ikx} \; e^{-x^2} = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}k^2} \qquad \text{(problem 4 sheet 12).}$$

Obtain PI for $y'' + 2y' + 2y = e^{ikx}$. Trying $y = Ce^{ikx}$ gives $C(-k^2 + 2ik + 2)e^{ikx} = e^{ikx}$ giving

$$C = \frac{1}{-k^2 + 2ik + 2}.$$

PI of full problem

$$y_p(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dk \; \frac{e^{ikx} e^{-\frac{1}{4}k^2}}{(-k^2 + 2ik + 2)}.$$

Returning to the general form of (*) $a(x)y''(x)+b(x)y'(x)+c(x)y(x)=f(x)$. If two solutions, y_1 and y_2, of the homogeneous equation are known then a particular solution of the full equation, y_p, can be found.
Approach to proof (not examinable)

Consider the equation

$$a(x)y''(x)+b(x)y'(x)+c(x)y(x)=\delta(x-x').$$

A solution of this equation is called a Green's function and denoted $G(x|x')$. A solution of (*) is then

$$y_p(x)=\int_{-\infty}^{\infty} dx'\ f(x')\ G(x|x').$$

There is a formula for $G(x|x')$ (note that $G(x|x')$ is not unique)

$$G(x|x')=\theta(x-x')\frac{y_1(x)y_2(x')-y_2(x)y_1(x')}{y_1(x')y_2'(x')-y_2(x')y_1'(x')}.$$

The object in the denominator is called the Wronksian of y_1 and y_2.

Returning to the homogeneous form of (*) $a(x)y''(x)+b(x)y'(x)+c(x)y(x)=0$ no general solution is known but many specific cases have been solved. Some examples

$$\alpha x^2y''+\beta xy'+\gamma y=0 \qquad \text{(Euler's equation)}$$

α, β, γ constants.

$$x^2y''+xy'+(\nu^2-x^2)y=0 \qquad \text{(Bessel's equation)}$$

where ν is a constant.

$$(1-x^2)y''-2xy'+2\alpha y=0 \qquad \text{(Legendre's equation)}$$

where α is a constant.

$$y''-2xy'+2\alpha y=0 \qquad \text{(Hermite's equation)}$$

where α is a constant.

Solutions of Euler's, Bessel's and Legendre's equations are required in the solution of Laplace's equation (more in part IV).

Euler's Equation

This can be transformed to the constant coefficient case through the change of variable $x = e^z$. This gives

$$x\frac{dy}{dx} = e^z \frac{dy}{dz}\frac{dz}{dx} = e^z \frac{dy}{dz}\left(\frac{dz}{dx}\right)^{-1} = \frac{dy}{dz}$$

$$x^2\frac{d^2y}{dx^2} = x^2\frac{d}{dx}\frac{dy}{dx} = x^2\frac{d}{dx}\frac{1}{x}\frac{dy}{dz} = -\frac{dy}{dz} + x\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{dy}{dz} + \frac{d^2y}{dz^2}$$

so the Euler's equation becomes

$$\alpha\frac{d^2y}{dz^2} + (\beta - \alpha)\frac{dy}{dz} + \gamma y = 0$$

(constant coefficients). Auxiliary equation

$$\alpha\lambda^2 + (\beta - \alpha)\lambda + \gamma = 0.$$

The general solution is

$$y_c = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} = C_1 x^{\lambda_1} + C_2 x^{\lambda_2} \qquad (\lambda_1 \neq \lambda_2)$$

where λ_1 and λ_2 are roots of the auxiliary equation.

Euler case solved through change of variable -in general such a simple change of variable is not available. Other methods required.

Series Solutions

Idea: write y as a power series about $x = 0$ or some other point

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and determine a *recursion relation* for the a_n coefficients.

Example Consider Hermite's equation $y'' - 2xy' + 2\alpha y = 0$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$xy'(x) = \sum_{n=0}^{\infty} n a_n x^n,$$

$$\begin{aligned} x^2 y'(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \qquad (n = 0 \text{ and } n = 1 \text{ terms zero}) \\ &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m \qquad (\text{relabel } m = n+2) \end{aligned}.$$

This gives

$$y'' - 2xy' + 2\alpha y = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 2(\alpha - n) a_n \right] x^n = 0.$$

This implies that the content of the square bracket is zero for all n leading to the recursion relation

$$a_{n+2} = \frac{2(n-\alpha) a_n}{(n+1)(n+2)}.$$

From this two independent solutions can be obtained.

An even solution: Set $a_1 = 0$. Recursion relation $\rightarrow$ a_3, a_5, a_7 etc. all zero. Fix $a_0 = 1$ and apply recursion relation

$$\begin{aligned} a_2 &= -\frac{2\alpha a_0}{1 \cdot 2} = -\frac{2\alpha}{1 \cdot 2} \\ a_4 &= \frac{2(2-\alpha) a_2}{3 \cdot 4} = 2^2 \frac{(\alpha - 2)\alpha}{1 \cdot 2 \cdot 3 \cdot 4} \\ a_6 &= \frac{2(4-\alpha) a_4}{5 \cdot 6} = -\frac{2^3 (\alpha - 4)(\alpha - 2)\alpha}{6!} \end{aligned}$$

pattern clear

$$a_{2m} = \frac{(-2)^m}{(2m)!} (\alpha - 2m + 2)(\alpha - 2m + 4) \ldots \alpha$$

(define $0! = 1$). An even solution of Hermite's equation reads

$$y_{even}(x) \sum_{m=0}^{\infty} \frac{(-2)^m}{(2m)!}(\alpha - 2m + 2)(\alpha - 2m + 4)...\alpha \; x^{2m}.$$

This series is convergent (radius of convergence=∞; use ratio test to prove this). For special values of α (even and positive) the series terminates and the solution is a polynomial of degree α. For example, when $\alpha = 2$ a_4, a_6, a_8 etc. all zero and $y_{even}(x) = 1 - 2x^2$ (check that this satisfies Hermite's equation $y'' - 2xy' + 4y = 0$).

An odd solution: Set $a_0 = 1$, $a_1 = 1$. Recursion relation $\rightarrow$ a_2, a_4, a_6 et. all zero. Odd coefficients can be worked out via the recursion formula. If α is an odd integer the series will terminate producing a polynomial of degree α.

General solution of Hermite's equation $y = C_1 y_{even}(x) + C_2 y_{odd}(x)$.

Generating Function

If α is a positive integer *one* of the solutions to Hermite's equation is polynomial. Such functions are called Hermite polynomials. Remarkably, all the polynomials can be combined into a single *generating function.*

Consider $\Phi(x, h) = e^{2xh - h^2}$. Expanding this in powers of h:

$$\Phi(x, h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} H_n(x)$$

$H_n(x)$ are polynomial solutions of Hermite's equation (see problem sheet 18).

Method of Frobenius

Homogeneous form of (**) $y''(x) + p(x)y'(x) + q(x)y(x) = 0$. Series method worked in Hermite case where both p and q are smooth. If p and q have singularities the series method sometimes fails, e.g. Euler's equation $\alpha x^2 y'' + \beta x y' + \gamma y = 0$ or $p(x) = \beta/(\alpha x)$ and $q(x) = \gamma/(\alpha x^2)$. The explicit solution $y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$ *not* picked up by power series method (unless both roots λ_1 and λ_2 are positive integers).

Way out: expand about a point other than $x = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

where $p(c)$ and $q(c)$ are finite. However, a singular point can often be the 'most symmetric' point.

Frobenius (or generalised series) method allows one to expand about a *regular singularity* (more later) of p and q. Without loss of generality consider an expansion about $x = 0$. Consider a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \; x^{n+s}$$

where s is some real number. Unlike in the standard power series method a_0 is always taken to be non-zero (the odd solution of Hermite's equation would emerge as an $s = 1$ Frobenius series with $a_0 \neq 0$). Start with s arbitrary -consistency leads to a quadratic equation for s called the *indicial equation.*
Example Bessel's equation $x^2y''+xy'+(x^2-\nu^2)y = 0$ ($p = 1/x$, $q = 1-\nu^2/x^2$) where ν is a constant.

$$y(x) = \sum_{n=0}^{\infty} a_n \; x^{n+s}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n(n+s) \; x^{n+s-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n(n+s)(n+s-1) \; x^{n+s-2}$$

Collect contributions to ODE

$$x^2y(x) = \sum_{n=0}^{\infty} a_n \; x^{n+s+2}$$

This is the least singular term; its first term a_0x^{s+2} has a higher power of x than the first terms of x^2y'' etc. Relabel 'more singular' contributions

$$\begin{aligned} x^2y''(x) =& \sum_{n=0}^{\infty} a_n(n+s)(n+s-1) \; x^{n+s} \\ =& a_0s(s-1)x^s + a_1(1+s)sx^{s+1} \\ & + \sum_{m=0}^{\infty} a_{m+2}(n+s+2)(n+s+1) \; x^{m+s+2} \qquad \text{relabel } n = m+2 \end{aligned}$$

$$
\begin{aligned}
xy'(x) =& \sum_{n=0}^{\infty} a_n(n+s)\ x^{n+s} \\
=& a_0 s x^s + a_1(1+s)x^{s+1} \\
&+ \sum_{m=0}^{\infty} a_{m+2}(m+s+2)\ x^{m+s+2} \qquad \text{relabel } n = m+2
\end{aligned}
$$

$$
\begin{aligned}
y(x) =& \sum_{n=0}^{\infty} a_n\ x^{n+s} \\
=& a_0 x^s + a_1 x^{s+1} + \sum_{m=0}^{\infty} a_{m+2}\ x^{m+s+2} \qquad \text{relabel } n = m+2
\end{aligned}
$$

The ODE can be written

$$
\begin{aligned}
x^2y'' +& xy' + (x^2 - \nu^2)y \\
=& a_0 \left[s(s-1) + s - \nu^2\right] x^s + a_1 \left[(1+s)s + (1+s) - \nu^2\right] x^{s+1} \\
+& \sum_{m=0}^{\infty} \left[(m+s+2)(m+s+1)a_{m+2} + (m+s+2)a_{m+2} - \nu^2 a_{m+2} + a_m\right] x^{m+s+2} \\
=& 0.
\end{aligned}
$$

Since this is true for all x the coefficients of all powers of x must vanish so that

$$a_0(s^2 - \nu^2) = 0$$

$$a_1\left((1+s)^2 - \nu^2\right) = 0$$

$$\left[(m+s+2)^2 - \nu^2\right] a_{m+2} = -a_m.$$

Asuuming $a_0 \neq 0$ gives $s^2 - \nu^2 = 0$ (the indicial equation), $a_1 = 0$ and the recursion relation

$$\left[(a+s+2)^2 - \nu^2\right] a_{m+2} = -a_m.$$

The roots of the indicial equation are $s = \pm\nu$. When $\nu = 0$ the indicial equation is $s^2 = 0$ so the method of Frobenius only delivers one solution (see problem 1 on sheet 18).

Here we consider the case $\nu = \frac{1}{2}$ so that $s = \pm\frac{1}{2}$. The recursion relation can be written

$$a_{m+2} = -\frac{a_m}{(m+2+\frac{1}{2})^2 - \frac{1}{4}} = -\frac{a_m}{(m+2)(m+3)}.$$

Since $a_1 = 0$ the recursion relation implies that a_3, a_5, a_7 etc. are all zero. Fixing $a_0 = 0$ and applying the recursion relation gives

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 3} = -\frac{1}{2 \cdot 3} \\ a_4 &= -\frac{a_2}{4 \cdot 5} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{5!} \\ a_6 &= -\frac{1}{7!} \quad \text{etc.} \end{aligned}$$

The solution is

$$\begin{aligned} y(x) &= x^{\frac{1}{2}} - \frac{x^{\frac{5}{2}}}{3!} + \frac{x^{\frac{9}{2}}}{5!} - \dots \\ &= x^{-\frac{1}{2}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) \\ &= x^{-\frac{1}{2}} \sin x. \end{aligned}$$

(recall $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$)

The other root $s = -\frac{1}{2}$ leads to

$$y(x) = x^{-\frac{1}{2}} \cos x,$$

and so the general solution of the $\nu = \frac{1}{2}$ problem is

$$y(x) = x^{-\frac{1}{2}} \left(C_1 \cos x + C_2 \sin x \right).$$

The method of Frobenius gives a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \; (x-c)^{n+s}$$

where p and/or q are singular at $x = c$. Method does not always give the general solution (e.g. $\nu = 0$ case of Bessel's equation).

There is a theorem dealing with the applicability of the Frobenius method in the case of *regular singularities.*

$x = c$ is a regular singular point if $(x - c)p(x)$ and $(x - c)^2 q(x)$ can be expanded as a power series about $x = c$ (all the singular ODEs we have met have regular singularities, an example of an ODE with a non-regular singularity $x^3 y'' + y = 0$ since here $q(x) = 1/x^3$ so that $x^2 q(x) = 1/x$ cannot be expanded about $x = 0$).

If p and q are non-singular at $x = c$, $x = c$ is called an *ordinary point* of the ODE $y''(x) + p(x)y'(x) + q(x)y(x) = 0$.

Fuchs' Theorem

$x = c$ is a regular singular or ordinary point of the ODE
$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$ if and only if

i) The two solutions are Frobenius series.

or

ii) One solution is a Frobenius series, $S_1(x)$ and the other solution is of the form $y(x) = S_1(x)\log(x - c) + S_2(x)$ where $S_2(x)$ is another Frobenius series (it is not a solution on its own).

Proof not given.

Case ii) occurs when the indicial equation has equal roots and *sometimes* when the roots differ by an integer (e.g. $\nu = 1$ case of Bessel's equation).

Finding a Second Solution

If one solution of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ can be found another one can be constructed. Let $u(x)$ be a solution then try $y(x) = u(x)v(x)$ then a short calculation gives

$$y'' + py' + qy = (u'' + pu' + qu)v + (2u' + pu)v' + uv'' = 0.$$

Now since u is, by assumption, a solution the first term on the right hand side is zero giving

$$(2u' + pu)v' + uv'' = 0.$$

This is a *first order* linear ODE for $v'(x)$ (see problem sheet 17).

Eigenfunctions and Eigenfunction Expansions

There is a strong analogy between solving some of the 'named' ODEs and finding the eigenvectors (and eigenvalues) of a matrix.

Hermite's equation $y'' - 2xy' + 2\alpha y = 0$ can be written

$$Ly = \lambda y$$

where L is the differential operator

$$L = -\frac{d^2}{dx^2} + 2x$$

and $\lambda = 2\alpha$.

Legendre's equation can be written in the same way, with

$$L = -(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}, \qquad \lambda = \alpha.$$

Can think of the differential operator L as a matrix (albeit an infinite dimensional one) and the function it acts on, y, as a vector.

To make this more precise it is useful to recall some properties of certain finite, say $n \times n$, matrices.

A *symmetric* matrix S satisfies

$$S^T = S, \qquad T = \text{transpose} \quad [A^T]_{ij} = [A]_{ji}.$$

A *Hermitian* matrix is an $n \times n$ matrix with complex entries satisfying

$$H^\dagger = H \qquad \dagger = \text{adjoint}.$$

Adjoint $\dagger$ denotes the complex conjugate of the transpose

$$[A^\dagger]_{ij} = \overline{[A]_{ji}}.$$

Clearly a real symmetric matrix is Hermitian.

Let v be an n-component column vector with complex entries. v is an eigenvector of H if

$$Hv = \lambda v$$

for some complex number λ (the eigenvalue).

Theorem: The eigenvalues of a Hermitian matrix are real.
Proof Let v be an eigenvector of H with eigenvalue λ

$$v^\dagger Hv = \lambda v^\dagger v.$$

Since $v^\dagger v$ is real it suffices to prove that $v^\dagger Hv$ is real:

$$\overline{v^\dagger Hv} = \overline{(v^\dagger Hv)^T}$$

(since $v^\dagger Hv$ is a 1×1 matrix)

$$= (v^\dagger Hv)^\dagger = v^\dagger H^\dagger v = v^\dagger Hv$$

if H is Hermitian. Note that $(AB)^\dagger = B^\dagger A^\dagger$.

Theorem: Eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal, i.e. if $Hv_1 = \lambda_1 v_1 \qquad Hv_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$ then $v_1^\dagger v_2 = 0$ or $(v_1, v_2) = 0$ (inner product notation).
Proof We have

$$v_2^\dagger Hv_1 = \lambda_1 v_2^\dagger v_1 \quad (A) \qquad v_1^\dagger Hv_2 = \lambda_1 v_1^\dagger v_2 \quad (B).$$

Take the complex conjugate of (B)

$$v_2^\dagger Hv_1 = \lambda_2 v_2^\dagger v_1 \qquad (\text{using } H^\dagger = H,\ \bar{\lambda}_2 = \lambda).$$

Subtracting (A) gives

$$0 = (\lambda_2 - \lambda_1) v_2^\dagger v_1$$

so that $(v_2, v_1) = 0$ if $\lambda_1 \neq \lambda_2$.

If eigenvalues degenerate can choose eigenvectors to be orthogonal (Gram-Schmidt).

Can choose the n eigenvectors v_i $(i = 1, 2, ..., n)$ of an Hermitian matrix H to be *orthonormal*

$$v_i^\dagger v_j = \delta_{ij} \qquad (v_i, v_j) = \delta_{ij}.$$

Any (n-component) vector v can be written as a linear combination of the v_is

$$v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$$

where c_1, c_2,... c_n are complex numbers.

Using the orthonormal property

$$(v_i, v) = c_i \qquad \text{(c.f. Fourier analysis)}$$

Also

$$|v|^2 = (v, v) = |c_1|^2 + |c_2|^2 + ...+!c_n|^2,$$

which can be thought of as an n-dimensional version of Pythagoras or a finite dimensional version of Parseval.

Resolution of unity: The identity matrix can be written as

$$I = \sum_{i=1}^{n} v_i \ v_i^\dagger,$$

since acting on any vector, $v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$, will reproduce v.

The inverse of an Hermitian matrix can be written as

$$H^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i} v_i \ v_i^\dagger,$$

since (left or right) multiplication by H reproduces the identity matrix.

Back to Differential operators

A differential operator L acts on some vector space of functions. Require functions to be such that

$$\int dx \ \bar{u}(x)(Lu)(x) < \infty.$$

It is usual to impose further restrictions on the functions such as

i) square integrability $\int_{-\infty}^{\infty} dx \ \bar{u}(x)u(x) < \infty$.

or

ii) periodicity

or

iii) vanishing at the end points of an interval $[a, b] \subset R$; $u(a) = u(b) = 0$.

In these cases an inner product can be defined

i) $(u, v) = \int_{-\infty}^{\infty} dx \ \bar{u}(x)v(x)$

ii) $(u,v)=\int_{-\pi}^{\pi}\ dx\ \bar{u}(x)u(x) \qquad (l=2\pi)$

iii) $(u,v)=\int_{a}^{b}\ dx\ \bar{u}(x)v(x)$.

Analogue of a Hermitian matrix:

Suppose

$$(u,Lv)=(Lu,v) \qquad (***)$$

for any u, v in the chosen space of functions (finite dimensional case: $H^\dagger=H$ $\equiv (Hu)^\dagger v = u^\dagger(Hv)$ for any vectors u,v). An operator satisfying (***) is called symmetric. If some further technical requirements are met it is called self-adjoint or Hermitian. We will be sloppy and call an operator satisfying (***) Hermitian.

Example An operator satisfying (***) is

$$L=-\frac{d^2}{dx^2}$$

for any of the three conditions i) ii) iii). Here (***) is just

$$-\int\ dx\ \bar{u}(x)v''(x)=-\int\ dx\ \bar{u}''v(x).$$

To establish this integrate by parts twice, e.g. for the periodic case

$$\int_{-\pi}^{\pi}\ dx\ \bar{u}(x)v''(x)=\bar{u}(x)v'(x)|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\ dx\ \bar{u}'(x)v'(x).$$

$\bar{u}(x)v'(x)|_{-\pi}^{\pi}$ is zero since u and v are periodic. Integrating by parts once more gives the result.

$L=-d^2/dx^2$ can be viewed as an Hermitian matrix acting on the space of periodic functions ($l=2\pi$). The eigenvectors (or eigenfunctions) are the functions $v_n(x)=e^{inx}$ $(n\in Z)$ with corresponding eigenvalues $\lambda_n=n^2$. These are orthogonal (as expected since L is Hermitian)

$$(v_m,v_n)=\int_{-\pi}^{\pi}\ dx\ \bar{v}_m(x)v_n(x)=\int_{-\pi}^{\pi}\ dx\ e^{-imx}e^{inx}=0 \quad (m\neq n).$$

Can make them orthonormal

$$v_n(x)=\frac{1}{\sqrt{2\pi}}e^{inx} \qquad (v_m,v_n)=\delta_{mn}.$$

A periodic function, f, can be thought of as a vector in the space acted on by L. Expand f in eigenvectors of L:

$$f(x) = \sum_{n \in Z} c_n v_n(x)$$

aka Fourier analysis

$$c_m = (v_m, f) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dx\ e^{-imx} f(x).$$

Parseval

$$(f, f) = \sum_{n \in Z} |c_n|^2.$$

(absence of 2π due to normalisation of v_ns to unity with the $1/\sqrt{2\pi}$ factor).

Fourier analysis $\equiv$ expanding in eigenvectors of the Hermitian operator $L = -d^2/dx^2$. Can choose a *different* Hermitian operator - this leads to alternative expansions - Legendre series, Hermite series, etc.

Legendre Series

Based on Legendre's equation

$$(1 - x^2)y''(x) - 2xy'(x) + \alpha y(x) = 0$$

This ODE has (regular) singularities at $x = \pm 1$. If α is of the form $\alpha = n(n + 1)$ where n is a non-negative integer then the ODE has a polynomial solution (see problem sheet 17) which is well defined at $x = \pm 1$ (the other solutions blow up at $x = 1$ and/or $x = -1$).

Recast ODE as an eigenvalue problem:

$$Ly(x) = \lambda y(x), \qquad L = -(1 - x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}.$$

Boundary conditions? $y(x)$ finite at $x = \pm 1$ (this excludes all non-polynomial solutions). The inner product is taken as

$$(u, v) = \int_{-1}^{1} dx\ \bar{u}(x)v(x).$$

It is straightforward to prove that with these boundary conditions L is Hermitian; it helps to write L in the form

$$L = -\frac{d}{dx}(1-x^2)\frac{d}{dx}.$$

The eigenfunctions are the polynomial solutions of Legendre's equation $P_n(x)$ $(n = 0, 1, 2, ...)$ with corresponding eigenvalues $\lambda_n = n(n+1)$.

The first few Legendre polynomials are

$P_0(x) = 1,\ \ P_1(x) = x,\ \ P_2(x) = \frac{1}{2}(3x^2 - 1)$

For n even/odd P_n is even/odd (c.f. Hermite polynomials). There are various formulae for the Legendre polynomials, e.g. they can be combined into a generating function

$$\Phi(x, h) = (1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x).$$

Actually there is an explicit formula (Rodrigues' formula)

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

Since they are the eigenfunctions of an Hermitian operator they are orthogonal

$$(P_m, P_n) = \int_{-1}^{1} dx\ P_m(x)P_n(x) = 0 \text{ if } m \neq n.$$

Normalisation: they can be made orthonormal, but the following convention is standard

$$\int_{-1}^{1} dx\ \ (P_n(x))^2 = \frac{2}{2n+1}.$$

Let f be a function defined on $[-1, 1]$. Can expand in Legendre polynomials (i.e. in the eigenfunctions of the Hermitian operator L)

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x).$$

Much as in Fourier analysis

$$(P_m, f) = \sum_{n=0}^{\infty} c_n (P_m, P_n) = \sum_{n=0}^{\infty} c_n \frac{2\delta_{mn}}{2m+1} = \frac{2c_m}{2m+1},$$

so that

$$c_m = \left(m + \frac{1}{2}\right)(P_m, f) = \left(m + \frac{1}{2}\right)\int_{-1}^{1} dx\ P_m(x)f(x).$$

Examples of Legendre series

i) $f(x) = x \qquad -1 < x < 1$. By inspection $f(x) = P_1(x)$, i.e. $c_1 = 1$ and all other Legendre series coefficients zero.

ii) If f is a polynomial of degree n then the Legendre series terminates, i.e. $c_m = 0$ if $m > n$ (see problem 3 on sheet 18).

iii) $f(x) = 1$ for $-1 < x < 0$ and $f(x) = 0$ for $-1 < x < 0$
Here

$$c_m = \left(n + \frac{1}{2}\right)\int_{-1}^{1} dx\ P_m(x)f(x) = \left(n + \frac{1}{2}\right)\int_{0}^{1} dx\ P_m(x).$$

Now c_m vanishes for all even m except $m = 0$ (can you see why this is the case without computing an integral?) $c_0 = \frac{1}{2}$. To determine the c_m for m odd requires the integral

$$\int_0^1 dx\ P_m(x)$$

This can be computed using Rodrigues' formula or the generating function (compute the integral $\int_0^1\ dx\ \Phi(x,h)$). The result is

$$\int_0^1 dx\ P_{2s+1}(x) = (-1)^s \frac{(2s-1)!!}{(2s+2)!!}.$$

Here we have used *double factorial notation:*

$$(2s+1)!! = (2s+1)(2s-1)(2s-3)...1 \qquad (2s)!! = (2s)(2s-2)(2s-4)...2.$$

In the even case the double factorial can be rewritten as $(2s)! = 2^s s!$.

Hermite's Equation

Try to do the same for Hermite's equation

$$y''(x) - 2xy'(x) + 2\alpha y(x) = 0 \qquad \text{or } Ly(x) = \lambda y(x),$$

where

$$L = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}, \qquad \lambda = 2\alpha.$$

Problem: L is not Hermitian!

$(u, Lv) \neq (Lu, v)$ regardless of boundary conditions. A quick way to prove Hermiticity of L in the Legendre case is to write L in the form

$$L = -\frac{d}{dx}(1-x^2)\frac{d}{dx}.$$

Claim: Any L of the form

$$L = -\frac{d}{dx}p(x)\frac{d}{dx},$$

where p is some real function is Hermitian (provided suitable boundary conditions are imposed).

Hermite L is not of this form. But it can be written

$$L = -e^{x^2}\frac{d}{dx}e^{-x^2}\frac{d}{dx},$$

which is not Hermitian due to the e^{x^2} prefactor. Now *change the definition of the inner product*

$$(u,v)_{new} = \int dx\ e^{-x^2}\ \bar{u}(x)v(x).$$

With this definition

$$\begin{aligned}
(u, Lv)_{new} &= -\int dx\ \frac{d}{dx}e^{-x^2}\frac{d}{dx}v(x) \\
&= -\int dx\ \left(\frac{d}{dx}e^{-x^2}\frac{d}{dx}\bar{u}(x)\right)v(x) \qquad \text{int. by parts twice} \\
&= (Lu, v)_{new},
\end{aligned}$$

(here boundary terms were dropped).

An operator of the form

$$L = -\frac{1}{w(x)}\left(\frac{d}{dx}p(x)\frac{d}{dx} - q(x)\right)$$

where $w(x) > 0$, p and q are real is Hermitian with respect to the inner product

$$(u, v)_w = \int dx\ w(x)\ \bar{u}(x)v(x)$$

provided suitable boundary conditions are imposed. $w(x)$ is called a *weight function.*

Check Hermiticity in an interval $[a, b] \subset R$ so that

$$(u, v)_w = \int_a^b dx\ w(x)\ \bar{u}(x)v(x).$$

Have to check that $(u, Lv)_w = (Lu, v)_w$ Integrating by parts twice (this time retaining the boundary terms)

$$\begin{aligned}(u, Lv)_w =& -\ \bar{u}(x)p(x)v'(x)\big|_a^b + \bar{u}'(x)p(x)v(x)\big|_a^b - \int_a^b \left(\frac{d}{dx}p(x)\frac{d}{dx}\bar{u}(x)\right)v(x)\\ =&(Lu, v)_w\end{aligned}$$

if the boundary terms vanish. This condition can be written

$$p(x)\ (\bar{u}(x)v'(x) - \bar{u}'(x)v(x))\big|_a^b = 0.$$

Any boundary conditions satisfying this condition (Lagrange's identity) give Hermiticity with respect to the weight function w. For example impose the boundary conditions $y(a) = y(b) = 0$ then Lagrange's identity is trivially satisfied (since u and v should be taken as zero at the end-points). In the Legendre case $p(x) = 1 - x^2$ which vanishes at $x = \pm 1$; this is why the Hermiticity of L only requires $y(1)$ and $y(-1)$ to be finite.

The eigenvalue equation, $Ly = \lambda y$, is often written

$$\left(\frac{d}{dx}p(x)\frac{d}{dx} - q(x) + \lambda w(x)\right)y(x) = 0. \qquad (****)$$

This is called the Sturm-Liouville equation. The eigenfunctions of L are real and eigenfunctions corresponding to distinct eigenvalues are orthogonal (c.f. finite dimensional Hermitian matrix).

Completeness

A function f satisfying the appropriate boundary conditions can be expanded in the eigenfunctions of L

$$f(x) = \sum_n c_n v_n(x).$$

Another way of expressing this is via

$$\sum_n v_n(x)\bar{v}_n(x') = \frac{1}{w(x)}\delta(x - x'),$$

(here the v_n are assumed to be orthonormal) which can be viewed as the infinite dimensional analogue of

$$\sum_{i=1}^{n} v_i v_i^\dagger = I.$$

<u>Example</u> Fourier case $L = -d^2/dx^2$, $w(x) = 1$. Periodic boundary conditions

$$v_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}.$$

$$\sum_{n\in Z} v_n(x)\bar{v}_n(x') = \frac{1}{2\pi}\sum_{n\in Z} e^{in(x-x')} = \delta(x - x').$$

Setting $x' = 0$ gives the standard expression for the periodic delta fumction $(l = 2\pi)$

$$\delta(x) = \frac{1}{2\pi}\sum_{n\in Z} e^{inx}.$$