1S1 Tutorial Sheet 6: Solutions[1]

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Questions

1. (4) Find the first four terms in the Taylor expansion of $\ln(x)$ about $x = 1$.

   Solution: So we are interest in the expansion of $\ln(1+h)$ in h:

$$\begin{aligned} \ln(1+h) &= \ln(1) + h\left.\frac{d\ln x}{dx}\right|_{x=1} + \frac{1}{2}h^2\left.\frac{d^2\ln x}{dx^2}\right|_{x=1} \\ &\quad +\frac{1}{6}h^3\left.\frac{d^3\ln x}{dx^3}\right|_{x=1} + \frac{1}{24}h^4\left.\frac{d^4\ln x}{dx^4}\right|_{x=1} + \ldots \end{aligned} \tag{1}$$

   Now

$$\begin{aligned} \frac{d\ln x}{dx} &= \frac{1}{x} \\ \frac{d^2\ln x}{dx^2} &= -\frac{1}{x^2} \\ \frac{d^3\ln x}{dx^3} &= \frac{2}{x^3} \\ \frac{d^4\ln x}{dx^4} &= -\frac{6}{x^3} \end{aligned} \tag{2}$$

   so

$$\ln(1+h) = h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + \ldots \tag{3}$$

2. (2) Use implicite differentiation to find dy/dx where

$$\ln x + 2xy + y^2 = 0 \tag{4}$$

   Solution: Differenciating across we get

$$\frac{1}{x} + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} \tag{5}$$

   so, solving for dy/dx we get

$$\frac{dy}{dx} = -\frac{1/x + 2y}{1+2y} = -\frac{1+2xy}{x+2xy} \tag{6}$$

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3. (2) Find

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x} \tag{7}$$

both by factorizing and by l'Hôpital's rule.

Solution: First by factorizing:

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x(x-1)}=\lim_{x\to 1}\frac{x+1}{x}=2 \tag{8}$$

and, we can see from the above that both numerator and denominator vanish at $x=1$ so we can use l'Hopital's rule:

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x}=\lim_{x\to 1}\frac{2x}{2x-1}=1 \tag{9}$$

Extra Questions

1. Find df/dx where $f(x)=\cos^{-1}x$.

   Solution: So, say $y=\cos^{-1}x$ and hence $x=\cos y$. Now differentiate

$$1=-\sin y\frac{dy}{dx} \tag{10}$$

   and hence

$$\frac{dy}{dx}=-\frac{1}{\sin y} \tag{11}$$

   and now it would be good to rewrite this in terms of x; $\sin y=\pm\sqrt{1-\cos^2 y}$ and choosing the positive square root, making the sine positive, as is appropriate for the interval zero to π.

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}} \tag{12}$$

   where we have used $\cos y=\cos\left(\cos^{-1}x\right)=x$.

2. Find dy/dx where $\ln y+\ln x=\exp xy$.

   Solution: Differentiating

$$\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}=ye^{xy}+x\exp xy\frac{dy}{dx} \tag{13}$$

   so, solving for dy/dx

$$\frac{dy}{dx}=\frac{ye^{xy}-\frac{1}{x}}{\frac{1}{y}-xe^{xy}}=\frac{xy^2e^{xy}-y}{x-xye^{xy}} \tag{14}$$

3. Differentiate $\ln \cos x$.

   Solution: This is done using the chain chain rule

$$\frac{d}{dx} \ln \cos x = -\frac{\sin x}{\cos x} = -\tan x \tag{15}$$

4. Differentiate $\ln x^3$ both by the chain rule and by $\ln x^3 = 3 \ln x$.

   Solution:

   So, by the chain rule we get

$$\frac{d}{dx} \ln x^3 = \frac{3x^2}{x^3} = \frac{3}{x} \tag{16}$$

   and

$$\frac{d}{dx} 3 \ln x = \frac{3}{x} \tag{17}$$

5. Find

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} \tag{18}$$

   Solution: So the top and bottom are both zero at the limit point $x = 0$, so, differentiating top and bottom

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{e^x}{\cos x} = 1 \tag{19}$$

6. Find

$$\lim_{x \to 1} \frac{\ln x}{1 - x} \tag{20}$$

   Solution: Again the top and bottom are both zero at $x = 1$ so

$$\lim_{x \to 1} \frac{\ln x}{1 - x} = -\lim_{x \to 1} \frac{1}{x} = -1 \tag{21}$$

7. Find

$$\lim_{x \to 1} \frac{\ln x}{\exp(x - 1)} \tag{22}$$

   Solution: Trick question, the denominator isn't zero at $x = 1$, in fact you kiust substitute

$$\lim_{x \to 1} \frac{\ln x}{\exp(x - 1)} = 0 \tag{23}$$

8. Find

$$\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1} \tag{24}$$

Solution: Again, and bottom are zero at $x=1$, so differentiate

$$\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1} = \lim_{x\to 1} \frac{2x}{2x-2} \tag{25}$$

so now the bottom is zero but the top isn't and the sign is different for x approaching from above and x approaching from below, so the limit is not defined.