The Bessel Equation[1]

13 April 2007

The Bessel equation is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \tag{1}$$

or, multiplying across by x^2,

$$x^2y'' + xy' + \left(x^2 - \nu^2\right)y = 0 \tag{2}$$

It is one of the important equation of applied mathematics and engineering mathematics because it is related to the Laplace operator in cylindrical coördinates. The Bessel equation is solved by series solution methods, in fact, to solve the Bessel equation you need to use the method of Fröbenius. It might be expected that Fr"obenius is needed because of the singularities at $x = 0$, however, lets pretend we hadn't noticed and try to use the ordinary series solution method:

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{3}$$

Now, by calculating directly

$$x^2y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^n \tag{4}$$

and

$$x^2y' = \sum_{n=0}^{\infty} na_n x^n \tag{5}$$

so the equation becomes

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^n = 0 \tag{6}$$

Hence, if we want to go up to the highest power we need to increase everything to the form x to the $n+2$. By letting $m+2 = n$ we get

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^{m+2} \tag{7}$$

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and

$$\sum_{n=0}^{\infty} n a_n x^n = \sum_{m=0}^{\infty} (m+2) a_{m+2} x^{m+2} \tag{8}$$

and, finally,

$$\sum_{n=0}^{\infty} n a_n x^n = a_0 + a_1 x + \sum_{m=0}^{\infty} a_{m+2} x^{m+2} \tag{9}$$

Putting this all back in to the equation, renaming m to n in the usual way, we get

$$a + 0 + a_1 x \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+2)a_{n+2}x^n - \nu^2 a_{n+2} + a_n \right] x^{n+2} = 0 \tag{10}$$

which gives recursion relation

$$a_{n+2} = -\frac{a_n}{(n+2)^2 - \nu^2} \tag{11}$$

along with $a_0 = a_1 = 0$. Thus, while we get a perfectly good two step recursion relation, the extra conditions, on a_0 and a_1 lead to the solution being trivial. Hence, the solution of the series form is trivial and, clearly, to find the actual solution, a more general series ansatz is needed.

Fröbenius means that you look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{12}$$

Now, in terms of this series we have

$$\begin{aligned}
x^2 y'' &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \\
x y' &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} \\
x^2 y &= \sum_{n=0}^{\infty} a_n x^{n+r+2} \\
\nu^2 y &= \sum_{n=0}^{\infty} \nu^2 a_n x^{n+r}
\end{aligned} \tag{13}$$

As usual, we move to the highest power, in this case $n + r + 2$, without going through the details, this gives

$$x^2 y'' = r(r-1)a_0 x^r + r(r+1)a_1 x^{r+1} + \sum_{n=0}^{\infty} a_{n+2}(n+r+2)(n+r+1)x^{n+r+2} \tag{14}$$

and

$$xy' = ra_0t^r + r(r+1)a_1x^{r+1} + \sum_{n=0}^{\infty} a_{n+2}(n+r+2)x^{n+r+2} \tag{15}$$

and finally

$$\nu^2 y = \nu^2 a_0 x^r + \nu^2 a_1 x^{r+1} + \sum_{n=0}^{\infty} a_{n+2}x^{n+r+2} \tag{16}$$

Now, if we put this all in one equation and set the x^r terms to zero, we have

$$[r(r-1) + r - v^2]a_0 = 0 \tag{17}$$

or, put another way, either $a_0 = 0$ or $r = \pm\nu$. The x^{r+1} term gives

$$[(r+1)^2 - \nu^2]a_1 = 0 \tag{18}$$

so, with $r = \pm nu\ a_1 = 0$

Now, the recusion relation is

$$[(n+r+2)(n+r+1) + (n+r+2) - \nu^2]a_{n+2} = -a_n \tag{19}$$

so, with $r = \pm\nu$ we have

$$a_{n+2} = -\frac{a_n}{(n \pm \nu + 2)^2 - \nu^2} \tag{20}$$

and so there are two solutions to the Bessel equation, one corresponding to $r = \nu$ and the other with $r = -\nu$.