1S1 Tutorial Sheet 6: Solutions<sup>1</sup>

## 7 February 2007

## Questions

(4) Find the first four terms in the Taylor expansion of ln(x) about x = 1.
 Solution: So we are interest in the expansion of ln (1 + h) in h:

$$\ln(1+h) = \ln(1) + h \left. \frac{d \ln x}{dx} \right|_{x=1} + \frac{1}{2} h^2 \left. \frac{d^2 \ln x}{dx^2} \right|_{x=1} \\ + \frac{1}{6} h^3 \left. \frac{d^3 \ln x}{dx^3} \right|_{x=1} + \frac{1}{24} h^4 \left. \frac{d^4 \ln x}{dx^4} \right|_{x=1} + \dots$$
(1)

Now

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

$$\frac{d^2 \ln x}{dx^2} = -\frac{1}{x^2}$$

$$\frac{d^3 \ln x}{dx^3} = \frac{2}{x^3}$$

$$\frac{d^4 \ln x}{dx^4} = -\frac{6}{x^3}$$
(2)

 $\mathbf{SO}$ 

## $\ln(1+h) = h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + \dots$

2. (2) Use implicite differentiation to find dy/dx where

$$\ln x + 2xy + y^2 = 0 \tag{4}$$

(3)

Solution: Differenciating across we get

$$\frac{1}{x} + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} \tag{5}$$

so, solving for dy/dx we get

$$\frac{dy}{dx} = -\frac{1/x + 2y}{1 + 2y} = -\frac{1 + 2xy}{x + 2xy} \tag{6}$$

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3. (2) Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$$

both by factorizing and by l'Hôpital's rule.

Solution: First by factorizing:

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x(x - 1)} = \lim_{x \to 1} \frac{x + 1}{x} = 2$$
(8)

and, we can see from the above that both numerator and denominator vanish at x = 1 s we can use l'Hopital's rule:

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x \to 1} \frac{2x}{2x - 1} = 1 \tag{9}$$

Extra Questions

1. Find df/dx where  $f(x) = \cos^{-1} x$ .

Solution: So, say  $y = \cos^{-1} x$  and hence  $x = \cos y$ . Now differentiate

$$1 = -\sin y \frac{dy}{dx} \tag{10}$$

and hence

$$\frac{dy}{dx} = -\frac{1}{\sin y} \tag{11}$$

and now it would be good to rewrite this in terms of x;  $\sin y = \pm \sqrt{1 - \cos^2 y}$  and choosing the positive square root, making the sine positive, as is appropriate for the interval zero to  $\pi$ .

$$\frac{iy}{dx} = -\frac{1}{\sqrt{1-x^2}} \tag{12}$$

where we have used  $\cos y = \cos(\cos^{-1} x) = x$ .

2. Find dy/dx where  $\ln y + \ln x = \exp xy$ .

Solution: Differentiating

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x} = ye^{xy} + x\exp xy\frac{dy}{dx} \tag{1}$$

so, solving for dy/dx

$$\frac{dy}{dx} = \frac{ye^{xy} - \frac{1}{x}}{\frac{1}{y} - xe^{xy}} = \frac{xy^2 e^{xy} - y}{x - xy e^{xy}} \tag{14}$$

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## 3. Differentiate $\ln \cos x$ .

Solution: This is done using the chain chain rule

$$\frac{d}{dx}\ln\cos x = -\frac{\sin x}{\cos x} = -\tan x \tag{15}$$

4. Differentiate  $\ln x^3$  both by the chain rule and by  $\ln x^3 = 3 \ln x$ . Solution:

So, by the chain rule we get

$$\frac{d}{dx}\ln x^3 = \frac{3x^2}{x^3} = \frac{3}{x}$$
 (16)

 $\quad \text{and} \quad$ 

$$\frac{d}{dx}3\ln x = \frac{3}{x}\tag{17}$$

5. Find

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} \tag{18}$$

Solution: So the top and bottom are both zero at the limit point x = 0, so, differentiating top and bottom

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{e^x}{\cos x} = 1$$
(19)

6. Find

$$\lim_{x \to 1} \frac{\ln x}{1 - x} \tag{20}$$

Solution: Again the top and bottom are both zero at x = 1 so

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$$\lim_{x \to 1} \frac{\ln x}{1 - x} = -\lim_{x \to 1} \frac{1}{x} = -1 \tag{21}$$

7. Find

$$\lim_{x \to 1} \frac{\ln x}{\exp(x-1)} \tag{22}$$

Solution: Trick question, the denominator isn't zero at x = 1, in fact you kiust substitute

$$\lim_{x \to 1} \frac{\ln x}{\exp(x-1)} = 0$$
(23)

8. Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x + 1} \tag{2}$$

Solution: Again, and bottom are zero at x = 1, so differentiate

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{2x}{2x - 2} \tag{25}$$

so now the bottom is zero but the top isn't and the sign is different for x approaching from above and x approaching from below, so the limit is not defined.

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