1S1 Tutorial Sheet 4[1]

1 December 2006

Questions

1. (4) Work out f' for

$$f(x) = x^2 \sin 3x, \qquad f(x) = \sin x^2, \qquad f(x) = \frac{\cos x}{\sin^2 x}, \qquad f(x) = x \cos \sqrt{x} \tag{1}$$

Solution: So, differentiating x^2 gives $2x$, and differentiating $\sin 3x$ gives $3\cos 3x$ using the chain rule: $f = \sin u$ and $u = 3x$ then

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = 3\cos u = 3\cos 3x \tag{2}$$

and now use the product rule

$$\frac{d}{dx}x^2 \sin 3x = 2x \sin 3x + 3x^2 \cos 3x \tag{3}$$

Here, again, use the chain rule with $f = \sin x^2 = \sin u$ and $u = x^2$ hence

$$\begin{aligned} \frac{df}{du} &= \cos u \\ \frac{du}{dx} &= 2x \\ \frac{df}{dx} &= 2x \cos x^2 \end{aligned} \tag{4}$$

The next one is a quotient rule

$$\frac{df}{dx} = \frac{\sin^2 x \frac{d}{dx} \cos x - \cos x \frac{d}{dx} \sin^2 x}{\sin^4 x} \tag{5}$$

Now we just have to use the chain rule to differentiate sine squared: let $f = u^2$ with $u = \sin x$ so

$$\begin{aligned} \frac{df}{du} &= u^2 \\ \frac{du}{dx} &= \cos x \\ \frac{df}{dx} &= 2\sin x \cos x \end{aligned} \tag{6}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/1S1

and putting it all together

$$\frac{df}{dx} = \frac{\sin^3 x - 2\cos^2 x \sin x}{\sin^4 x} = \frac{\sin^2 x - 2\cos^2 x}{\sin^3 x} \tag{7}$$

Finally, for the last one, lets do the $\cos\sqrt{x}$ bit first using the chain rule

$$\frac{d}{dx}\cos\sqrt{x} = -\frac{1}{2\sqrt{x}}\sin\sqrt{x} \tag{8}$$

where I let $u = \sqrt{x}$. Now, using the product rule

$$f' = \cos\sqrt{x} - x\frac{1}{2\sqrt{x}}\sin\sqrt{x} = \cos\sqrt{x} - \frac{1}{2}\sqrt{x}\sin\sqrt{x} \tag{9}$$

where I neatened it up at the end by using $x/\sqrt{x} = \sqrt{x}$.

2. (2) Find and classify the stationary points

$$f = 2x^4 - 4x^2 + 6 \tag{10}$$

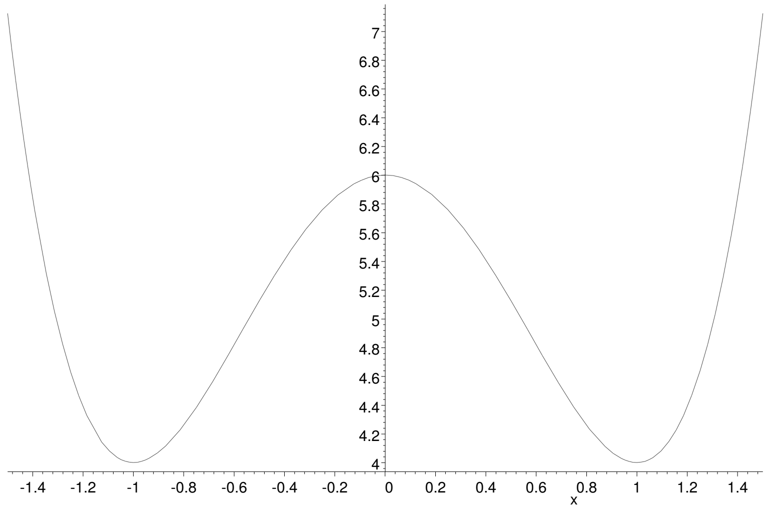
Solution: To find the stationary points we solve $f' = 0$, giving

$$f' = 8x^3 - 8x = 8x(x^2 - 1) = 0 \tag{11}$$

so the stationary points are $x = 0$, $x = 1$ and $x = -1$. To classify them, lets find f''

$$f'' = 24x^2 - 8 \tag{12}$$

so $f''(-1) = 16$, hence this is a minimum, as is $x = 1$ because $f''(1) = 16$ too. On the otherhand $f''(0) = -8$ and therefore this is a maximum. Here is a plot:



3. (1) Show $f(x) = \tan x$ has a point of inflection at $x = 0$.

   Solution: Well $f'(x) = \sec^2 x$ and, using the chain rule

$$f''(x) = 2 \sec x \frac{d}{dx} \sec x \tag{13}$$

   and, using either the chain rule, with $\sec x = (\cos x)^{-1}$ or the quotient rule

$$\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \frac{\sin x}{\cos^2 x} \tag{14}$$

   so

$$f''(x) = 2 \frac{1}{\cos x} \frac{\sin x}{\cos^2 x} = \frac{2 \sin x}{\cos^3 x} \tag{15}$$

   and, therefore, since $\cos x$ is positive for x small, $f''(x)$ is positive for x small and positive and negative for x small and negative. Hence $x = 0$ is a point of inflection.

4. (1) Classify the stationary points of $3x^4 - 4x^4$

   Solution: Obviously this question has a misprint in it, but we will answer it anyway:

$$3x^4 - 4x^4 = -x^4 \tag{16}$$

   and so if $f = -x^4$, $f' = -4x^3$ and the only stationary point is at $x = 0$. It is easy to see that although $f''(0) = 0$ $f'(x)$ is negative for small positive x and positive for small negative x, hence this is a maximum.

Extra Questions

1. Differentiate

$$\frac{1}{\sqrt{1+x}}, \quad \sin \cos x, \quad \sin \frac{1}{1+x^2}, \quad \sqrt{\sin x}, \quad \tan 2x, \quad \sin \frac{1}{x} \tag{17}$$

   Solution: More chain rule, let $f = 1/\sqrt{u}$ where $u = 1 + x$ hence

$$f' = \frac{df}{du} \frac{du}{dx} = -\frac{1}{2u^3/2} = -\frac{1}{2}(1+x)^{3/2} \tag{18}$$

   For the next one, let $f = \sin u$ and $u = cosx$ so

$$f' = \frac{df}{du} \frac{du}{dx} = \cos u(-\sin x) = -\sin x \cos \cos x \tag{19}$$

   and again, for the next, let $f = \sin u$ and $u = (1 + x^2)^{-1}$ giving

$$f' = \frac{df}{du} \frac{du}{dx} = \cos u \left(\frac{-2x}{(1+x^2)^2} \right) = \frac{2x}{(1+x^2)^2} \cos \frac{1}{1+x^2} \tag{20}$$

where I also used the chain rule to work out du/dx, letting $u = v^{-1}$ where $v = 1+x^2$, hence

$$\frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx} = -u^{-2}(2x) = \frac{-2x}{(1+x^2)^2} \tag{21}$$

The next one you should let $u = \sin x$ to give

$$\frac{d}{dx}\sqrt{\sin x} = \frac{\cos x}{2\sqrt{\sin x}} \tag{22}$$

and the next one is easier, using $u = 2x$ we get

$$\frac{d}{dx}\tan 2x = 2\sec^2 2x \tag{23}$$

and, finally, using $u = 1/x$

$$\frac{d}{dx}\sin\frac{1}{x} = -\frac{1}{x^2}\cos\frac{1}{x} \tag{24}$$

2. Classify the stationary points for

$$f(x) = x^3 + 3x^2 - 9x + 1, \qquad f(x) = x^{2/3}, \qquad f(x) = x^{1/3}(x+4), \qquad 2x^3 - 6x + 7 \tag{25}$$

Solution: First find the stationary points:

$$f' = 3x^2 + 6x - 9 = 3(x+3)(x-1) \tag{26}$$

so the stationary points are $x - 3$ and $x = 1$, next, the second derivative

$$f'' = 6x + 6 \tag{27}$$

so $f''(-3) = -12 < 0$, a maximum, and $f''(1) = 12 > 0$, a minimum. For the next one $f = x^{2/3}$ hence

$$f' = \frac{2}{3x^{1/3}} \tag{28}$$

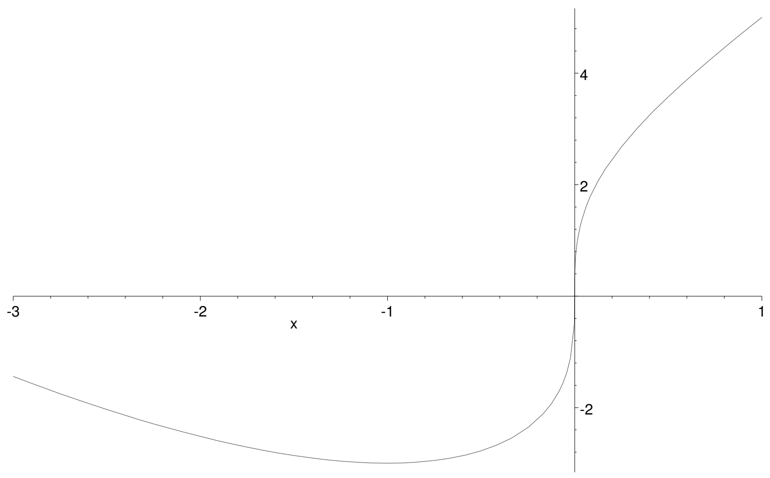
and this has no zeros, hence, no stationary points, next, $f(x) = x^{1/3}(x+4)$ gives, using the product rule

$$f' = \frac{1}{3x^{2/3}}(x+4) + x^{1/3} = \frac{4}{3}x^{1/3}\left(1+\frac{1}{x}\right) \tag{29}$$

which is zero when $x = -1$, in simplifying the expression I used $1/x^{2/3} = x^{1/3}/x$. Now, lets work out $f''(-1)$, well

$$f'' = \frac{4}{9x^{2/3}}\left(1+\frac{1}{x}\right) + \frac{4}{3}x^{1/3}\left(-\frac{1}{x^2}\right) \tag{30}$$

and sticking in $x = -1$ gives $f''(-1) = 4/3 > 0$ and therefore a minimum. This one was quite trick, here is a graph:



Finally, for $f = 2x^3 - 6x + 7$,

$$f'(x) = 6x^2 - 6 = 6(x-1)(x+1) \tag{31}$$

so the stationary points are $x = 1$ and $x = -1$, $f'' = 12x - 6$ so the first is a minimum and the second a maximum.

3. Find and classify the stationary points of $x^3 - 3x^2 + 3x - 1$. You may have to look at the sign of $f'(x)$ to do this.

   Solution: Take the derivative

   $$f' = 3x^2 - 6x + 3 = 3(x-1)^2 \tag{32}$$

   so the minimum is at $x = 1$ but $f'' = 6x - 6$ and this is zero at this point. However, $f' > 0$ for all values of x so this point cannot be a max or min and changes sign at $x = 1$, it is a point of inflection!