Soft Foundations for Geometric Computation

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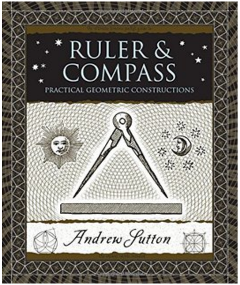
Overview

I. Introduction

Trouble with Computational Models

- Ancient Greek Geometry
  - Ruler and Compass Model



Impossibility of squaring a circle (Lindemann 1882)

Trouble with Computational Models

- Ancient Greek Geometry
  - $\boxed{\text{Ruler and Compass Model}}$
- General Models of Computation
  - $\boxed{\text{Turing Machine Model}}$ (Church's Thesis)
- Models for Geometric Computing
  - $\boxed{\text{Real RAM model}}$ (not Church Equivalent!)
- ... the trouble begins

The Numerical Nonrobustness Phenomenon

- The trouble according to Numerical Analysts

  "pitfalls"

- The trouble according to Computational Geometers

  "crashes, loops, topological errors"

- Computational Geometry attacks (1980-2000)
- ... but what about Exact Computation?

Exact Geometric Computation (EGC)

- The EGC prescription
  - $\boxed{\text{Ensure all branches are error-free}}_{Rx}$
- "Most general/successful solution"
  - Encoded in libraries such as `CGAL`, `LEDA`, `CORE`
- ... therein lies the seed of our next challenge

Barriers to EGC

- EGC algorithms may not be Turing-computable
  - "the Zero Problem"
- EGC may be too inefficient
- EGC requires full degeneracy analysis
- Exact computation is unnecessary/inappropriate
- ...beyond EGC?

Towards an alternative Computational Model

- ...but which model?
  - Before developing top-down abstract models,
    we propose a bottom-up look at examples!
- 2 classes of problems:
  (A) algebraic
  (B) combinatorial

Towards an alternative Computational Model

A.1 $\boxed{\text{Root isolation and clustering}}$

– [ISSAC'06,'09,'11,'12,'16,'18; SNC'11, CiE'13, ICMS'18]

with V.Sharma, A.Eigenwillig, M.Sagraloff, R.Becker, J.Xu

A.2 $\boxed{\text{Isotopic approximation of surfaces}}$

– [ISSAC'08,SoCG'09,'12, SPM'12, ICMS'14,'18]

with V.Sharma, G.Vegter, M.Burr, S.Choi, L.Lin

B.1 $\boxed{\text{Robot motion planning}}$

– [SoCG'13, WAFR'14, FAW'15, WAFR'16]

Y.-J. Chiang, C.Wang, J.-M.Lien, Z.Luo, C.-H.Hsu, J.Ryan

B.2 $\boxed{\text{Voronoi diagrams}}$

– [ISVD'13, SGP'16]

V.Sharma, J.-M.Lien, E.Papadopoulou, H.Bennett

Towards an alternative Computational Model

- What is new and common?
  - all subdivision algorithms!
  - $\boxed{\text{Soft}}$ Predicates ("Soft but not mush")
  - Local formulation ("search in a box")
  - Adaptive complexity (not worst case)
  - Implementable (usually implemented)
  - Practical (may match state of art)
  - New theoretical foundations ("resolution-exactness")
- Escape from the Zero Problems!

II. Soft Tools

> *"The history of the zero recognition problem is somewhat confused by the fact that many people do not recognize it as a problem at all."*
>
> — Daniel Richardson (1996)

Numerical and Interval Methods

- Let $f : \mathbb{R}^n \to \mathbb{R}$

(1) Set extension of f:

$$\boxed{S \subseteq \mathbb{R}^n \mapsto f(S) \subseteq \mathbb{R}}$$

E.g., $f([-1,1] \times [3,4]) = \{f(x,y) : x \in [-1,1], y \in [3,4]\}$

(2) Interval extension of f:

$$\boxed{\square f : \square\mathbb{R}^n \to \square\mathbb{R}}$$

satisfying two properties:

- Inclusion: $f(B) \subseteq \square f(B)$
- Convergence: $\lim_{i\to\infty} \square f(B_i) = f(\lim_{i\to\infty} B_i) = f(p)$

Numerical and Interval Methods

- Question of Effectivity

  <u>Need for approximate real numbers</u>

  – Use **dyadic numbers** (“bigFloats”):

  $$\mathbb{F} := \{m2^n : m, n \in \mathbb{Z}\}$$

  – Effective intervals: $\square\mathbb{F}$

Subdivision Algorithms

- What are subdivision algorithms?
  - Generalized binary search, organized as a quadtree.

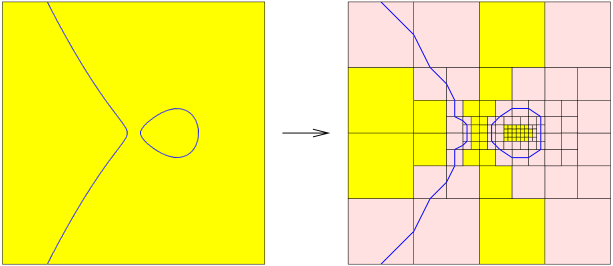


Figure: Mesh approximation of curve $f(X, Y) = Y^2 - X^2 + X^3 + 0.02 = 0$

Generic Subdivision Algorithm

- Basic form

```
Input:    (B_0, ε, ...)
Output:   G
      Initialize queue Q_0 ← {B_0}
      Phase I.
          Q_1 ← SUBDIVIDE(Q_0)
      Phase II.
          Q_2 ← REFINE(Q_1)
      Phase III.
          G ← CONSTRUCT(Q_2)
```

– Each Phase is a WHILE-LOOP, controlled by a queue of boxes

– Most of our algorithms can be put into a similar framework!

Generic Subdivision Algorithm

- What controls Subdivision (Phase I)?

  A small number of predicates!

Exclusion Predicate	$C_0(B) \equiv 0 \notin f(B)$
Normal Variation Predicate	$C_1(B) \equiv 0 \notin f_x(B)^2 + f_y(B)^2$
Parametrizability Predicate	$C_{xy}(B) \equiv 0 \notin \square f_x(B)$ or $0 \notin f_y(B)$
Pellet Test	$T_k(B) \equiv \lvert f^{[k]}(m_B)\rvert r_B^k > \sum_{i \neq k} \lvert f^{[i]}(m_B)\rvert r_B^i$
Motion Planning predicates	"feature-based methods"
Voronoi Diagram predicates	"feature-based methods"
$\vdots$	$\vdots$

Generic Subdivision Algorithm

- Three Levels of Abstractions

  Exact Level: $C_0(B) \equiv 0 \notin f(B)$

  Interval Level: $\square C_0(B) \equiv 0 \notin \square f(B)$

  Approximate Level: $\widetilde{\square} C_0(B) \equiv 0 \notin \widetilde{\square} f(B)$

- In general: :

  $$\widetilde{\square} C(B) \quad \Rightarrow \quad \square C(B) \quad \Rightarrow \quad C(B)$$

Thus we can control numerical precision
and produce rigorously justified implementation.

What is a "Soft Predicate"?

- They are approximations of exact (or "hard") predicates.

– Suppose the exact box predicate C is

$$B \mapsto C(B) \in \{-1, 0, +1\},$$

– Call $\widetilde{C}$ a soft version of C if

$$B \mapsto \widetilde{C}(B) \in \{-1, 0, +1\}$$

such that

(Conservative) $\widetilde{C}(B) \neq 0$ implies $\widetilde{C}(B) = C(B)$

(Convergent) $\lim_{i \to \infty} \widetilde{C}(B_i) = C(\lim_{i \to \infty} B_i) = C(p)$

III. Soft Problems

"Eventually, the topic [...of proving non-zeroness...] takes over the whole subject [...of Transcendental Number Theory...]"

— David Masser (2000)

Relaxed Correctness Criteria

- What do our "soft tools" achieve?
  - – Subdivision reduces global correctness criteria

    to local correctness criteria
  - – Our soft tools to achieve some "relaxed" local criteria.
  - – The relaxed local criteria are synthesized into a

    (possibly "relaxed") global criteria.

Relaxed Correctness Criteria

- 3 Examples

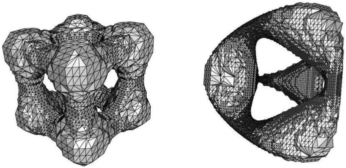
(Eg 1) Meshing of Curves/Surfaces:

(Eg 2) Root Isolation:

(Eg 3) Motion Planning:

Eg 1: Meshing Problem

- Meshing curves and surfaces:



```
GIVEN:    a function f(x,y,z)
TO FIND: an approximation S̃ to the surface S = f^{-1}(0)
          such that:
             A. S̃ ≃ S            (ambient isotopic)
             B. d_H(S̃, S) ≤ ε    (geometric accuracy)
```

GIVEN: a function $f(x, y, z)$

TO FIND: an approximation $\widetilde{S}$ to the surface $S = f^{-1}(0)$ such that:

A. $\widetilde{S} \simeq S$ (ambient isotopic)

B. $d_H(\widetilde{S}, S) \leq \varepsilon$ (geometric accuracy)

Eg 1: Meshing Problem

- Relaxed Local Criteria
  - Standard: "Local Isotopy implies Global Isotopy"

    (E.g., [Snyder], [Collins-Krandick], etc)
  - Soft idea [Plantinga-Vegter]:

    (i.e., allow small incursions and excursions)

"do not take boxes too seriously"

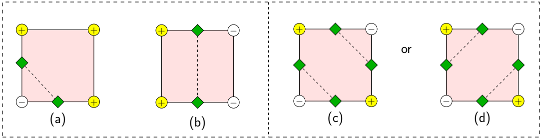


Figure: Marching Cube Construction

Root Isolation and Clustering

- Root Isolation Problem:

GIVEN: $f \in \mathbb{Z}[z]$,
TO COMPUTE: a set $\{\Delta_1, \ldots, \Delta_m\}$

where $\Delta_i \subseteq \mathbb{C}$ are pairwise disjoint ε-discs,
each containing a unique root.

Root Isolation and Clustering

- Relaxation and Generalization:

  Root Clustering Problem:

  > GIVEN: $f \in \mathbb{C}[z]$,
  > TO COMPUTE: a set $\{(\Delta_1, m_1), \ldots, (\Delta_m, m_k)\}$
  > where $\Delta_i \subseteq \mathbb{C}$ are pairwise disjoint ε-discs,
  > each $\#(\Delta_i) = \#(3\Delta_i) = m_k \geq 1$.

  – Why this is essential: solving polynomials systems

$$
\boxed{\begin{array}{l}
f_1(z_1) = 0 \\
f_2(z_1, z_2) = 0 \\
f_3(z_1, z_2, z_3) = 0
\end{array}}
$$

Root Isolation and Clustering

- The set $\mathtt{Zero}(\Delta)$ is called a natural cluster if $\#(\Delta) = \#(3\Delta)$

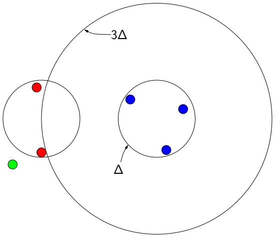


Figure: Red cluster is unnatural, Blue cluster is natural

– Natural clusters are disjoint or has inclusion relation

– They form a cluster tree of size $< 2n$.

Motion Planning

- Demo of Rod and Ring in 3D

  (see other Demos in Gallery)

Motion Planning

- Demo of Rod and Ring in 3D

  (see other Demos in Gallery)

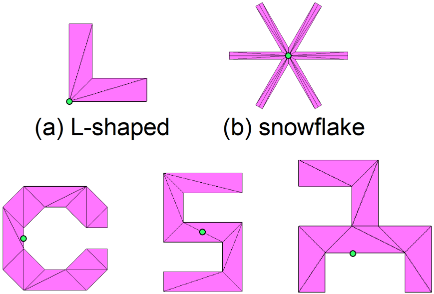
- Motion Planning Problem (for a robot R_0):

```
GIVEN: (Ω, α, β),
TO FIND: either
         an Ω-avoiding path from α to β,
      or return NO-PATH.
```

Search in configuration space $\mathcal{C}space(R_0, \Omega)$

Motion Planning

- Some rigid complex robots in 2D



(a) L-shaped (b) snowflake

(c) C-shaped (d) S-shaped (e) 3-legged

Motion Planning

- Relaxed Correctness Criteria

```
A path planner is ε-exact
     if there is a K > 1 such that
     (1) it returns a path if the maximum
        clearance of paths from α to β is > Kε,
     (2) if returns NO-PATH if the maximum
        clearance is < K/ε,
```

Indeterminacy if maximum clearance is in $[K/\varepsilon, K\varepsilon]$.

IV. Conclusion

Conclusion

- WHAT HAVE WE DONE?
  - given up exact model (Real RAM Model)
  - developed an effective numerical model
  - main algorithmic paradigm: subdivision/iteration

Conclusion

- WHAT HAVE WE DONE?
- WHAT HAVE WE ACHIEVED?
  - – state-of-art in motion planning

    First exact and complete 5DOF

    realtime implementation
  - – state-of-art results in root isolation

    First near-optimal root isolation algorithm

    implementation (cf. [Schönhage-Pan (1981-1992)])

Conclusion

- WHAT HAVE WE DONE?
- WHAT HAVE WE ACHIEVED?
- BROAD CONSEQUENCES?
  - scope of computational geometry vastly broadened
  - non-linear geometry becomes accessible
  - implementable algorithms that are also practical

Conclusion

- WHAT HAVE WE DONE?
- WHAT HAVE WE ACHIEVED?
- BROAD CONSEQUENCES
- FUTURE WORK
  - develop new algorithms for old CG problems
  - produce complexity analysis of such algorithms
  - theory of real computation and continuous complexity

Thanks for Listening!

"Algebra is generous,
she often gives more than is asked of her."

— Jean Le Rond D'Alembert (1717-83)

"To Generalize is to be an Idiot. To Particularize is the Alone Distinction of Merit – General Knowledges are those Knowledges that Idiots possess."

— William Blake (1757 – 1827)

Annotations to Sir Joshua Reynolds's Discourses, pp. xvii – xcviii