1S1 Tutorial Sheet 3: Solutions[1]

17 November 2006

Questions

1. (2) Work out f', f'' and keep going differentiating until you get zero.

$$f(x) = x^7 + 6x^4 + 4x - 1 \tag{1}$$

Solution: So, you just use the formula

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

so

$$\begin{aligned} f(x) &= x^7 + 6x^4 + 4x - 1 \\ f'(x) &= 7x^6 + 24x^3 + 4 \\ f''(x) &= 42x^5 + 72x^2 \\ f'''(x) &= 210x^4 + 144x \\ f^{(4)}(x) &= 840x^3 + 144 \\ f^{(5)}(x) &= 2520x^2 \\ f^{(6)}(x) &= 5040x \\ f^{(7)}(x) &= 5040 \\ f^{(8)}(x) &= 0 \end{aligned} \tag{3}$$

2. (2) Differentiate

$$f = \frac{1}{x^2} \tag{4}$$

and

$$f = x + \frac{1}{x} \tag{5}$$

Solution: So

$$f = \frac{1}{x^2} = x^{-2} \tag{6}$$

and hence

$$f' = -2x^{-3} = -\frac{2}{x^3} \tag{7}$$

and, similarily

$$f = x + \frac{1}{x} = x + x^{-1} \tag{8}$$

and

$$f' = 1 - x^{-2} = 1 - \frac{1}{x^2} \tag{9}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/1S1`

3. (2) Use the product and quotient rules to differentiate

$$f = \sqrt{x}(x^2 + 1) \tag{10}$$

and

$$f = \frac{\sqrt{x}}{x^2 + 1} \tag{11}$$

Solution: So, splitting this up as $\sqrt{x}$ multiplied by $x^2 + 1$ and using

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{2}} \tag{12}$$

we have

$$f' = 2\sqrt{x}x + \frac{x^2 + 1}{2\sqrt{x}} = \frac{5}{2}x\sqrt{x} + \frac{1}{2\sqrt{2}} \tag{13}$$

where to tidy it up a bit I used

$$\frac{x^2}{\sqrt{x}} = x\sqrt{x} \tag{14}$$

and, for the second one, we use the quotient rule

$$\begin{aligned} f' &= \frac{(x^2+1)\frac{d}{dx}\sqrt{x} - \sqrt{x}\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{\frac{1}{2\sqrt{x}}(x^2+1) - 2x\sqrt{x}\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{x^2 + 1 - 4x^2}{2\sqrt{x}(x^2+1)^2} = \frac{1 - 3x^2}{2\sqrt{x}(x^2+1)^2} \end{aligned} \tag{15}$$

where, at the end, I have moved a $2\sqrt{x}$ to the bottow to tidy it up.

4. (2) The tan is defined as

$$\tan x = \frac{\sin x}{\cos x} \tag{16}$$

and the secant as

$$\sec x = \frac{1}{\cos x} \tag{17}$$

Show using the quotient rule that

$$\frac{d}{dx}\tan x = \sec^2 x \tag{18}$$

Solution: So, again using the quotient rule

$$\begin{aligned} \frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\frac{d\sin x}{dx} - \sin x\frac{d\cos x}{dx}}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned} \tag{19}$$

where I have used the Pythagorous theorem at the end to replace sine squared plu cos squared with one.

Extra Questions

1. Differentiate

$$\frac{1}{1+x}, \quad \frac{x^2-x-6}{x-3}, \quad \frac{3x^2-4}{x^2-x+16}, \quad \frac{x^2-4}{\sqrt{x}}, \quad \sin x \cos x, \quad x \sin x \tag{20}$$

Solution: For the first one use the quotient rule and the fact that the differentiation of a constant is zero.

$$\frac{d}{dx}\frac{1}{1+x} = \frac{-1}{(1+x)^2} \tag{21}$$

we will see later that the chain rule also works here. Next

$$\begin{aligned} \frac{d}{dx}\frac{x^2-x-6}{x-3} &= \frac{(x-3)\frac{d}{dx}(x^2-x-6)-(x^2-x-6)\frac{d}{dx}(x-3)}{(x-3)^2} \\ &= \frac{(x-3)(2x-1)-(x^2-x-6)}{(x-3)^2} = \frac{x^2-6x+9}{(x-3)^2} = 1 \end{aligned} \tag{22}$$

where the last bit might look mysterious but, in fact, $(x-3)^2 = x^2-6x+9$. Again, for the next one, we use the quotient rule

$$\begin{aligned} \frac{d}{dx}\frac{3x^2-4}{x^2-x+16} &= \frac{(x^2-x+16)\frac{d}{dx}(3x^2-4)-(3x^2-4)\frac{d}{dx}(x^2-x+16)}{(x^2-x+16)^2} \\ &= \frac{6x(x^2-x+16)-(3x^2-4)(2x-1)}{(x^2-x+16)^2} = \frac{-3x^2+104x+5}{(x^2-x+16)^2} \end{aligned} \tag{23}$$

So, for the next one

$$\frac{d}{dx}\frac{x^2-4}{\sqrt{x}} = \frac{-\frac{1}{2\sqrt{x}}(x^2-4)-2x\sqrt{x}}{x} \tag{24}$$

and the next uses the product rule

$$\frac{d}{dx}\sin x \cos x = \sin x \frac{d}{dx}\cos x + \cos x \frac{d}{dx}\sin x = \cos^2 x - \sin^2 x \tag{25}$$

and finally

$$\frac{d}{dx} x \sin x = \sin x + x \cos x \tag{26}$$

2. $f(x) = \sin^2 x + \cos^2 x$, we know by the Pythagorous theorem that $f(x) = 1$, but check that $f'(x) = 0$ from the original formula with sines and cosines. If $f'(x) = 0$ then $f(x)$ is a constant, check the value of the constant by working out $f(0)$. Is the answer consistent with the Pythagorous theorem?

Solution: So just do the differenciation, by the product rule,

$$\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x \sin x) = \sin x \frac{d}{dx}\sin x + \sin x \frac{d}{dx}\sin x = 2\sin x \cos x \tag{27}$$

with a similar calculation for the cosine.

$$\frac{d}{dx}\cos^2 x = -2\sin x \cos x \tag{28}$$

Again, we used the product rule here, we will see how to use the chain rule later. Now

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = 2\sin x \cos x - 2\sin x \cos x = 0 \tag{29}$$

and hence $f(x)$ =constant. Putting $x = 0$ and using $\cos 0 = 1$ and $\sin 0 = 0$ we see that the constant is one.

3. Differentiate $\sec x$ using the quotient rule.

Solution: More quotient rule

$$\frac{d}{dx}\frac{1}{\cos x} = \frac{\sin x}{\cos^2 x} = \tan x \sec x \tag{30}$$