1S1 Tutorial Sheet 1: Solutions[1]

22 October 2006

Questions

1. (2) What is the natural domain of
$$f(x) = \sqrt{-x^2 + 4x - 3} \tag{1}$$
where you can note the factorization: $-x^2+4x-3 = -(x-1)(x-3) = (x-1)(3-x)$.

   Solution: The natural domain covers all the places where the function is real. The function here is real provided the thing in the square root is positive. Now, we can factorize
$$-x^2 + 4x - 3 = -(x-1)(x-3) = (x-1)(3-x) \tag{2}$$
and this is positive if $x \geq 1$ and $x \leq 3$. Put another way, the quadratic is zero when $x = 1$ or $x = 3$, you can get this using the usual minus b plus or minus the square root of b squared minus four ac all over two a formula. Now it can only change sign at these two places and you can see for super large positive or negative x it is negative, hence, it must be positive between $x = 1$ and $x = 3$. Either way, the domain is $1 \leq x \leq 3$.

2. (2) We define the function
$$f(x) = x^2 \tag{3}$$
on the domain $x \geq 0$. Is this function invertible? If so, what is its inverse?

   Solution: So, the function is invertible if it is strictly increasing or decreasing and x^2 is strictly increasing for $x > 0$, hence, this is an invertible function. To invert it
$$y = x^2 \tag{4}$$
hence
$$x = \sqrt{y} = f^{-1}(y) \tag{5}$$
or $f^{-1}(x) = \sqrt{x}$. This is real for $x \geq 0$.

3. Consider the piecewise function
$$f(x) = \begin{cases} x & x < 2 \\ 3 & x = 2 \\ 1 & 2 < x \leq 3 \\ 4 - x & x > 3 \end{cases} \tag{6}$$

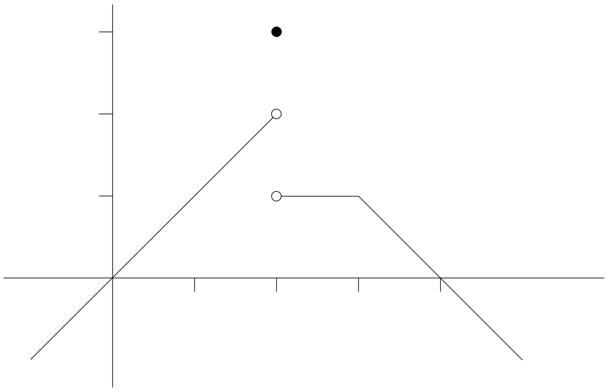
[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/1S1`

(a) (2) Graph the function.

(b) (2) Calculate

$$\lim_{x\to 2-} f(x), \qquad \lim_{x\to 2+} f(x), \qquad f(2), \qquad \lim_{x\to 3} f(x) \tag{7}$$

Solution: So, again, the stuff on the right hand of the curly bracket gives a list of instructions as to what $f(x)$ is for different intervals. It looks like this



and looking that this graph is is clear that

$$\lim_{x\to 2-} f(x) = 2, \qquad \lim_{x\to 2+} f(x) = 1, \qquad f(2) = 3 \qquad \lim_{x\to 3} f(x) = 1 \tag{8}$$

Extra Questions

1. What is the natural domain of

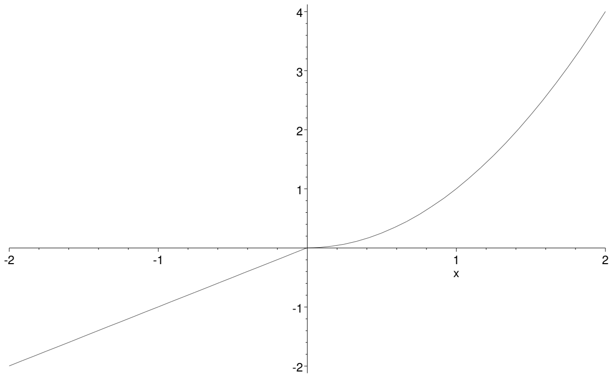
$$f(x) = \sqrt{-x} \tag{9}$$

Solution: Again, $f(x)$ if what is inside the square root sign is real, which means $x \leq 0$.

2. Graph and invert

$$f(x) = \begin{cases} x^2 & x > 0 \\ x & x < 0 \end{cases} \tag{10}$$

Solution: Well the graph looks like



and because the function is defined peicewise we invert is piecewise using the same method for each bit as before to get

$$f^{-1}(x) = \begin{cases} \sqrt{x} & x > 0 \\ x & x < 0 \end{cases} \tag{11}$$

3. Find

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} \tag{12}$$

Solution: So, it seems that the function is not defined at $x = 1$, at $x = 1$ but numerator and denominator are zero, however, if we factorize that top we find

$$f(x) = \frac{x^2 + x - 2}{x - 1} = \frac{(x-1)(x+2)}{x-1} = x + 2 \tag{13}$$

provided $x \neq 1$, hence

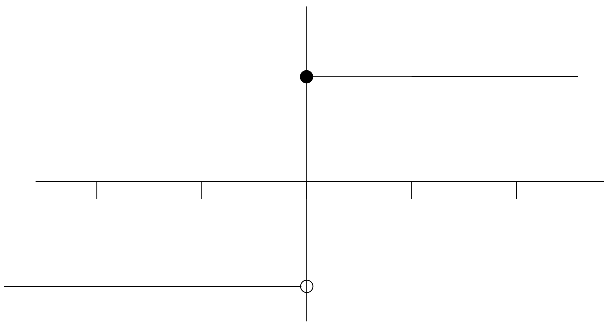
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3 \tag{14}$$

4. Graph the function

$$f(x) = \frac{|x|}{x} \tag{15}$$

What is its domain? What are the one-sided limits at $x = 0$?

Solution: So, $|x| = x$ for $x \geq 0$ and $-x$ for $x < 0$, hence the function is zero over zero at $x = 1$ and is not define. Otherwise, it is plus one for $x > 0$ and minus one for $x < 0$ and the graph looks like



It is easy to see

$$\lim_{x \to 0-} f(x) = -1, \qquad \lim_{x \to 0+} f(x) = 1 \tag{16}$$