1S1 Tutorial Sheet 5[1]

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Useful facts:

- **Taylor Series**: for a function $f(t)$ the Taylor expansion about $t = 0$ is given by
$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \frac{1}{6}f'''(0)t^3 + \frac{1}{24}f^{(4)}(0)t^4 + O(t^5) \quad (1)$$
where $O(t^5)$ means terms where all the powers of t are t^5 and higher. Another way of puttin this is
$$f(t) = f(0) + f'(0)t + \frac{1}{2}f''(0)t^2 + \frac{1}{6}f'''(0)t^3 + \frac{1}{24}f^{(4)}t^4 + \cdots + \frac{1}{n!}f^{(n)}t^n + \cdots \quad (2)$$
so the nth term is $\frac{1}{n!}f^{(n)}t^n$ and $n!$ is n factorial,
$$n! = n(n-1)(n-2)(n-3)\cdots 1 \quad (3)$$
with, for example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. More generally the expansion about a point t is given by
$$f(t+h) = f(t) + f'(t)h + \frac{1}{2}f''(t)h^2 + \frac{1}{6}f'''(t)h^3 + \frac{1}{24}f^{(4)}(t)h^4 + O(h^5) \quad (4)$$

- **The Growth Equation** is
$$\frac{df}{dt} = rf \quad (5)$$
with $f(0)$ usually given as an initial condition. The solution is then
$$f(t) = f(0)e^{rt} \quad (6)$$

- The main property of the **exponential** is
$$\frac{d}{dt}e^t = e^t \quad (7)$$
and $e = 2.71828183$.

- The Taylor expansion of the exponential can easily be calculated because the exponential can be differentiated easily and
$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + \cdots \quad (8)$$

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- If a bank offers an interest rate of r, so for 4% $r = .04$ and adds your interest n times a year, then if you start with P your total after one year is

$$T = \left(1 + \frac{r}{n}\right)^n P \tag{9}$$

and after t years $T = \left(1 + \frac{r}{n}\right)^{nt} P$

- The relationship between the growth equation and the growth of money is explained by

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right) = e^r \tag{10}$$

Questions
The numbers in brackets give the numbers of marks available for the question.

1. (2) What is the Taylor expansion of $\sqrt{1+x^2}$ about $x = 0$ up to $O(x^4)$, that is you can stop after the x^3 terms.
2. (2) What is the Taylor expansion of $1/x^2$ about $x = 1$ up to $O(4)$; hence work out $f(1+h)$ to $O(h^4)$ where $f(x) = 1/x^2$.
3. (2) Solve

$$\frac{dy}{dt} = -5y \tag{11}$$

where $y(0) = 8$.

4. (2) Using the chain rule calculate $f'(x)$ where

$$f = e^{x^2} \tag{12}$$

and

$$f = \frac{1}{1+\exp(x)} \tag{13}$$

Extra Questions
The questions are extra; you don't need to do them in the tutorial class.

1. Find the Taylor expansion of $\exp(-x)$ about $x = 0$.
2. Find the first three terms of the Taylor expansion of $x\exp(x)$
3. The limit as $x \to -\infty$ of $\exp(x)$ is zero, for $f(x) = \exp(-1/x)$ find $f'(0)$ and $f''(0)$.
4. Solve

$$\frac{dy}{dt} = 3y \tag{14}$$

where $y(0) = 2$.

5. What is

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{nt} \tag{15}$$