1S1 Tutorial Sheet 2: Solutions[1]

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Questions

1. (2) Evaluate the limits

$$\lim_{x\to 2}\frac{x^2-x-2}{x-2}, \qquad \lim_{x\to 4}\frac{x^2-8x+16}{x-4} \tag{1}$$

Solution: So in the case of the first one putting in $x=2$ gives zero above and below the line; the first thing to do is to see if anything cancels. We factorize the top:

$$x^2-x-2=(x-2)(x+1) \tag{2}$$

hence

$$\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}=\lim_{x\to 2}(x+1)=3 \tag{3}$$

In the second limit the top and the bottom are again zero at the limit point, but when we factorize

$$x^2-8x+16=(x-4)^2 \tag{4}$$

and hence

$$lim_{x\to 4}\frac{x^2-8x+16}{x-4}=lim_{x\to 4}\frac{(x-4)^2}{x-4}=lim_{x\to 4}(x-4)=0 \tag{5}$$

2. (3) Evaluate the limits

$$\lim_{x\to 2}\frac{x-2}{x^2-x-2}, \qquad \lim_{x\to 4-}\frac{x-4}{x^2-8x+16} \quad \lim_{x\to 4+}\frac{x-4}{x^2-8x+16} \tag{6}$$

Solution: For the first one we are in the familiar situation where the top and bottom are zero at the limit point, so again we factorize, in fact have already seen that $x^2-x-2=(x-2)(x+1)$ so we get

$$\lim_{x\to 2}\frac{x-2}{x^2-x-2}=\lim_{x\to 2}\frac{1}{x+1}=\frac{1}{3} \tag{7}$$

For the next one we have again already done the factorization in the previous question

$$\lim_{x\to 4-}\frac{x-4}{x^2-8x+16}=\lim_{x\to 4-}\frac{1}{x-4} \tag{8}$$

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but now we are getting zero on the bottow but one on top, hence this limit will be plus or minus infinity, the question is which. Now we are taking the limit as $x \to 4-$ so we are coming to four from below, that is x is getting close to four but is less than it, hence $x - 4$ is negative and one over a negative number is negative and

$$\lim_{x\to 4-} \frac{x-4}{x^2-8x+16} = \lim_{x\to 4-} \frac{1}{x-4} = -\infty \tag{9}$$

The last one is very similar

$$\lim_{x\to 4+} \frac{x-4}{x^2-8x+16} = \lim_{x\to 4+} \frac{1}{x-4} \tag{10}$$

but now we are approaching four from above and hence $x - 4$ is positive and s

$$\lim_{x\to 4-} \frac{x-4}{x^2-8x+16} = \lim_{x\to 4-} \frac{1}{x-4} = \infty \tag{11}$$

3. (2) Evaluate the limits

$$\lim_{x\to\infty} \frac{x^2-2}{x^2-x-2}, \qquad \lim_{x\to\infty} \frac{x^3-4}{x^2-8x+16} \tag{12}$$

Solution: So divide above and below by the highest power on the bottom

$$\lim_{x\to\infty} \frac{x^2-2}{x^2-x-2} = \lim_{x\to\infty} \frac{1-2/x^2}{1-1/x-2/x^2} = 1 \tag{13}$$

since all the other bits go to zero. And, for the second limit, same again

$$\lim_{x\to\infty} \frac{x^3-4}{x^2-8x+16} = \lim_{x\to\infty} \frac{x-4/x^2}{1-8/x+16/x^2} = \infty \tag{14}$$

since the $x \to \infty$.

4. (1) Evaluate the limit

$$\lim_{x\to\infty} (\sqrt{x^4+2x^2} - x^2) \tag{15}$$

Solution: Here we need to use the trick of moving the aquare root sign to the bottom

$$\begin{aligned} \sqrt{x^4+2x^2} - x^2 = \sqrt{x^4+2x^2} - x^2)\frac{-\sqrt{x^4+2x^2}-x^2}{-\sqrt{x^4+2x^2}-x^2} \\ &= \frac{x^4-x^4-2x^2}{-\sqrt{x^4+2x^2}-x^2)} \\ &= \frac{2x^2}{\sqrt{x^4+2x^2}+x^2)} \end{aligned} \tag{16}$$

where in the last term I have just neatened everything a bit. Now we divide above and below by x^2, the highest power on the bottom

$$\lim_{x\to\infty} (\sqrt{x^4+2x^2} - x^2) = \lim_{x\to\infty} \frac{2x^2}{\sqrt{x^4+2x^2}+x^2)} = \lim_{x\to\infty} \frac{2}{\sqrt{1+2/x^2}+1)} = 1 \tag{17}$$

Extra Questions

1. Evaluate the limits

$$\lim_{x\to\infty}\frac{1+\sqrt{x^2+1}}{1+x}, \quad \lim_{x\to 3}\frac{x^2-x-6}{x-3}, \quad \lim_{x\to\infty}\frac{3x^2-4}{x^2-x+16}, \quad \lim_{x\to-\infty}\frac{x-4}{x^2-8x+16} \tag{18}$$

Solution: So for the first one divide above and below by $|x|$ and use that $\sqrt{x^2}=|x|$ and that in this limit, where x is going to plus infinity, $x=|x|$

$$\lim_{x\to\infty}\frac{1+\sqrt{x^2+1}}{1+x}=\lim_{x\to\infty}\frac{1/x^2+\sqrt{1+1/x^2}}{1/x+1}=1 \tag{19}$$

where the detail left out is

$$\frac{\sqrt{x^2+1}}{|x|}=\frac{\sqrt{x^2+1}}{\sqrt{x^2}}=\sqrt{\frac{x^2+1}{x^2}}=\sqrt{1+1/x^2} \tag{20}$$

and, as I said,

$$\frac{x}{|x|}=1 \tag{21}$$

for x positive. The next limit is another example where you factorize since you have the top and the bottom both going to zero at the limit point,

$$\lim_{x\to 3}\frac{x^2-x-6}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+2)}{x-3}=\lim_{x\to 3}(x+2)=5 \tag{22}$$

The next one is a limit at infinity and we divide above and below by the highest power on the bottow

$$\lim_{x\to\infty}\frac{3x^2-4}{x^2-x+16}=\lim_{x\to\infty}\frac{3-4/x^2}{1-1/x+16/x^2}=3 \tag{23}$$

and for the last one

$$\lim_{x\to-\infty}\frac{x-4}{x^2-8x+16}=\lim_{x\to-\infty}\frac{1/x-4/x^2}{1-8/x+16/x^2}=0 \tag{24}$$

2. Evaluate the limits

$$\lim_{x\to\infty}(\sqrt{x^4+3x^2}-x^2), \qquad \lim_{x\to\infty}(\sqrt{x^6+2x^4+5x^3}-x^3-x), \quad \lim_{x\to 0}\frac{x(x-1)}{x^2+x} \tag{25}$$

Solution: More messing with square roots

$$\sqrt{x^4+3x^2}-x^2 = \frac{\sqrt{x^4+3x^2}-x^2}{1}\,\frac{-\sqrt{x^4+3x^2}-x^2}{-\sqrt{x^4+3x^2}-x^2}$$

$$
\begin{aligned}
&= \frac{-3x^2}{-\sqrt{x^4+3x^2}-x^2} \\
&= \frac{3x^2}{\sqrt{x^4+3x^2}+x^2} \qquad (26)
\end{aligned}
$$

so

$$
\lim_{x\to\infty}(\sqrt{x^4+3x^2}-x^2) = \lim_{x\to\infty}\frac{3x^2}{\sqrt{x^4+3x^2}+x^2} = \lim_{x\to\infty}\frac{3}{\sqrt{1+3/x^2}+1} = 3 \qquad (27)
$$

and again

$$
\begin{aligned}
\sqrt{x^6+2x^4+5x^3} \qquad -x^3-x & \\
&= \frac{\sqrt{x^6+2x^4+5x^3}-x^3-x}{1}\frac{-\sqrt{x^6+2x^4+5x^3}-x^3-x}{-\sqrt{x^6+2x^4+5x^3}-x^3-x} \\
&= \frac{x^6+2x^4+x^2-(x^6+2x^4+5x^3)}{-\sqrt{x^6+2x^4+5x^3}-x^3-x^2} \\
&= \frac{5x^3-x^2}{\sqrt{x^6+2x^4+5x^3}+x^3+x^2} \qquad (28)
\end{aligned}
$$

leading to

$$
\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^6+2x^4+5x^3}-x^3-x) &= \lim_{x\to\infty}\frac{5x^3-x^2}{\sqrt{x^6+2x^4+5x^3}+x^3+x^2} \\
&= \lim_{x\to\infty}\frac{5-1/x}{\sqrt{1+2/x^2+5/x^3}+1+1/x} \\
&= \frac{5}{2} \qquad (29)
\end{aligned}
$$

Finally, for the last one divide above and below by x to see that answer is minus one.