1S1 Tutorial Sheet 5: Solutions[1]

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Questions
Overall this was quite a hard problem sheet.

1. (2) What is the Taylor expansion of $\sqrt{1+x^2}$ about $x=0$ up to $O(x^4)$, that is you can stop after the x^3 terms. *Solution:* So,

$$f(x)=\sqrt{1+x^x} \tag{1}$$

so to work out the Taylor expansion we need the derivatives.

$$f'(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\sqrt{1+x^2}=\frac{d}{du}\sqrt{u}\frac{du}{dx} \tag{2}$$

with $u=1+x^2$, hence

$$f'(x)=\frac{1}{2\sqrt{u}}2x=\frac{x}{\sqrt{1+x^2}} \tag{3}$$

and so, in fact, $f'(0)=0$. Next, using the quotient rule

$$f''(x)=\frac{d}{dx}\frac{x}{\sqrt{1+x^2}}=\frac{\sqrt{1+x^2}-x\frac{d}{dx}\sqrt{1+x^2}}{1+x^2} \tag{4}$$

and, since we know already that

$$\frac{d}{dx}\sqrt{1+x^2}=\frac{x}{\sqrt{1+x^2}} \tag{5}$$

we have, by multiplying top and bottom by $\sqrt{1+x^2}$

$$f''(x)=\frac{1+x^2-x^2}{(1+x^2)\sqrt{1+x^2}}=(1+x^2)^{-3/2} \tag{6}$$

This means $f''(0)=1$, finally,

$$f'''(x)=\frac{d}{dx}(1+x^2)^{-3/2}=-\frac{1}{3}x(1+x^2)^{5/2} \tag{7}$$

using the chain rule. This shows $f'''(0)=0$ and

$$\sqrt{1+x^2}=1+\frac{1}{2}x^2+O(x^4) \tag{8}$$

where we have used $f(0)=1$.

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/1S1`

2. (2) What is the Taylor expansion of $1/x^2$ about $x=1$ up to $O(4)$; hence work out $f(1+h)$ to $O(h^4)$ where $f(x)=1/x^2$.

Solution: So we are interested in $f(x)=x^{-2}$, hence

$$\begin{aligned} f'(x) &= -2x^{-3} && (9)\\ f''(x) &= 6x^{-4} && (10)\\ f'''(x) &= -24x^{-5} && (11) \end{aligned}$$

so

$$\frac{1}{(1+h)^2}=1-2h+3h^2-4h^3+O(h^4) \tag{12}$$

3. (2) Solve

$$\frac{dy}{dt}=-5y \tag{13}$$

where $y(0)=8$.

Solution: So the equation is of the form $y'=ry$ and hence the solution is

$$y(t)=Ce^{-5t} \tag{14}$$

with C a constant. It is easy to see this works by differentiating. To fix C put $t=0$ and put in the known value of $y(0)$

$$8=y(0)=Ce^0=C \tag{15}$$

hence $C=8$ and

$$y(t)=8e^{-5t} \tag{16}$$

4. (2) Using the chain rule calculate $f'(x)$ where

$$f=e^{x^2} \tag{17}$$

and

$$f=\frac{1}{1+\exp(x)} \tag{18}$$

Solution: For the first one, use the chain rule

$$\frac{d}{dx}e^{x^2}=\frac{d}{du}e^u\frac{du}{dx} \tag{19}$$

where $u=x^2$. Therefore

$$\frac{d}{dx}e^{x^2}=2xe^{x^2} \tag{20}$$

and, for the second, use the quotient rule

$$\frac{d}{dx}\frac{1}{1+\exp(x)}=\frac{-e^x}{(1+e^x)^2} \tag{21}$$

Extra Questions
The questions are extra; you don't need to do them in the tutorial class.

1. Find the Taylor expansion of $\exp(-x)$ about $x = 0$.

   Solution: Well the easiest thing would be to send x to $-x$ in the expansion of $\exp(x)$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{22}$$

   so

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \tag{23}$$

   Alternatively

$$\frac{d}{dx} e^{-x} = -e^{-x} \tag{24}$$

   Hence

$$\frac{d^n}{dx^n} e^{-x} = (-1)^n e^{-x} \tag{25}$$

   and use the usual formula for the Taylor series with $f(x) = \exp(-x)$ and hence $f^{(n)}(0) = (-1)^n$.

2. Find the first three terms of the Taylor expansion of $x \exp(x)$

   Solution: Well let $f(x) = x \exp(x)$ so, by the product rule

$$f'(x) = e^x + xe^x \tag{26}$$

   and

$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x \tag{27}$$

   and

$$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x \tag{28}$$

   and, the pattern is getting clear now

$$f^{(4)} = 4e^x + xe^x \tag{29}$$

   This means

$$xe^x = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \dots \tag{30}$$

   where I got carried away and did four terms rather than three.

3. The limit as $x \to -\infty$ of $\exp(x)$ is zero, for $f(x) = \exp(-1/x)$ find $f'(0)$ and $f''(0)$.

   Solution: The point of this exercise is to begin to see that there are functions whose Taylor series are zero. Using the chain rule gives

$$f'(x) = \frac{d}{dx} e^{-1/x} = -\frac{1}{x^2} e^{-1/x} \tag{31}$$

   and, using the product rule

$$f''(x) = \frac{2}{x^3} e^{-1/x} + \frac{1}{x^4} e^{-1/x} \tag{32}$$

   Now, working out the value at $x = 0$ is hard, you need to take the limits as x goes to zero, harder than I intended, one way would be to use the method of l'Hopital which we hadn't done when this problem sheet came out, another is to change from x to another variable $y = 1/x$, for example

$$f'(0) = -\lim_{x \to 0} \frac{e^{-1/x}}{x^2} = -\lim_{y \to \infty} y^2 e^{-y} = 0 \tag{33}$$

   and

$$f''(0) = -\lim_{x \to 0} \frac{(2x+1)e^{-1/x}}{x^4} = \lim_{y \to \infty} (2y^3 + y^4) e^{-y} \tag{34}$$

   and again, this is tricky. Anyway, the point is $f^{(n)}(0) = 0$ for all n and this function has a zero Taylor series, even though it isn't zero.

4. Solve

$$\frac{dy}{dt} = 3y \tag{35}$$

   where $y(0) = 2$.

   Solution: So again the solution is $y(t) = y(0) \exp rt$ and here

$$y = 2e^{3t} \tag{36}$$

5. What is

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{nt} \tag{37}$$

   Solution: Well, we know from the properties of limits that

$$\lim_{x \to a} f^n(x) = \left(\lim_{x \to a} f(x)\right)^n \tag{38}$$

   and we have seen that

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e^r \tag{39}$$

   hence

$$\lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{nt} = (e^r)^t = e^{rt} \tag{40}$$