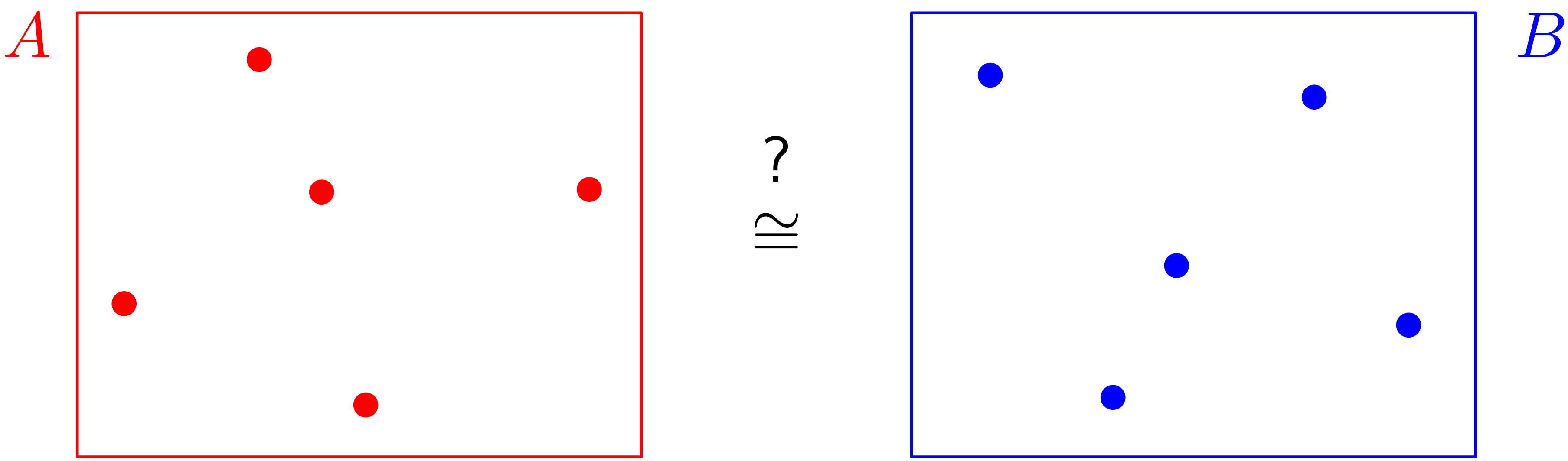
The Computational Geometry of Congruence Testing, Part II

Günter Rote
Freie Universität Berlin



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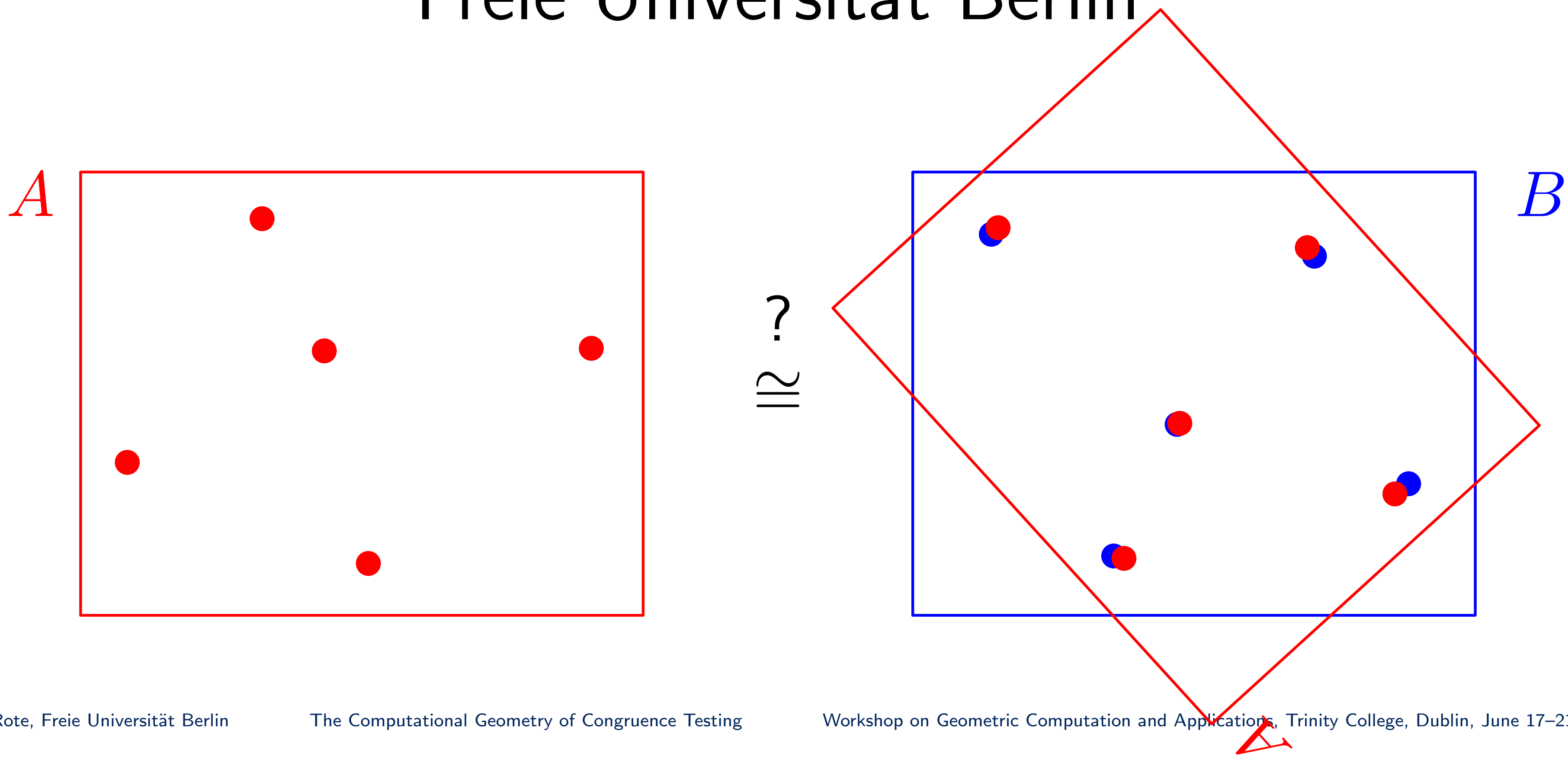

Overview

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- 1 dimension
- 2 dimensions
- 3 dimensions
- 4 dimensions

$\Big\}$ $O(n \log n)$ time

$\longleftarrow$ today (joint work with Heuna Kim)

- d dimensions $O(n^{\lceil d/3 \rceil} \log n)$ time [Brass and Knauer 2002]

$O(n^{\lfloor (d+2)/2 \rfloor / 2} \log n)$ Monte Carlo [Akutsu 1998/Matoušek]

$\downarrow O(n^{\lfloor (d+1)/2 \rfloor / 2} \log n)$ time

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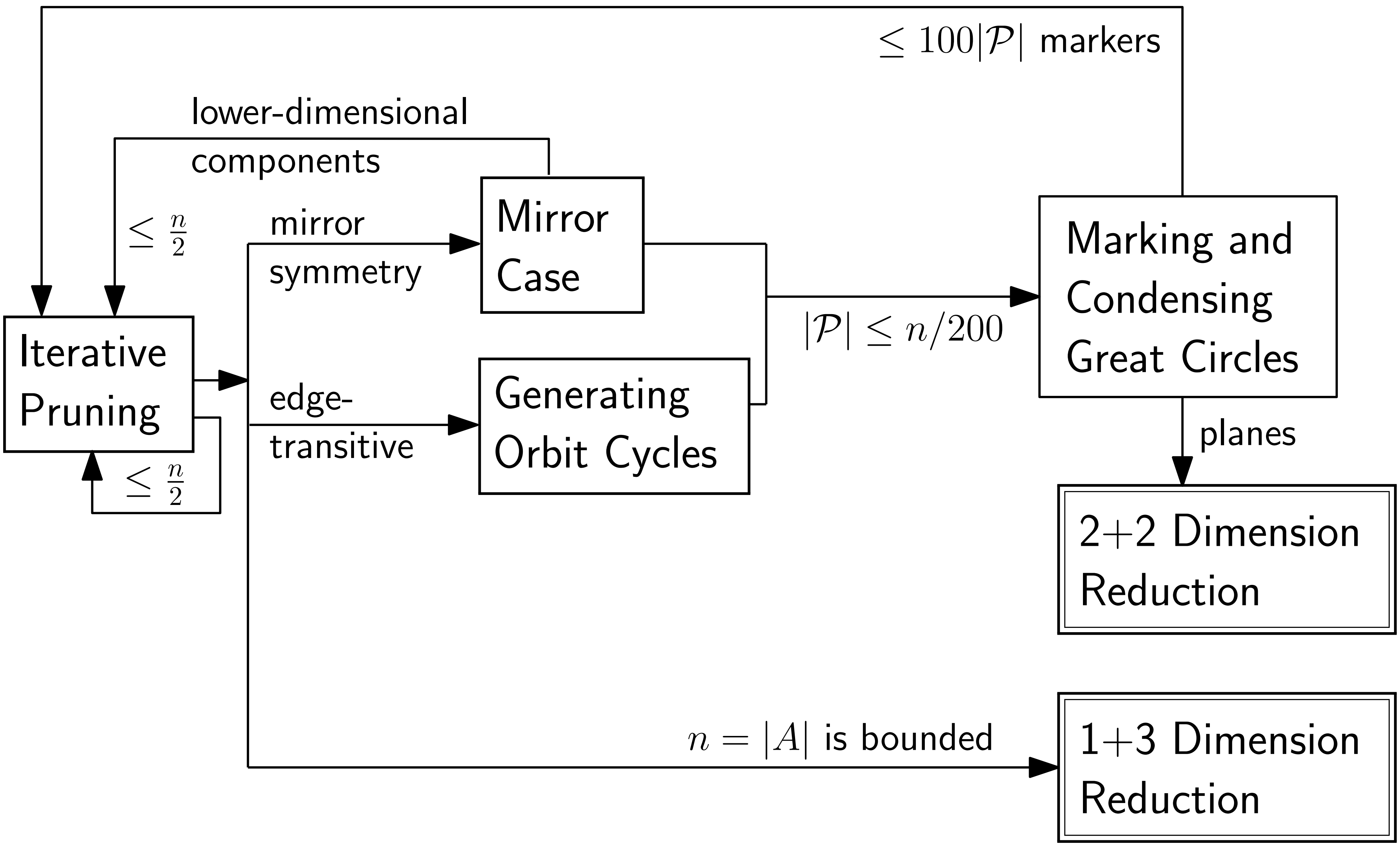
- Rotations in 4-space
- Plücker coordinates for 2-planes in 4-space
- The Hopf fibration of $\mathbb{S}^3$
- Closest pair graph
- 2+2 dimension reduction
- Coxeter classification of reflection groups

4 Dimensions: Algorithm Overview

joint work with Heuna Kim



4 Dimensions: Algorithm Overview

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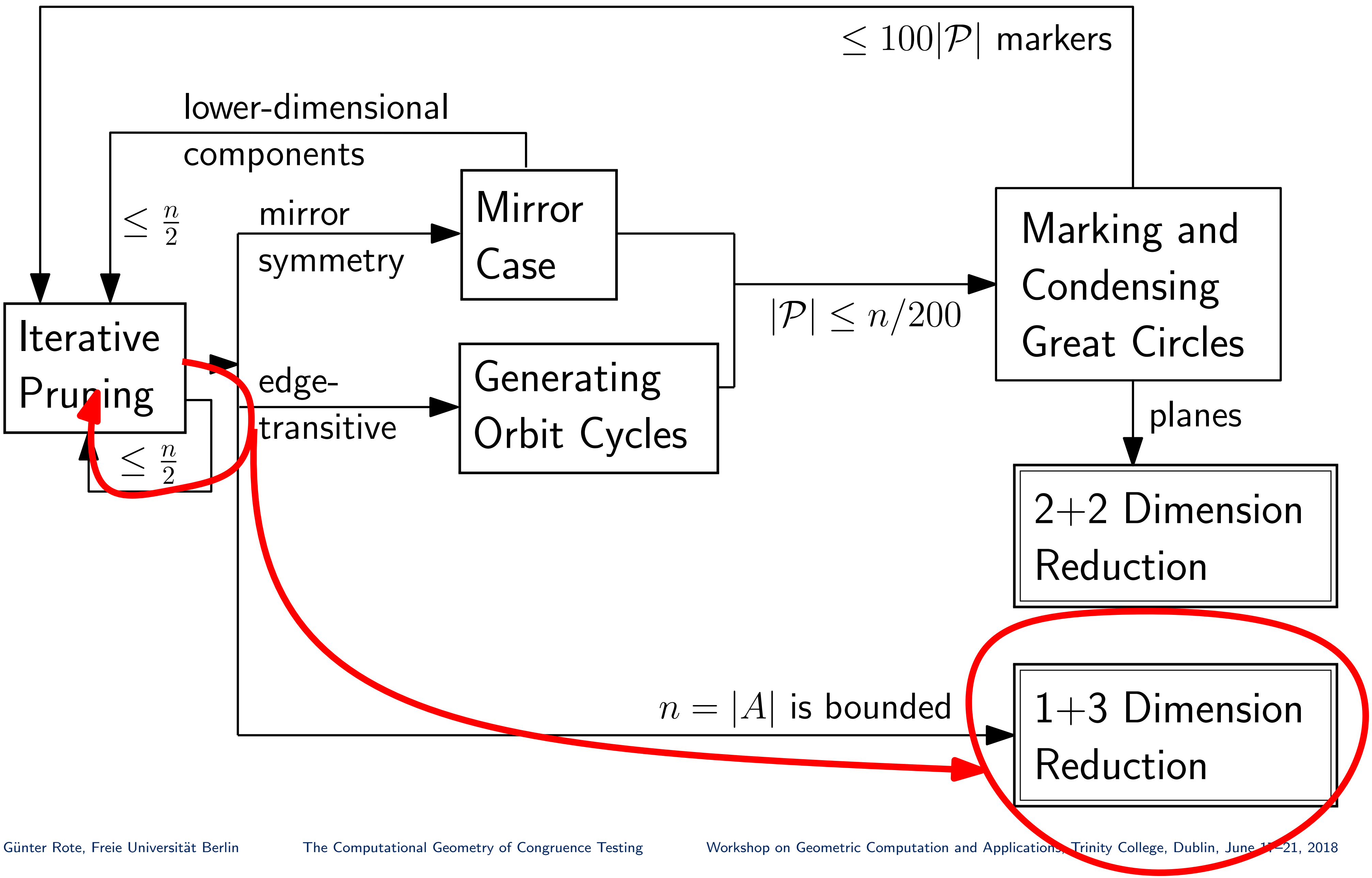


Initialization: Closest-Pair Graph

1) PRUNE by distance from the origin.

- $\implies$ we can assume that A lies on the 3-sphere $\mathbb{S}^3$.

2) Compute the closest pair graph

$$G(A) = (A, \{\, uv : \|u - v\| = \delta \,\})$$

where $\delta :=$ the distance of the closest pair, in $O(n \log n)$ time.

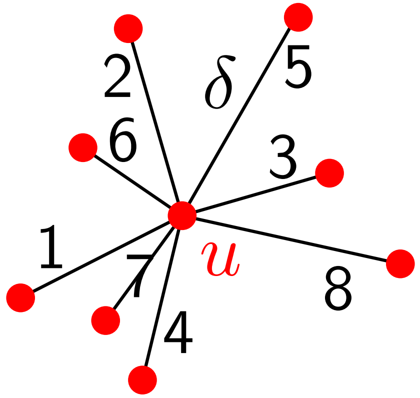
[Bentley and Shamos, STOC 1976]

- We can assume that δ is SMALL: $\delta \le \delta_0 := 0.0005$. (Otherwise, $|A| \le n_0$, by a packing argument.)

Everything looks the same!

By the PRUNING principle, we can assume that all points look locally the same:

- All points have congruent neighborhoods in $G(A)$.
  (The neighbors of u lie on a 2-sphere in $\mathbb{S}^3$;
  There are at most $K_3 = 12$ neighbors.)



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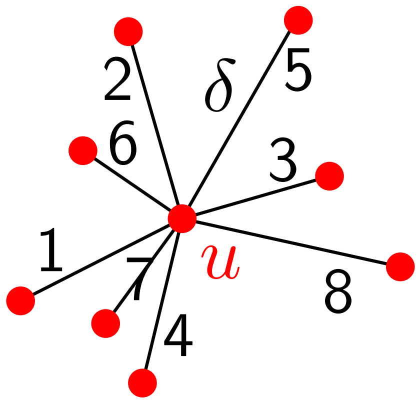
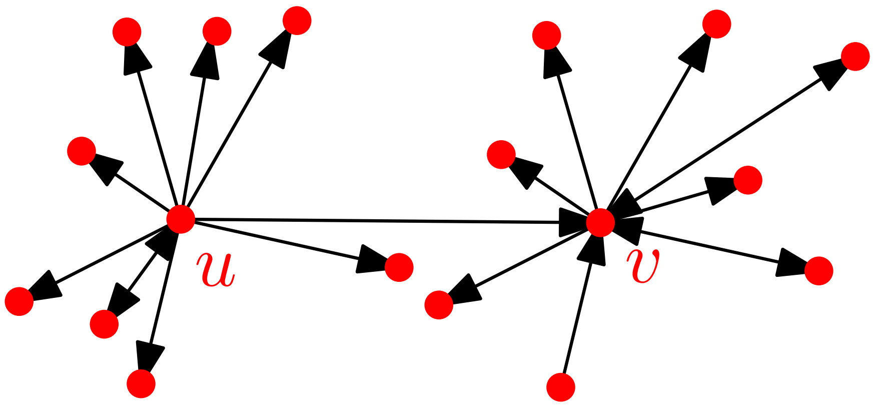


- Make a directed graph D from $G(A)$
  and PRUNE its arcs uv by
  the joint neighborhood of u and v.



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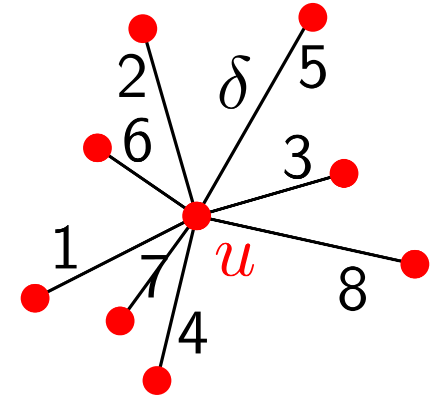

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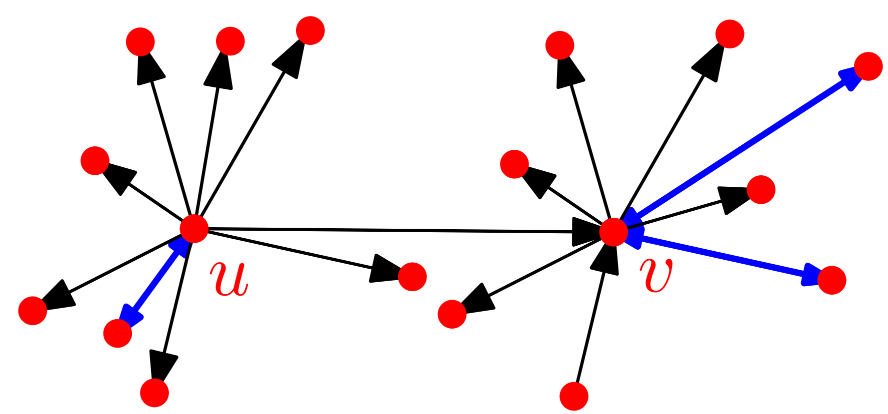

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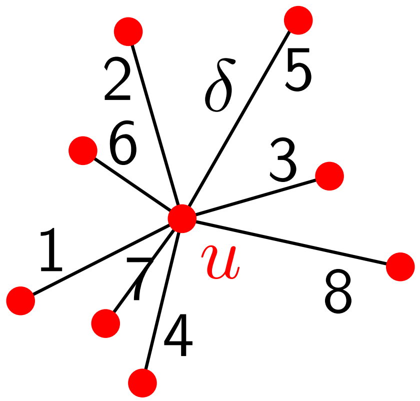

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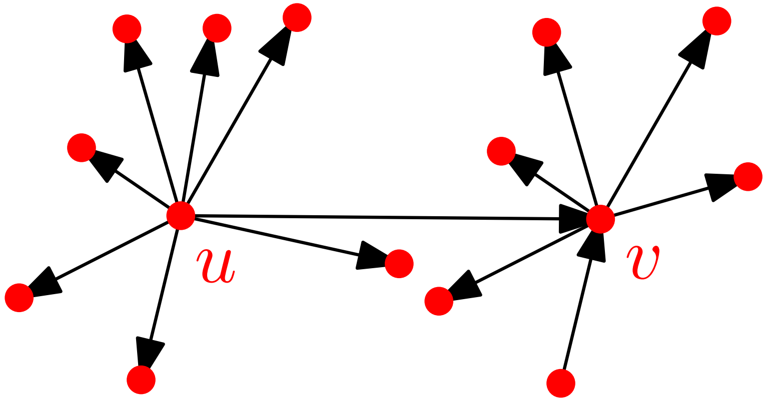

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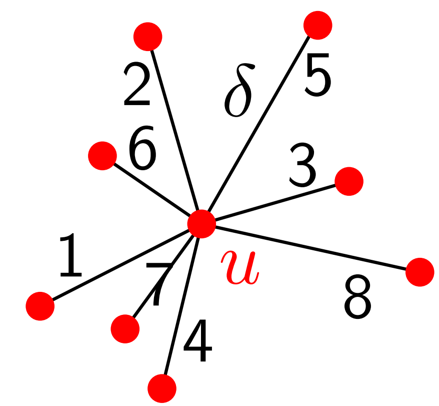

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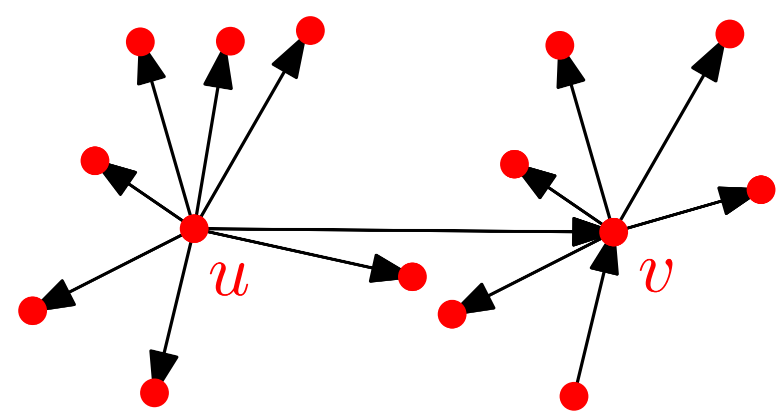
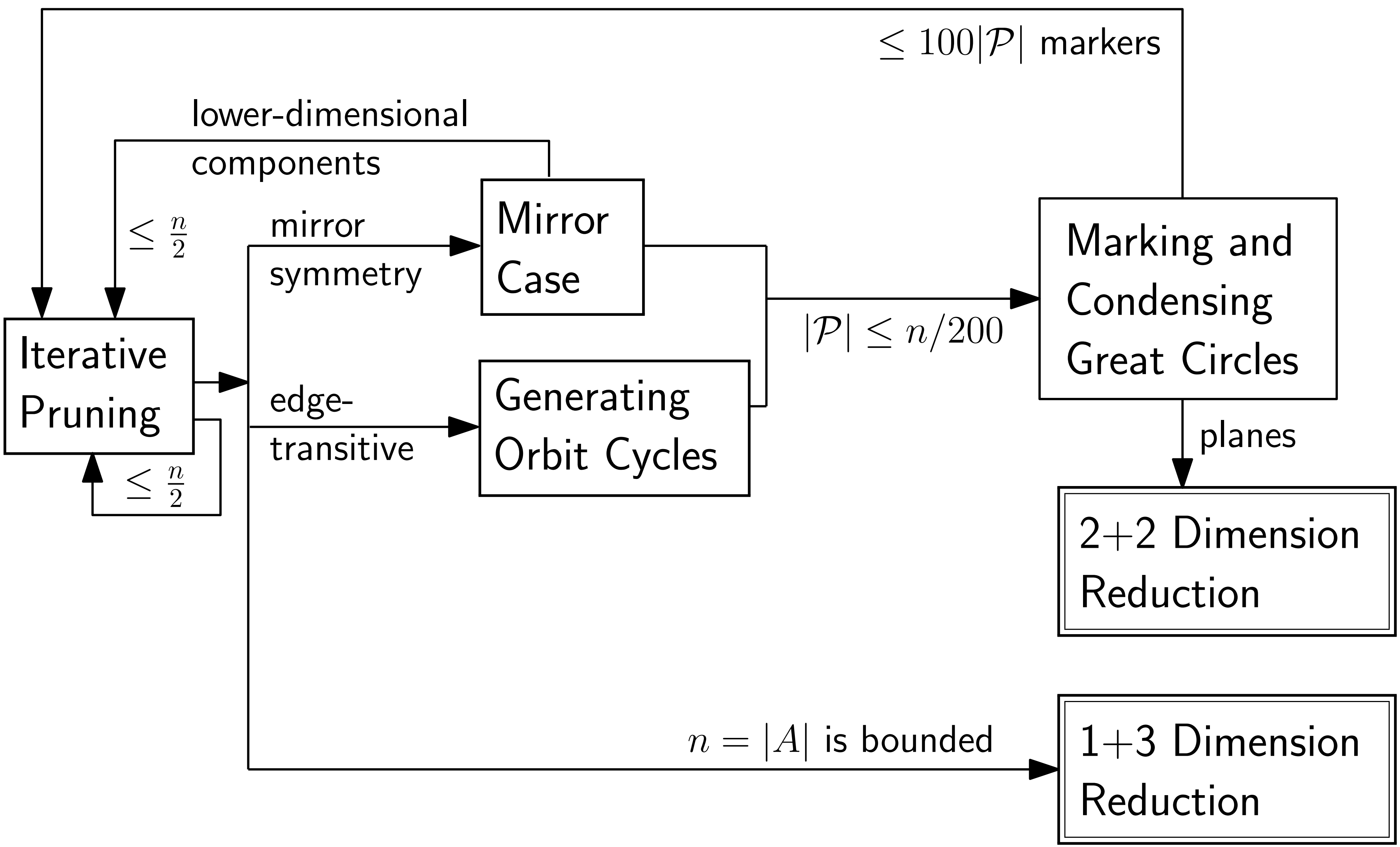


- . . . until all arcs uv
  look the same.

Algorithm Overview



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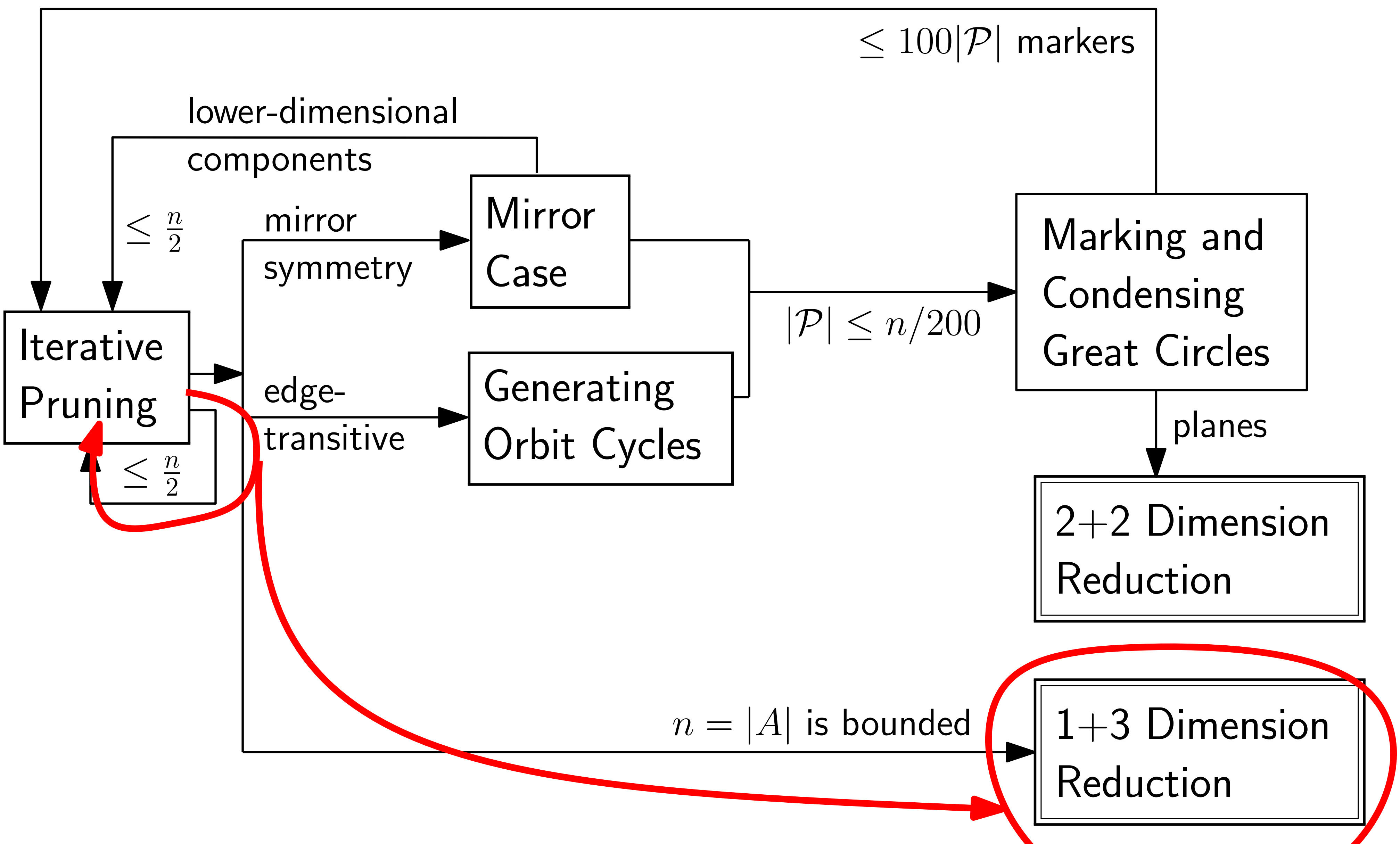

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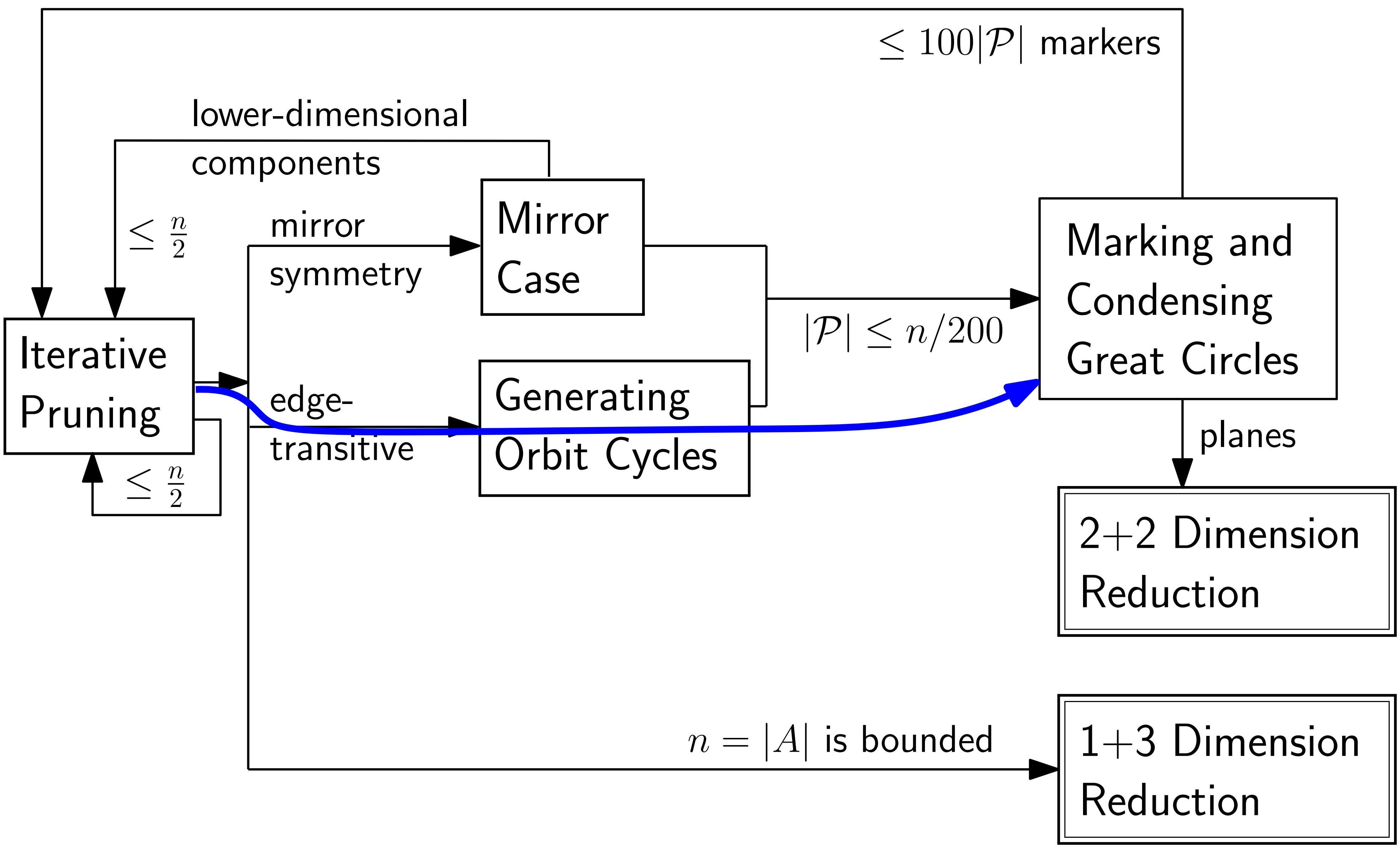
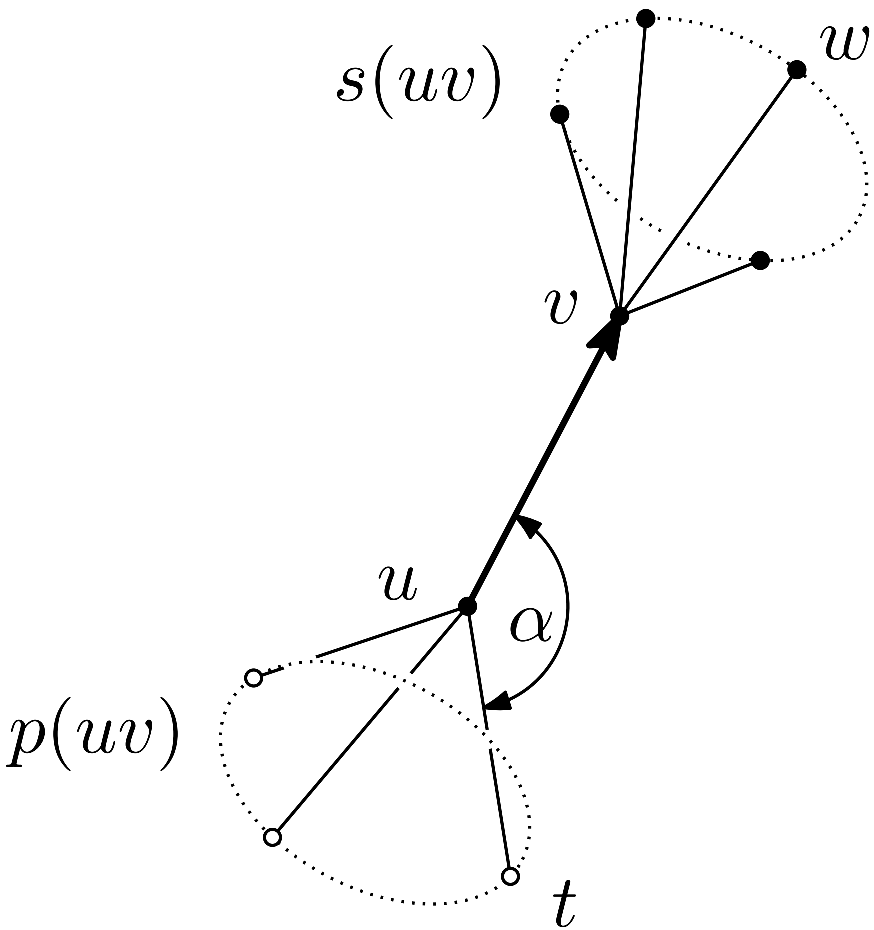


Predecessor-Successor Figure

Pick some α. $\quad s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$



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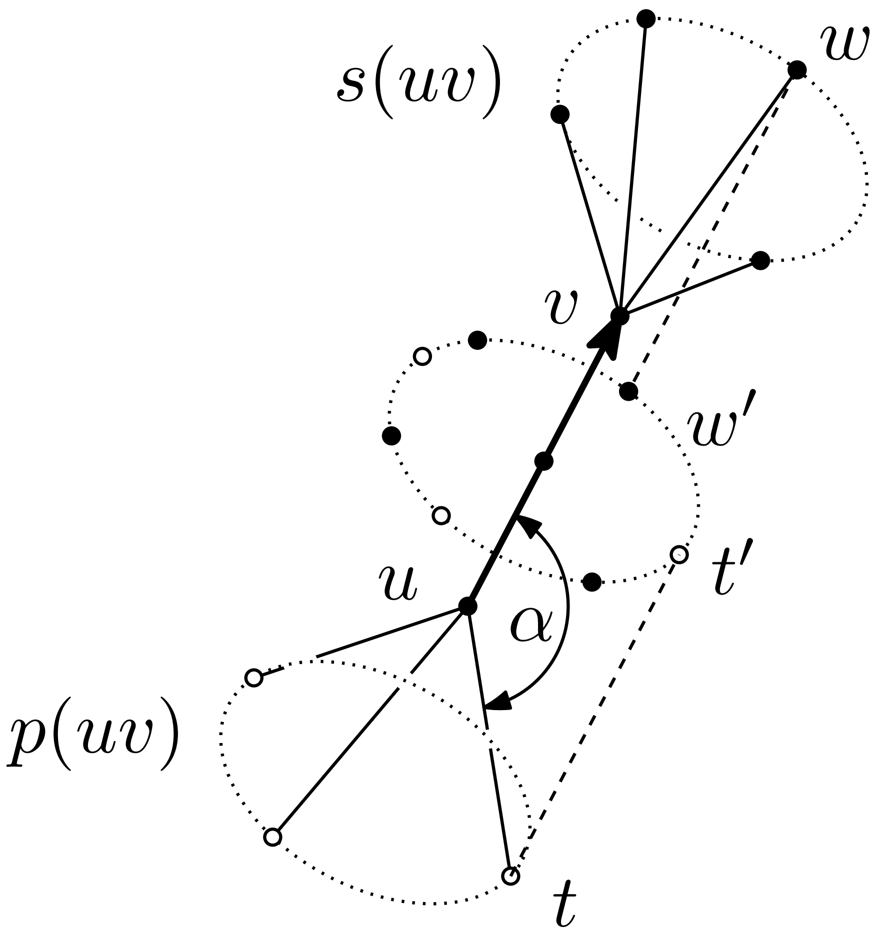

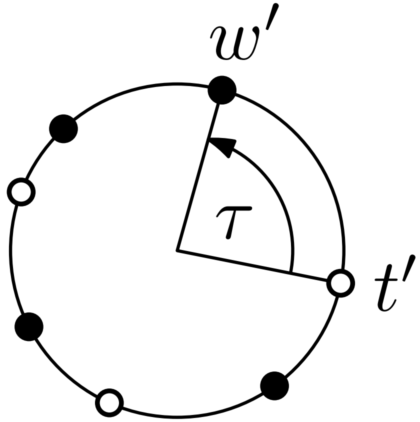

torsion angle τ

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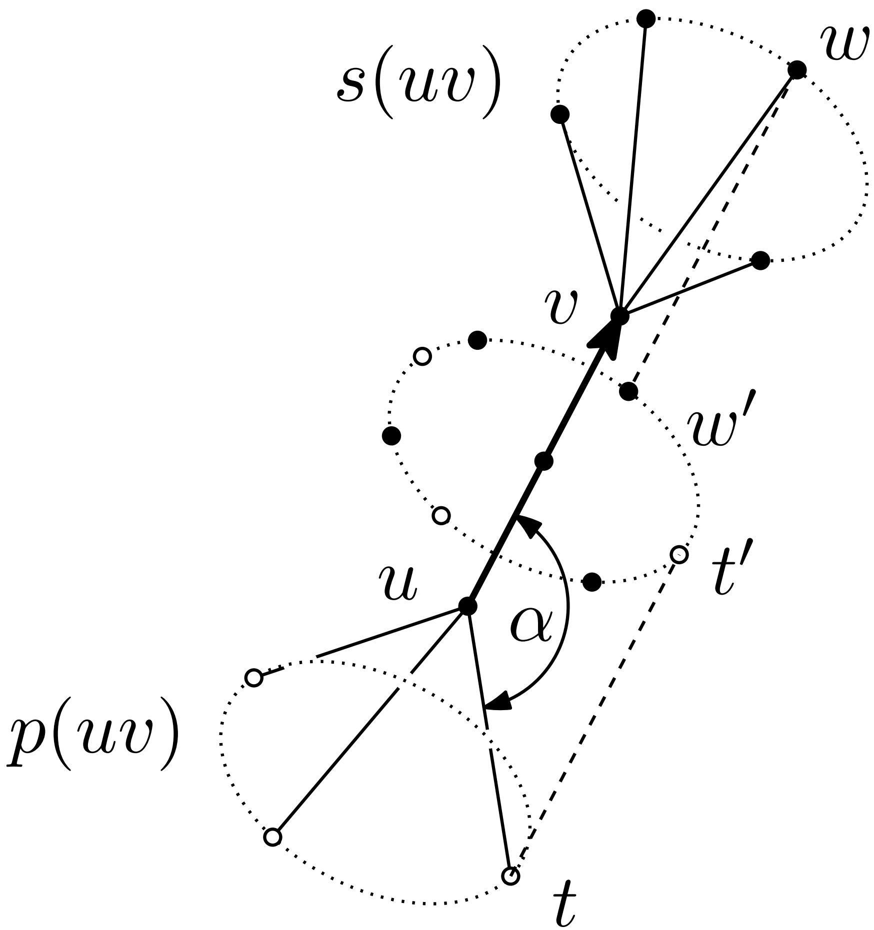

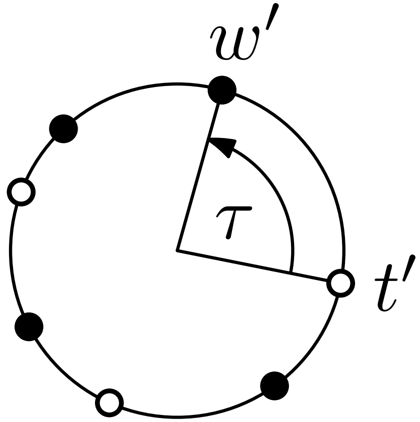

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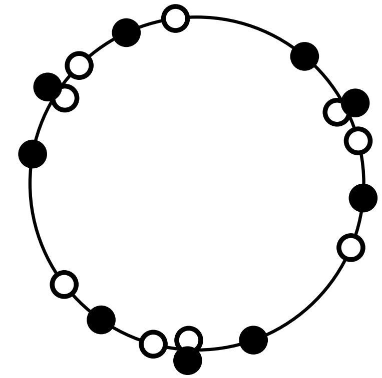

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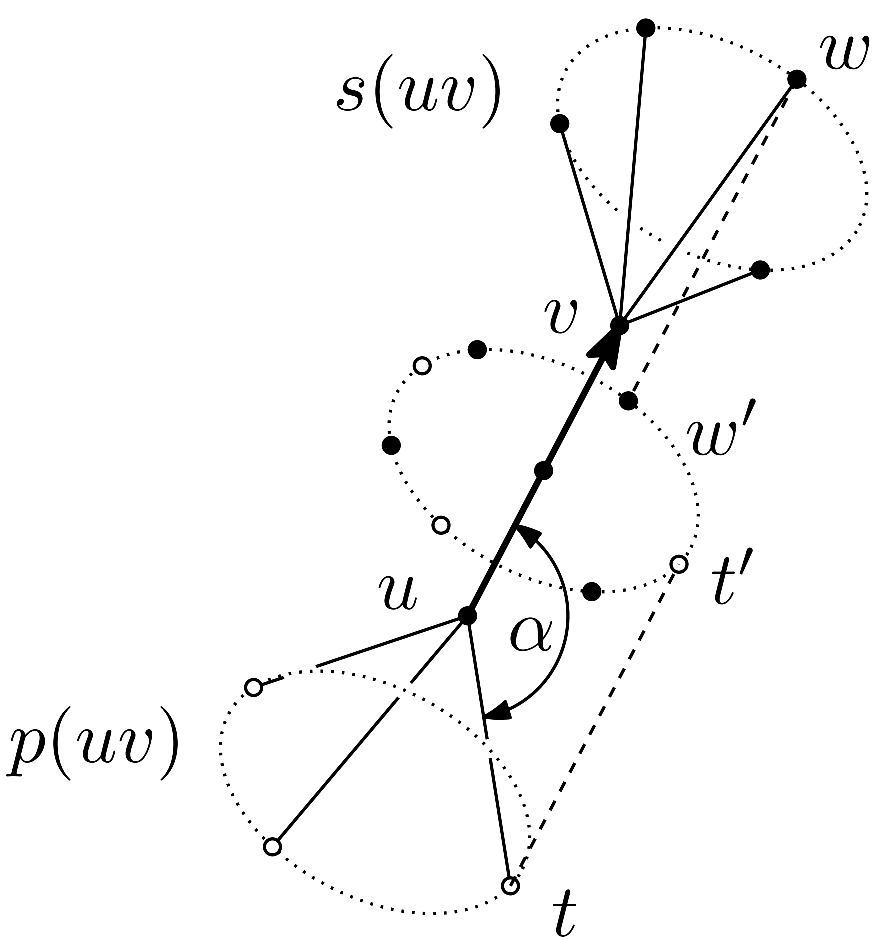
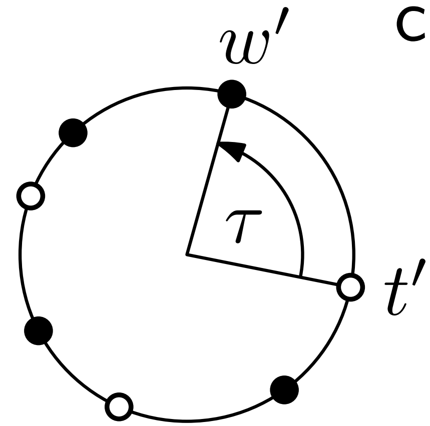
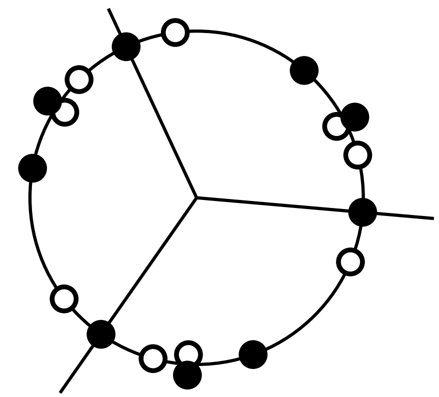




torsion angle τ

canonical directions



can PRUNE arcs from $s(u,v)$

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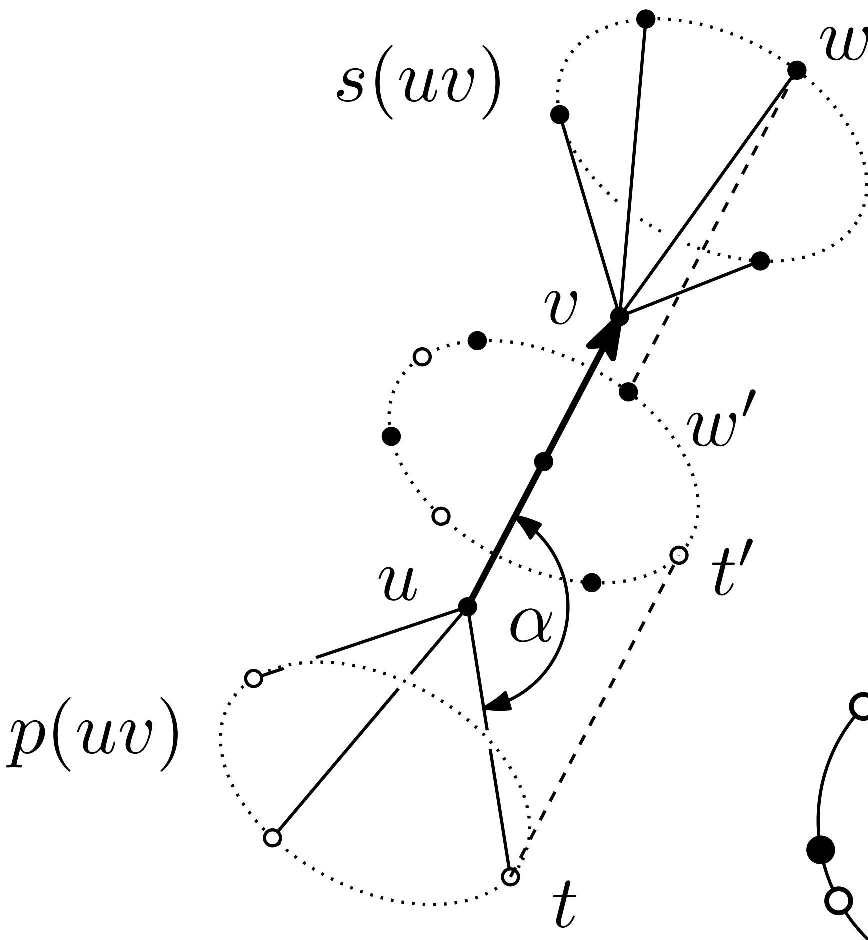

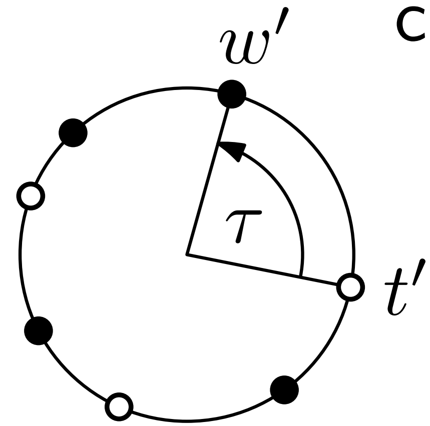

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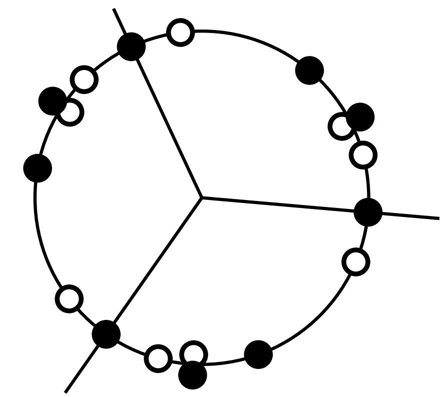

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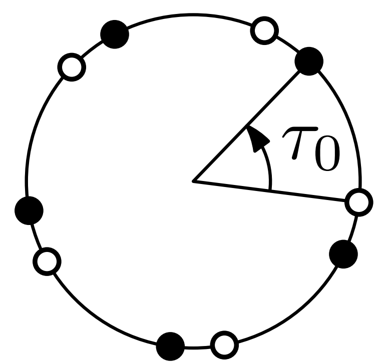
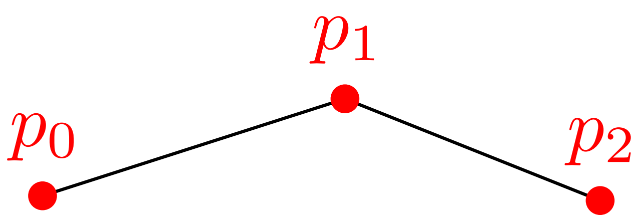


For every path tuv with $\angle tuv = \alpha$, $\exists\ vw$ with $\angle uvw = \alpha$ and torsion angle τ_0.

Orbit cycles

! For every path $p_i p_{i+1} p_{i+2}$ with $\angle p_i p_{i+1} p_{i+2} = \alpha$,
$\exists\ p_{i+3}$ with $\angle p_{i+1} p_{i+2} p_{i+3} = \alpha$ and torsion τ_0.



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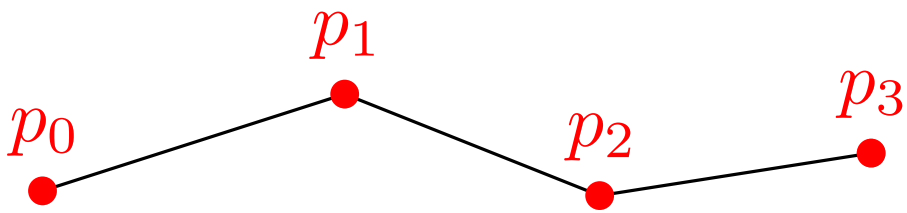

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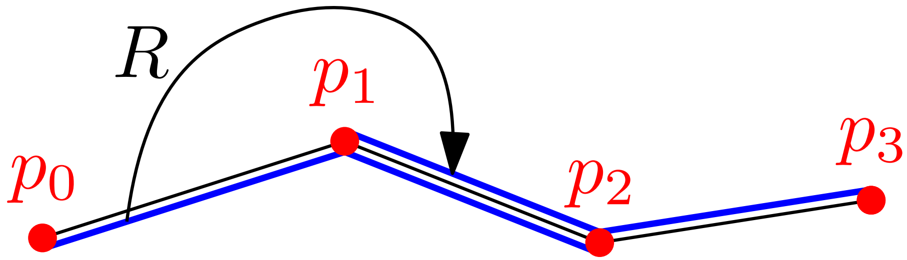

$$R(p_0, p_1, p_2) = (p_1, p_2, p_3)$$

Orbit cycles

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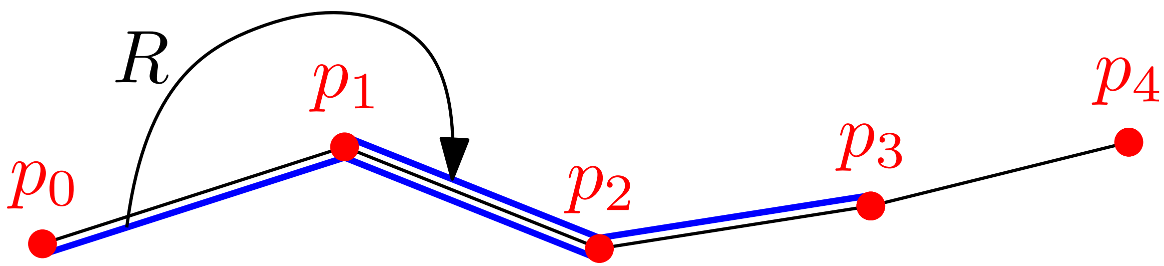

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$$R(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_4)$$

Orbit cycles

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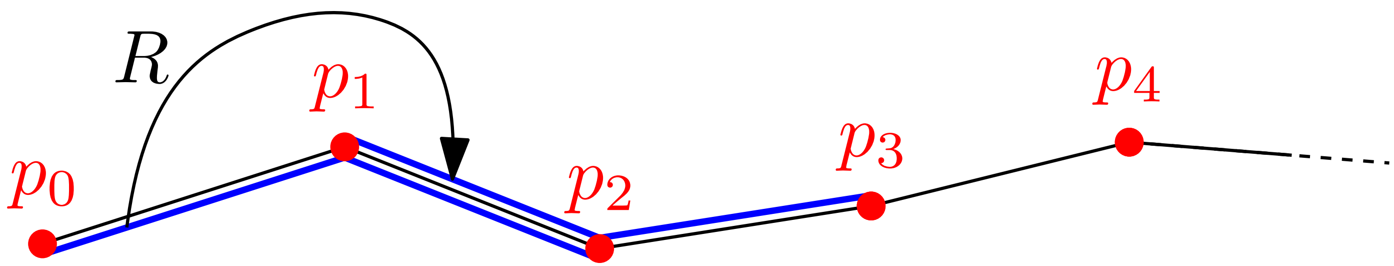

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$$R(p_1, p_2, p_3, p_4) = (p_2, p_3, p_4, p_5)$$
$$\ldots$$

$Rp_i = p_{i+1}$: The orbit of p_0 under R, a helix

Rotations in 4 dimensions

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{pmatrix} = \begin{pmatrix} R_\varphi & 0 \\ 0 & R_\psi \end{pmatrix}$$

in some appropriate coordinate system.

$\varphi \neq \pm\psi$: $\rightarrow$ unique decomposition $\mathbb{R}^4 = P \oplus Q$ into two completely orthogonal 2-dimensional *axis planes* P and Q

$\varphi = \pm\psi$: isoclinic rotations

The orbit of a point $p_0 = (x_1, y_1, x_2, y_2)$ lies on a *helix* on a *flat torus* $C_r \times C_s$, with $r = \sqrt{x_1^2 + y_1^2}$, $s = \sqrt{x_2^2 + y_2^2}$

circle with radius r

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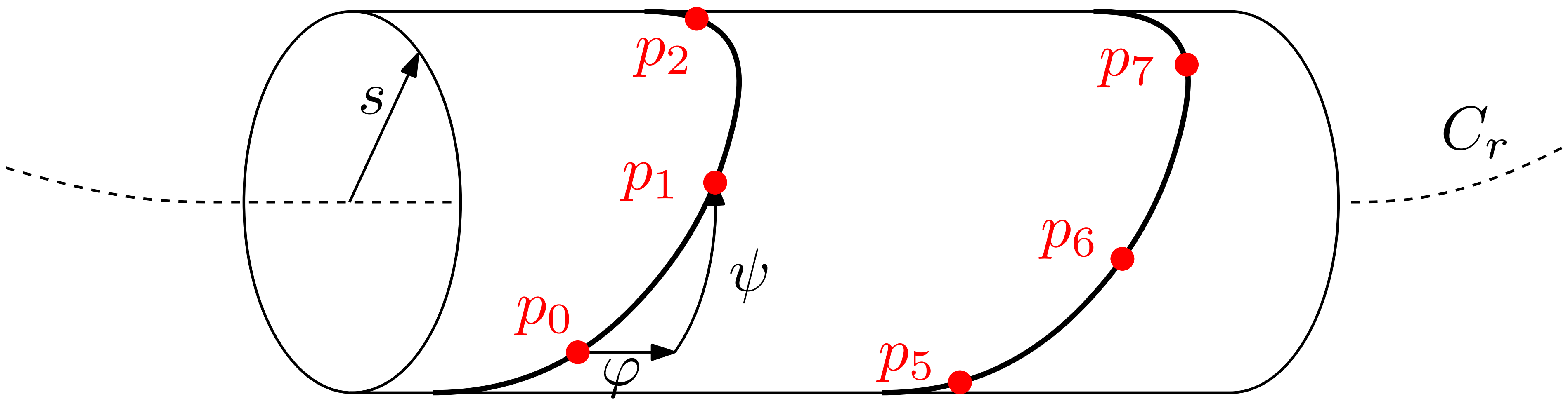
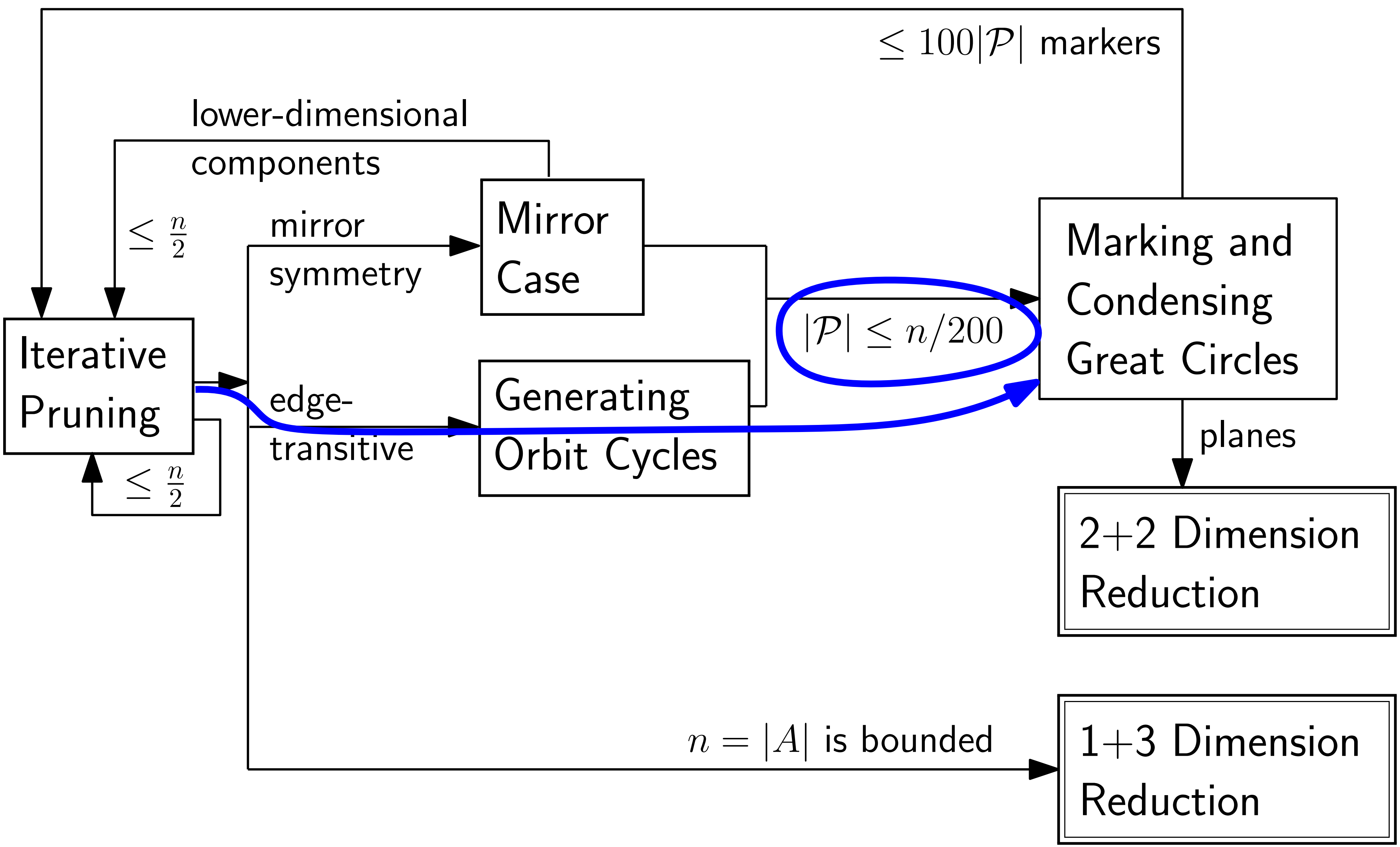


Planes in 4 dimensions

- Every point lies on ≤ 60 orbit cycles.
- Every orbit cycle contains ≥ 12000 points, because δ is small.
- Every orbit cycle generates 1 plane (corresponding to the smaller of φ and ψ.)

$\Longrightarrow$ a collection of $\leq n/200$ planes (or: great circles)

Algorithm Overview



Algorithm Overview

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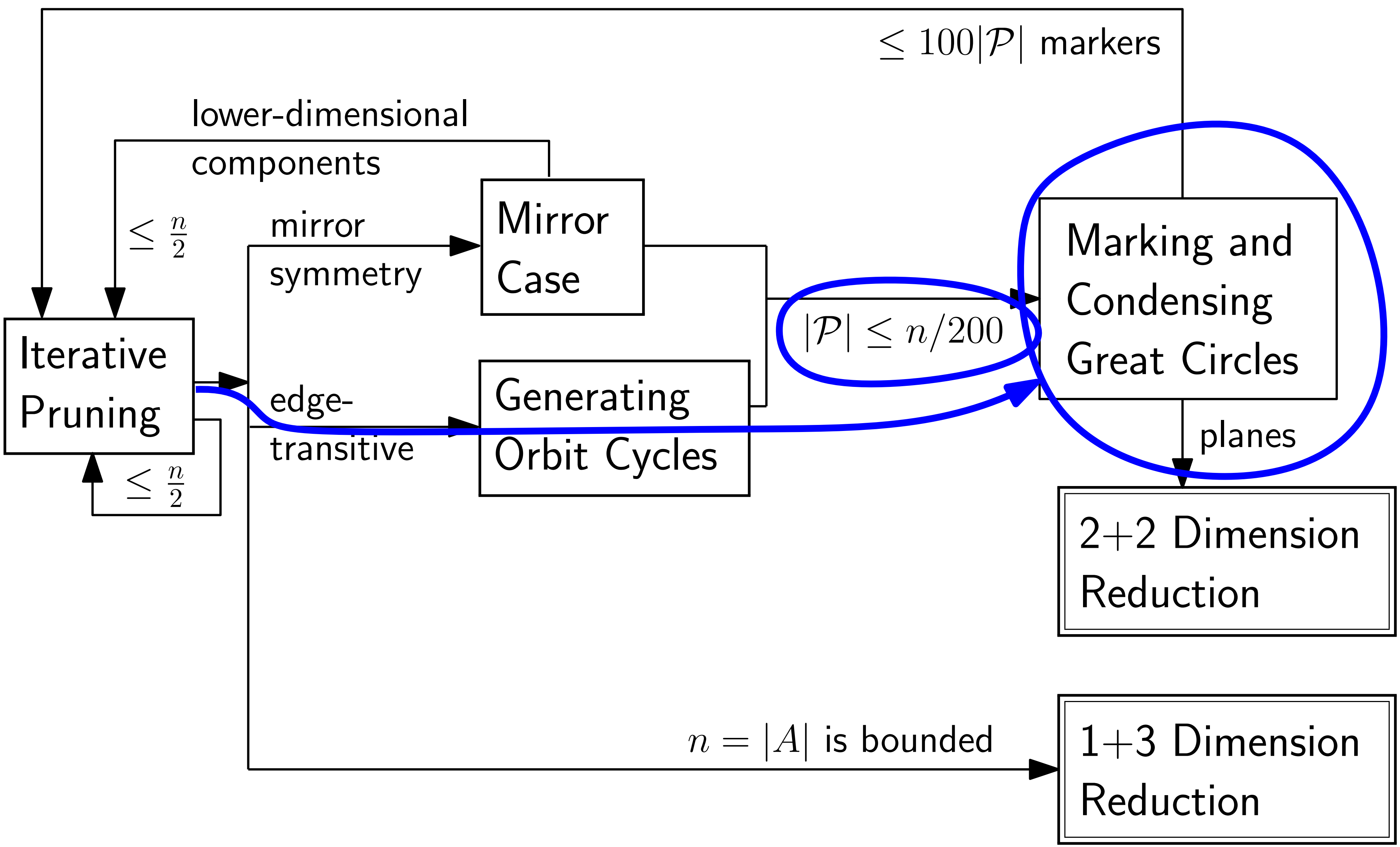
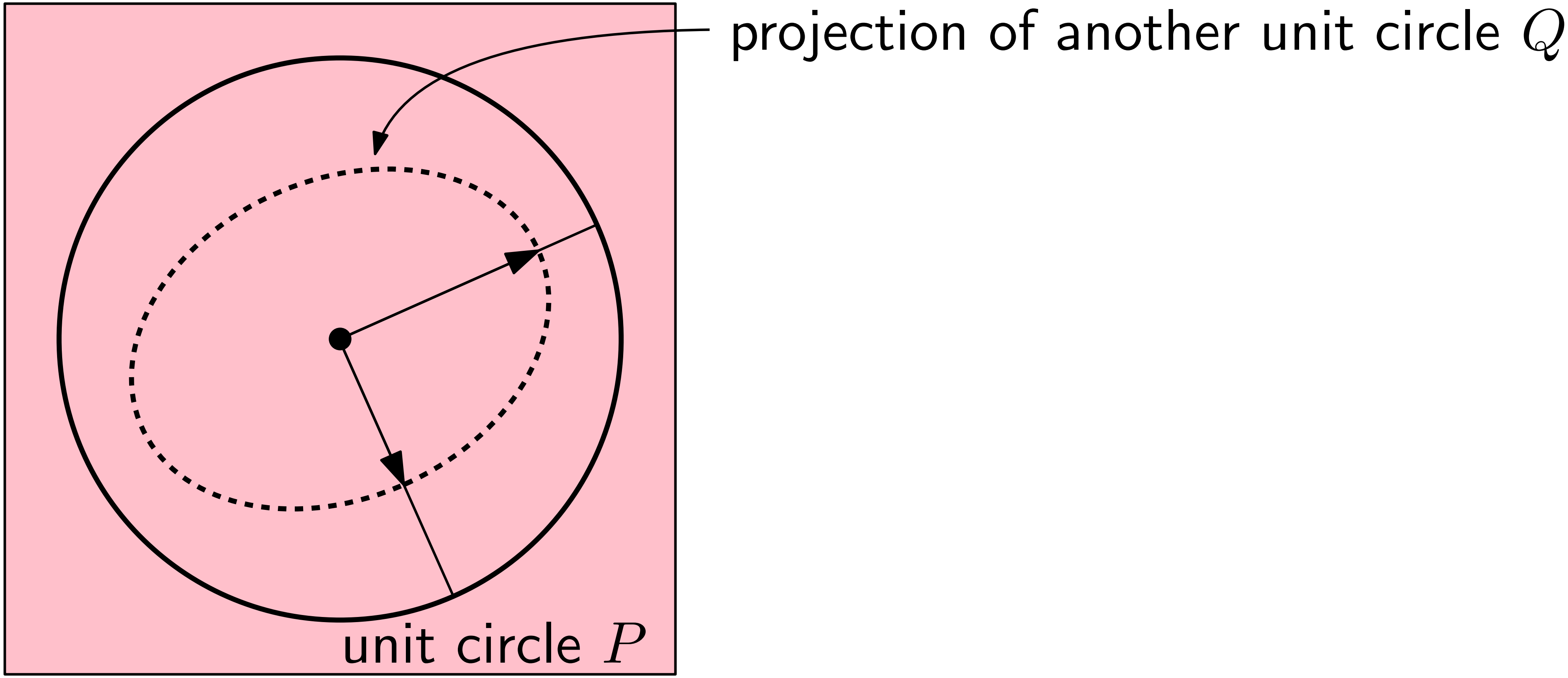


Marking Points on Great Circles



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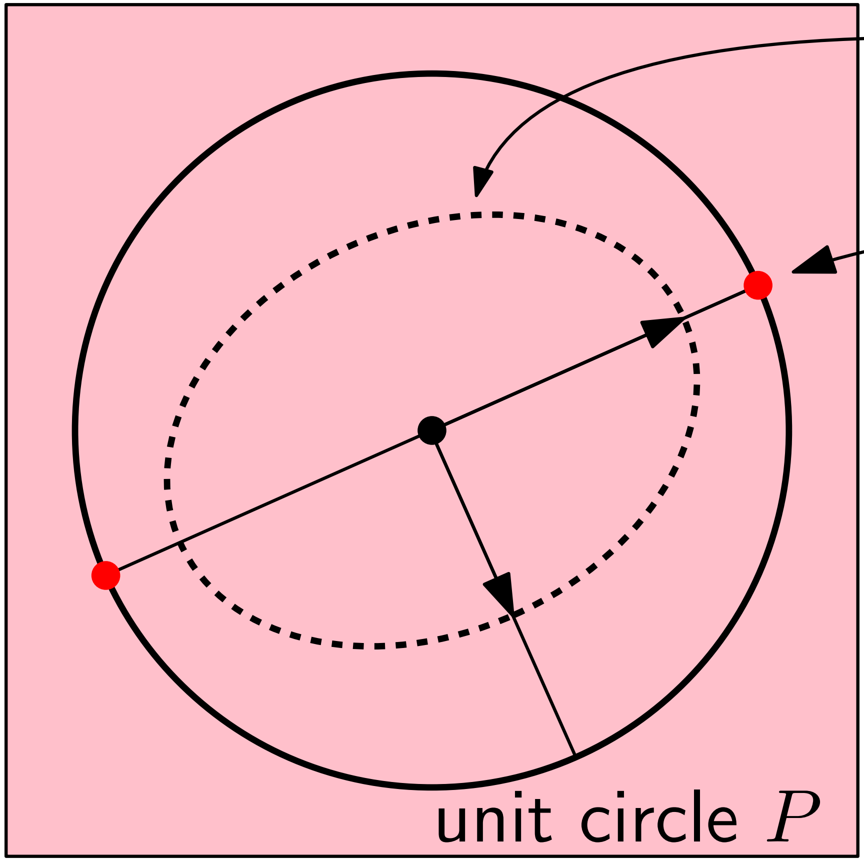
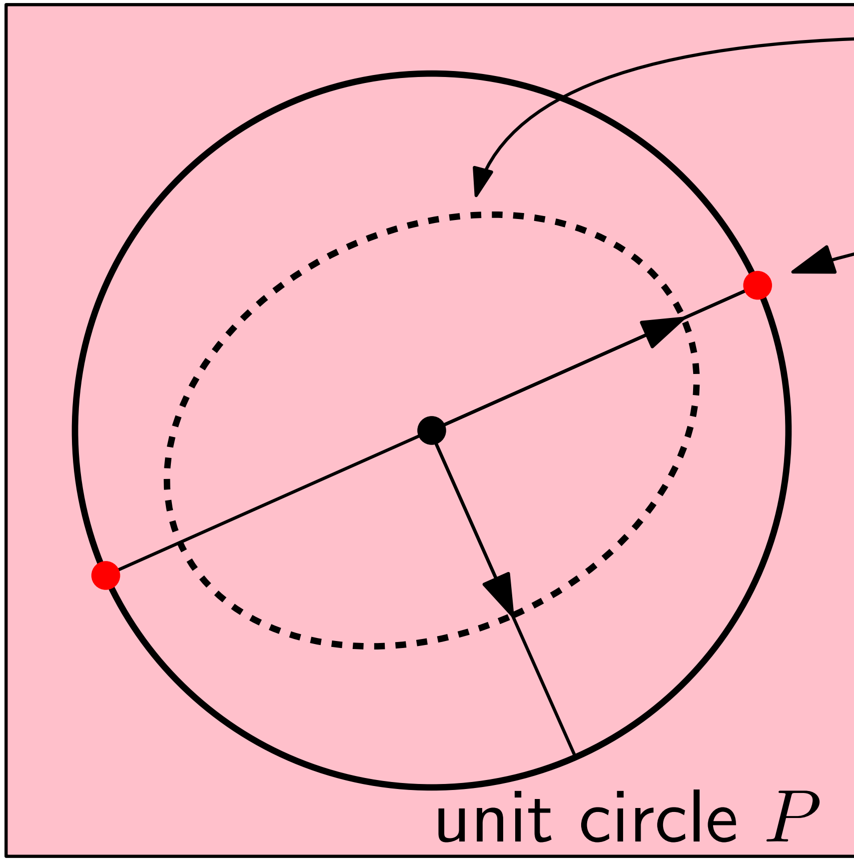


projection of another unit circle Q

IDEA: mark those two points in P

Marking Points on Great Circles



projection of ~~another unit circle~~ Q
a neighbor of P

IDEA: mark those two points in P

IDEA 2: Construct the closest-pair graph in the space of great circles, in $O(n \log n)$ time.

Plücker coordinates

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planes in 4-space $\Leftrightarrow$ great circles on $\mathbb{S}^3$ $\Leftrightarrow$ a.k.a. lines in $\mathbb{R}P^3$

plane through (x_1, y_1, x_2, y_2) and (x'_1, y'_1, x'_2, y'_2) :

$$(v_1, \ldots, v_6) = \left(\begin{vmatrix} x_1 & y_1 \\ x'_1 & y'_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_2 \\ x'_1 & y'_2 \end{vmatrix}, \begin{vmatrix} y_1 & x_2 \\ y'_1 & x'_2 \end{vmatrix}, \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}, \begin{vmatrix} x_2 & y_2 \\ x'_2 & y'_2 \end{vmatrix} \right)$$

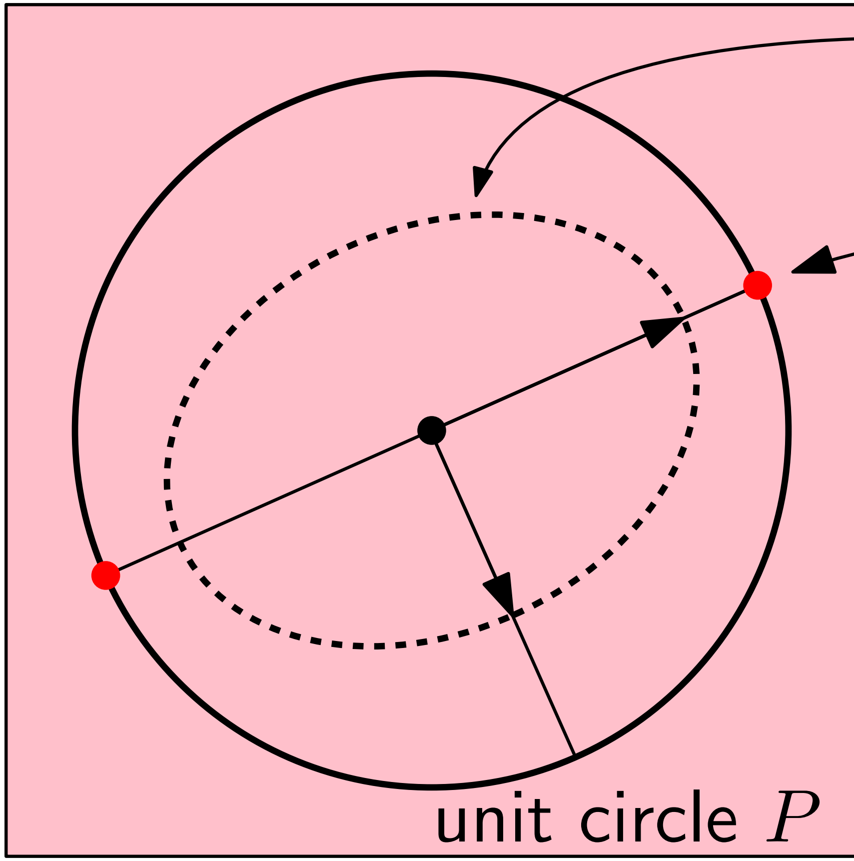
$(v_1, \ldots, v_6) \in \mathbb{R}P^5$. [Plücker relations $v_1 v_6 - v_2 v_5 + v_3 v_4 = 0$]

Normalize:
$\rightarrow$ A great circle is represented by two antipodal points on $\mathbb{S}^5$.

This representation is geometrically meaningful:
Distances on $\mathbb{S}^5$ are preserved under rotations of $\mathbb{R}^4$ / $\mathbb{S}^3$.

(Packings of 2-planes in 4-space were considered by [Conway, Hardin and Sloane 1996], with different distances.)

Marking Points on Great Circles



projection of ~~another unit circle~~ Q
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Every plane has at most $K_5 \le 44$ neighbors.

Marking Points on Great Circles

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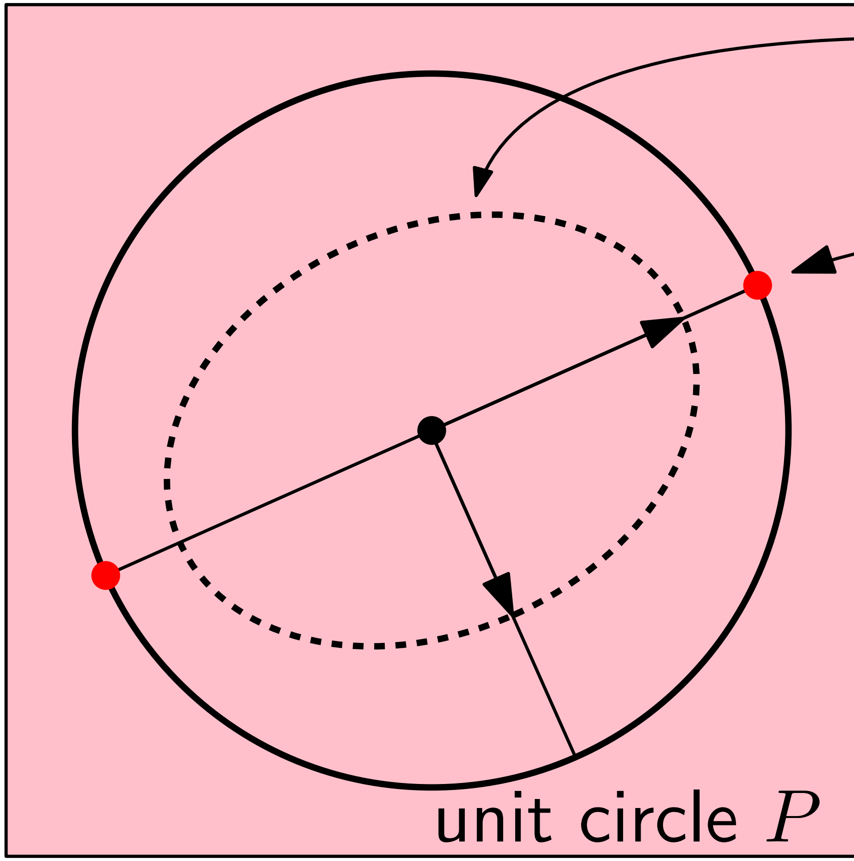

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$m \leq \frac{n}{200}$ great circles in $\mathbb{R}^4$ $\longrightarrow$ m point pairs on $\mathbb{S}^5$

At most 88 (≤ 100) points are marked on every great circle.

These points replace A. $\rightarrow$ successful CONDENSATION

Algorithm Overview

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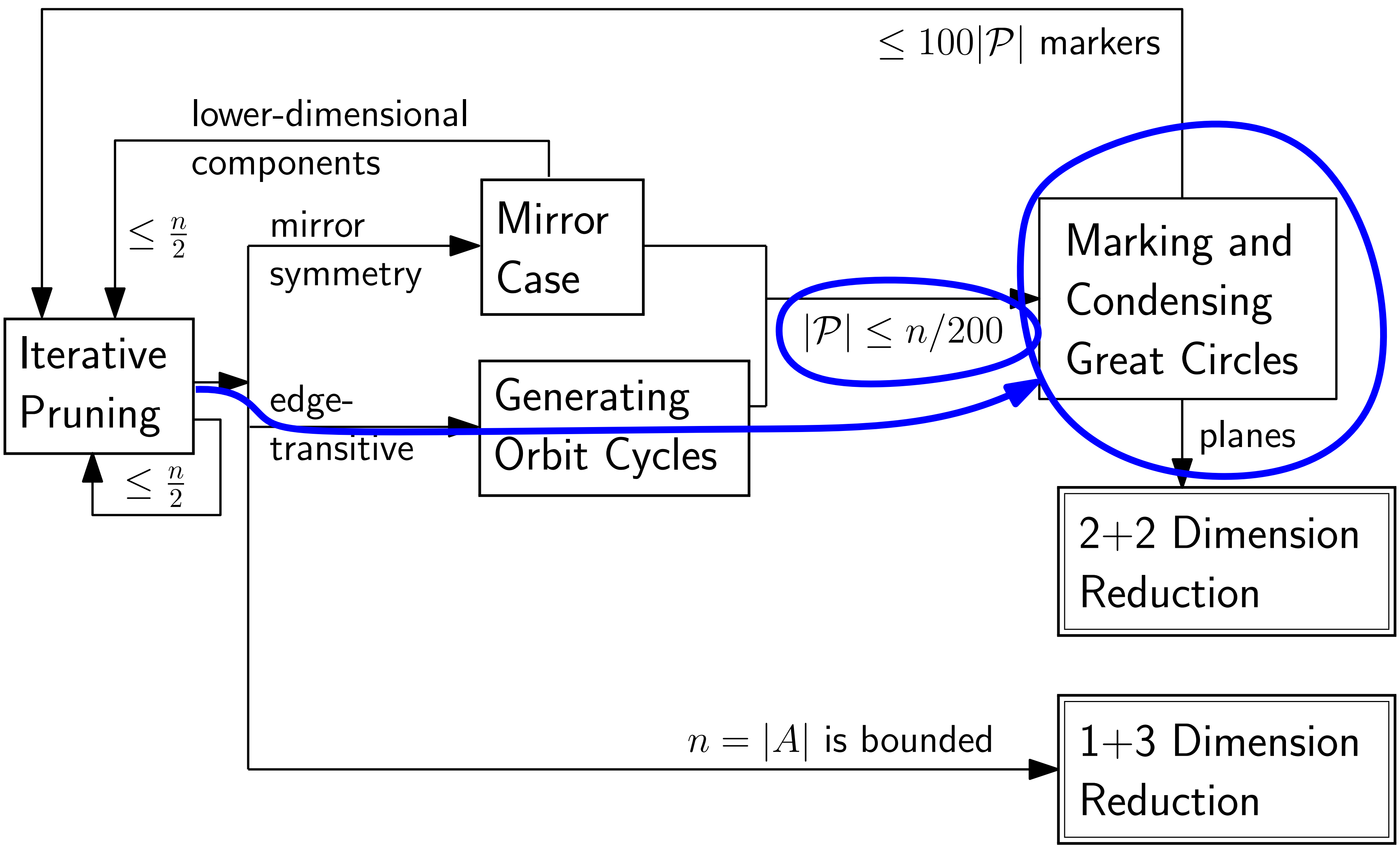

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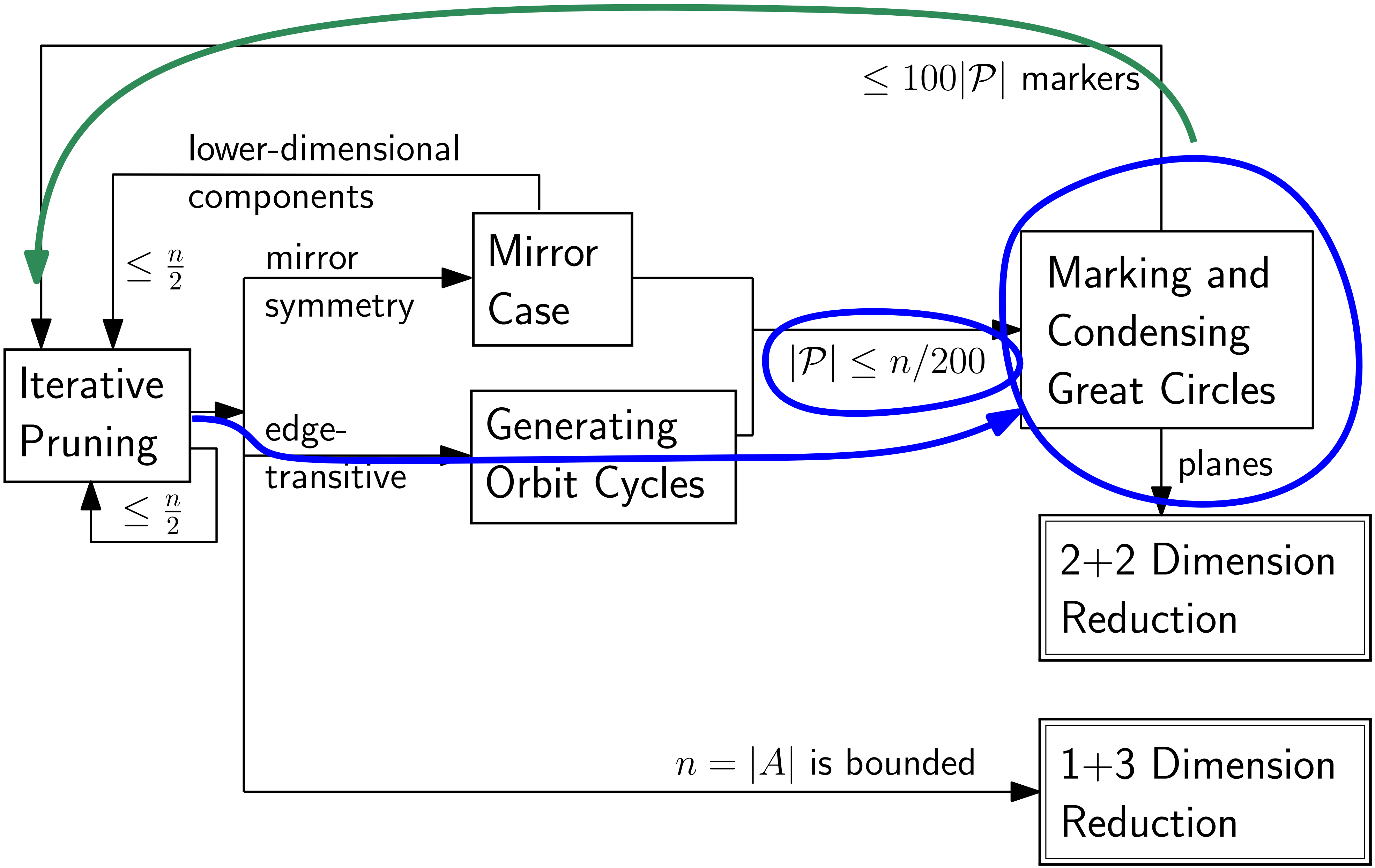
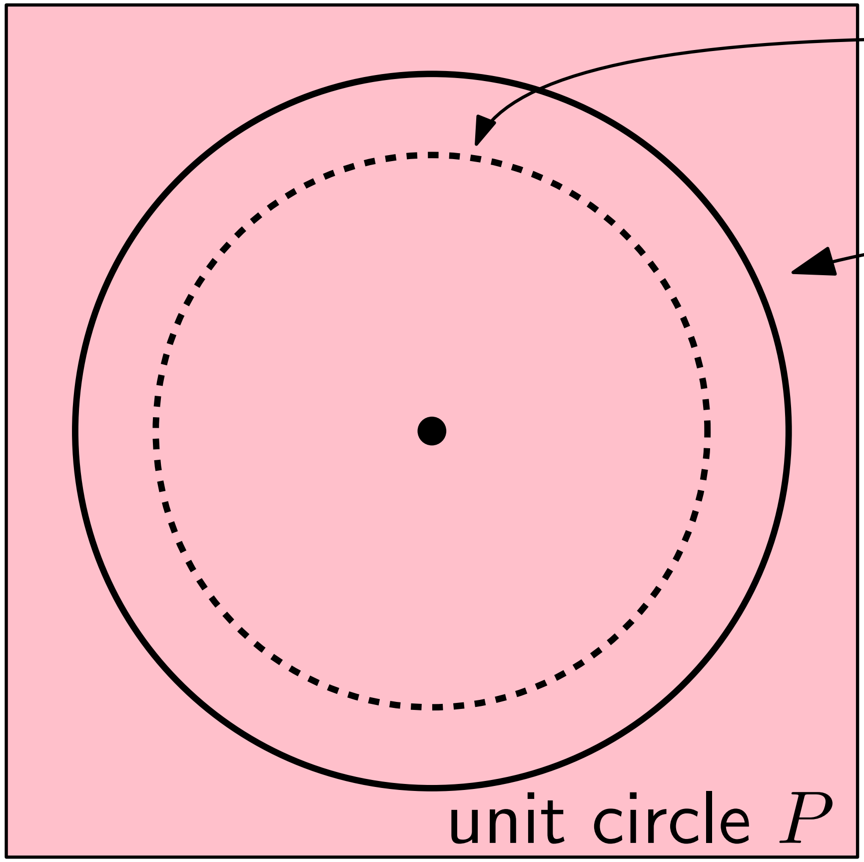


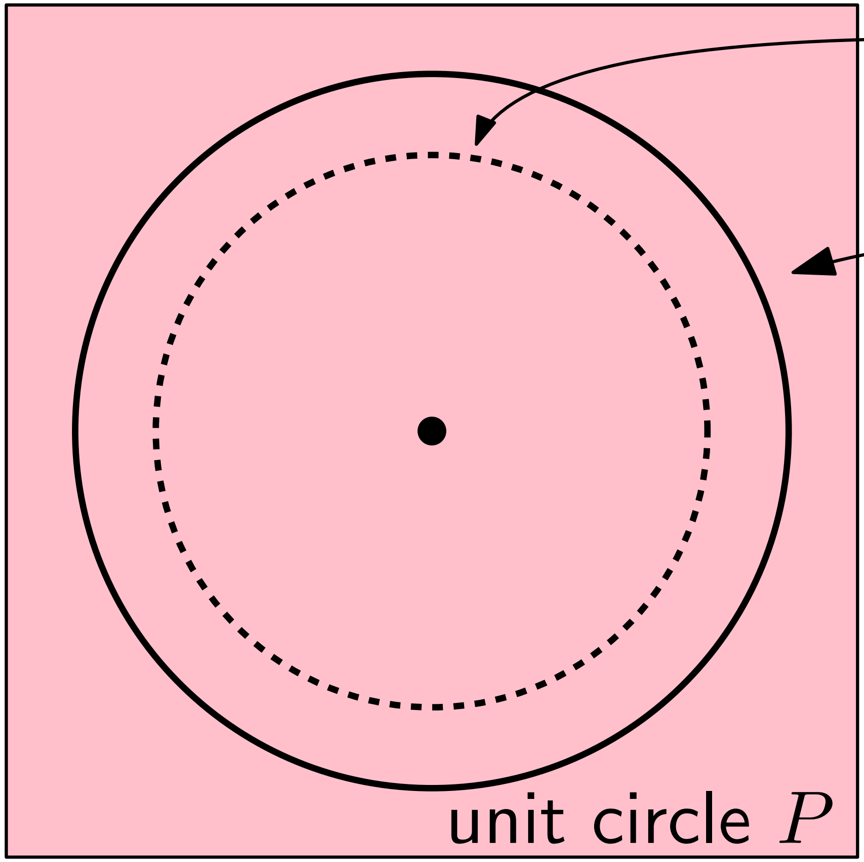
Isoclinic planes



projection of a neighbor Q of P

Where to mark??

Isoclinic planes

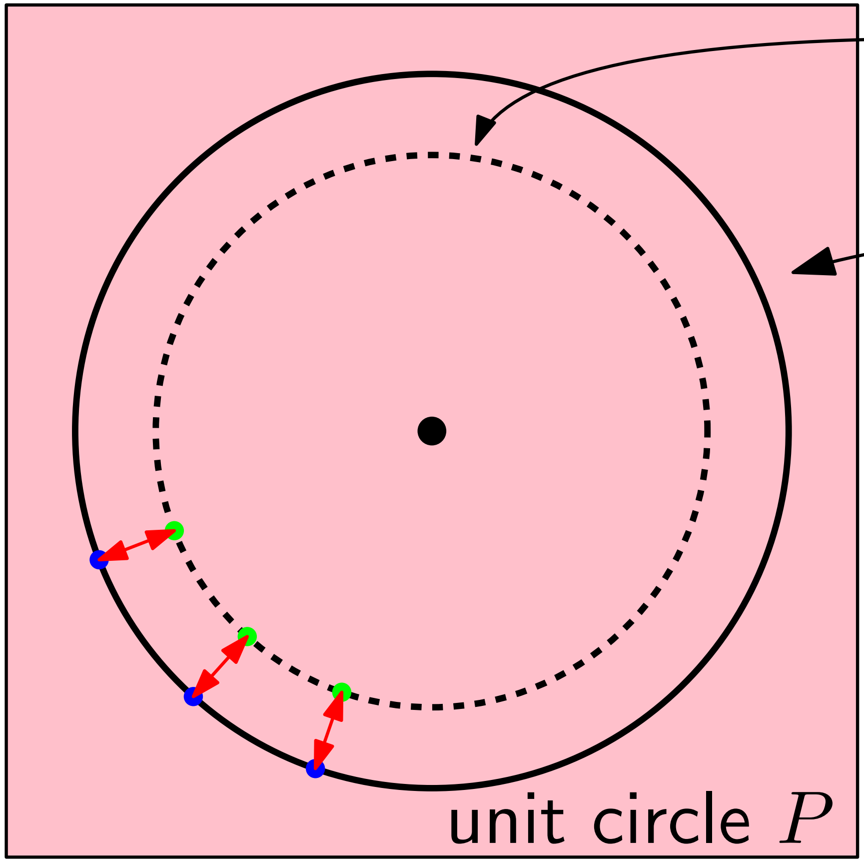


projection of a neighbor Q of P

Where to mark??

Problem if *all* closest pairs are isoclinic.

Isoclinic planes



projection of a neighbor Q of P

Where to mark??

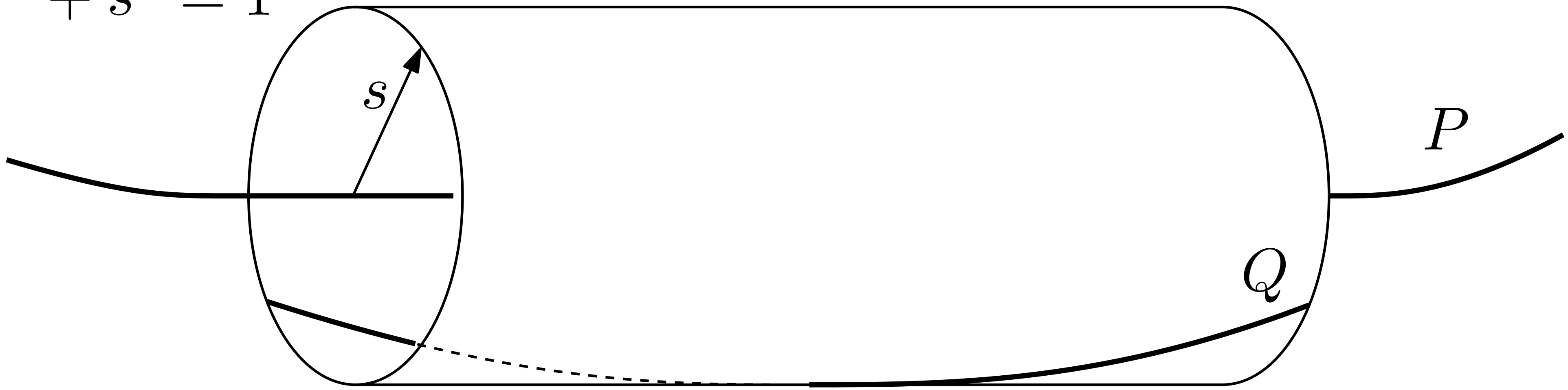
Problem if *all* closest pairs are isoclinic.

Constant distances from one circle to the other.
"Clifford-parallel" $\equiv$ isoclinic

Clifford-parallel circles

$$P\colon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha + t) \\ s\sin(\alpha + t) \end{pmatrix}$$

$$r^2 + s^2 = 1$$



Clifford-parallel circles

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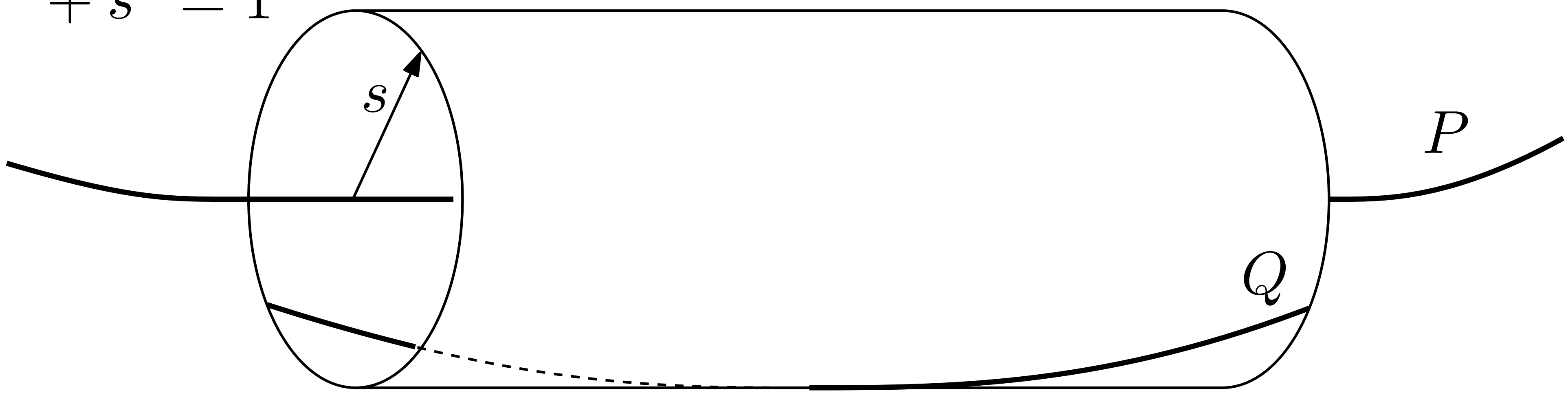
$$P\colon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, \quad Q\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha+t) \\ s\sin(\alpha+t) \end{pmatrix}$$

$r^2 + s^2 = 1$



$h(x_1, y_1, x_2, y_2) =$ the *right* Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$

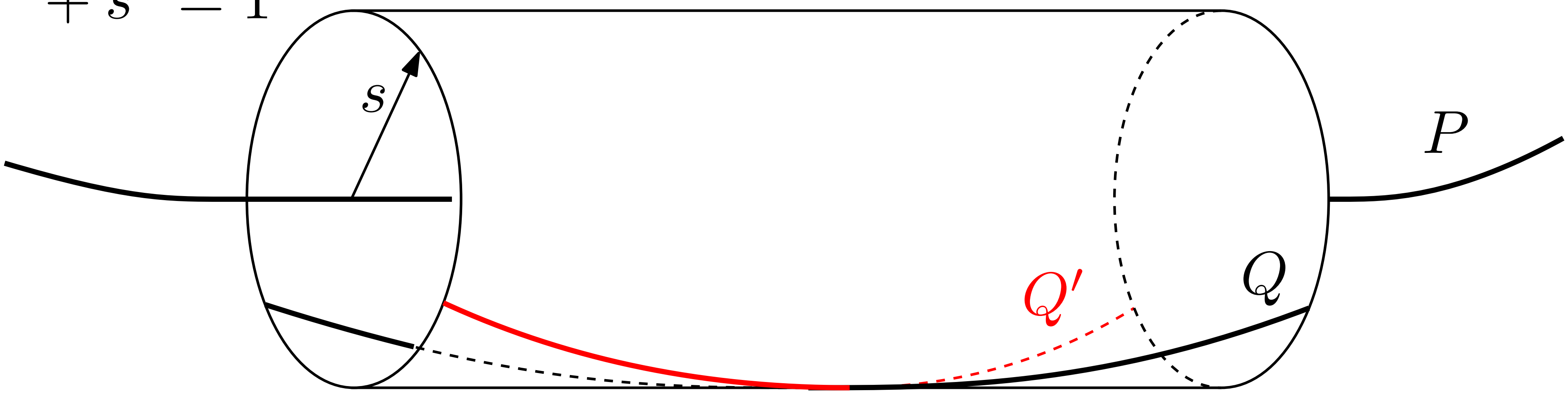
$$\Big(2(x_1y_2 - y_1x_2),\ 2(x_1x_2 + y_1y_2),\ 1 - 2(x_2^2 + y_2^2)\Big)$$

[Hopf 1931]

Clifford-parallel circles

$$P\colon \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix},\quad Q\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha + t) \\ s\sin(\alpha + t) \end{pmatrix} \qquad Q'\colon \begin{pmatrix} r\cos t \\ r\sin t \\ s\cos(\alpha - t) \\ s\sin(\alpha - t) \end{pmatrix}$$

$r^2 + s^2 = 1$



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[Hopf 1931]

The Hopf fibration

Right Hopf map $h\colon \mathbb{S}^3 \to \mathbb{S}^2$

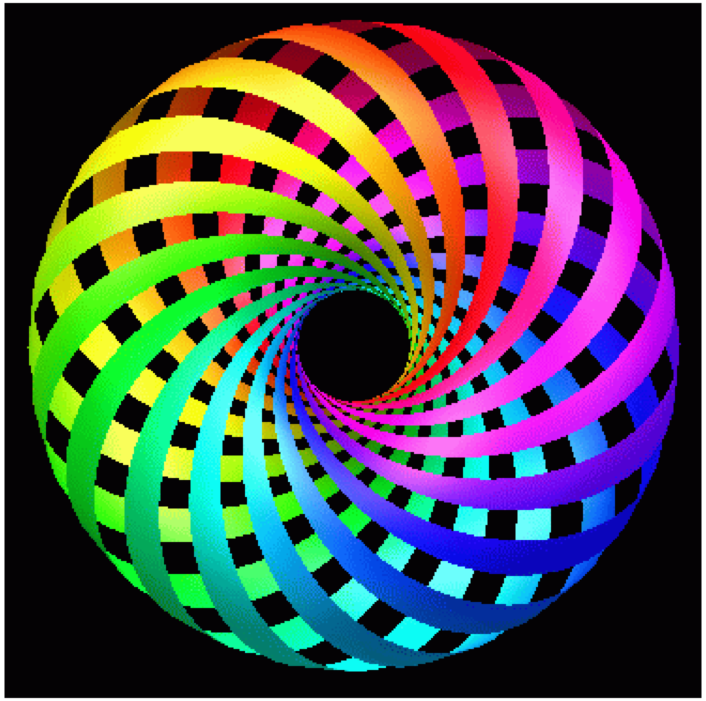
The fibers $h^{-1}(p)$ for $p \in \mathbb{S}^2$ are great circles: a *Hopf bundle*

Every great circle belongs to a unique right Hopf bundle.

Isoclinic $\equiv$ belong to the same Hopf bundle
This is a transitive relation.

stereographic projection $\mathbb{S}^3 \to \mathbb{R}^3$

(Villarceau circles)



http://www.geom.uiuc.edu/~banchoff/script/b3d/hypertorus.html

The Hopf fibration

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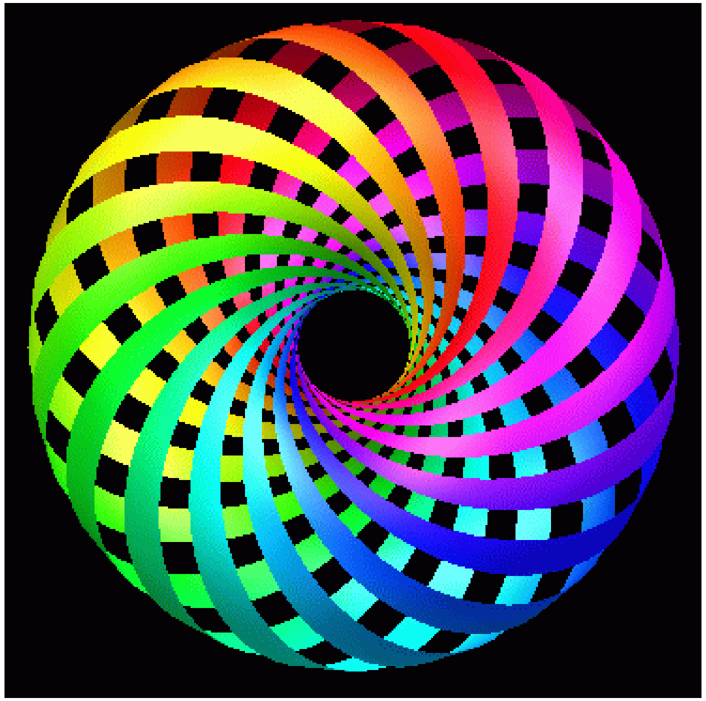
Every great circle belongs to a unique right Hopf bundle.

Isoclinic $\equiv$ belong to the same Hopf bundle
This is a transitive relation.

If all closest pairs are isoclinic
$\to$ all great circles in a connected component of the closest-pair graph belong to the same bundle.
$\to$ h maps them to points on $\mathbb{S}^2$.

We know how to deal with $\mathbb{S}^2$!



http://www.geom.uiuc.edu/~banchoff/script/b3d/hypertorus.html

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Equivariant condensation on the 2-sphere:

Input: $A \subseteq \mathbb{S}^2$.

Output: $A' \subseteq \mathbb{S}^2$, $|A'| \le \min\{|A|, 12\}$.

- $A' =$ vertices of a regular icosahedron
- $A' =$ vertices of a regular octahedron
- $A' =$ vertices of a regular tetrahedron
- $A' =$ two antipodal points, or
- $A' =$ a single point.

Condensing on the 2-Sphere

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- $A' =$ vertices of a regular tetrahedron
- $A' =$ two antipodal points, or
- $A' =$ a single point.

Condense each connected component of the closest-pair graph to ≤ 12 great circles.
Compute closest-pair graph (on $\mathbb{S}^5$) from scratch.
If no progress, distance between closest pairs is $\ge D_{\text{icosa}}$
$\to \le 829$ great circles $\to$ 2+2 DIMENSION REDUCTION

Algorithm Overview

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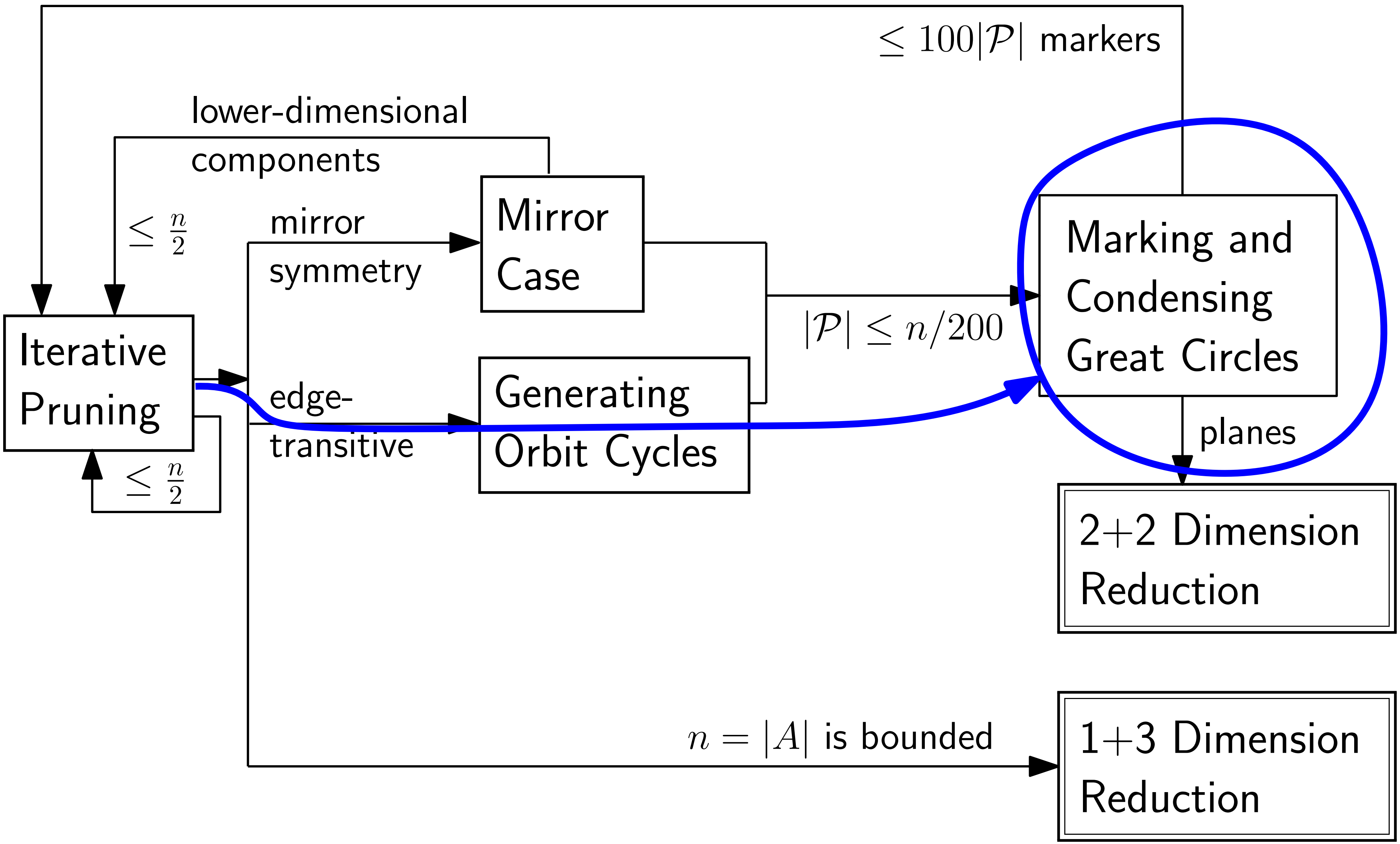
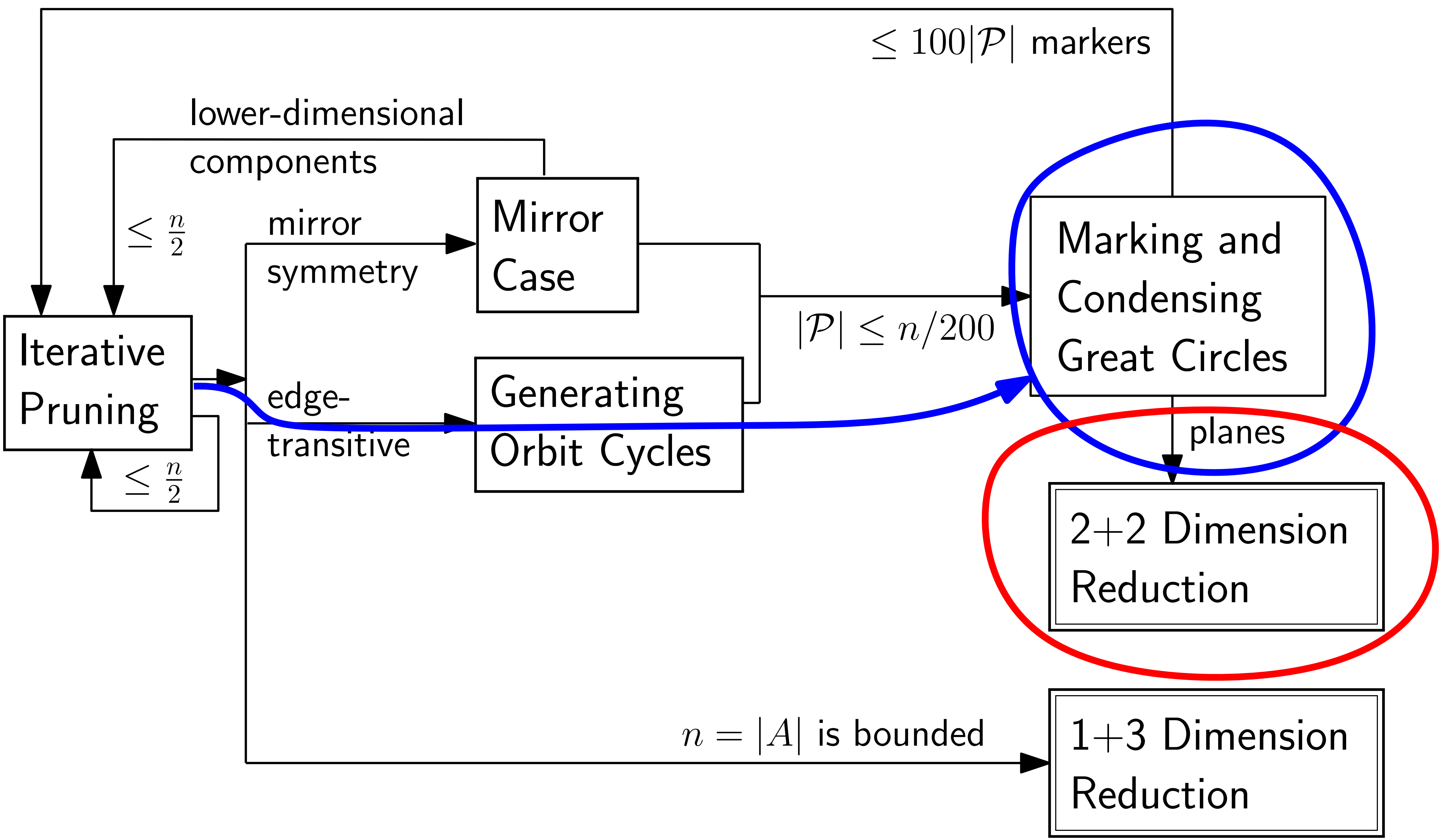


Algorithm Overview



2+2 Dimension Reduction

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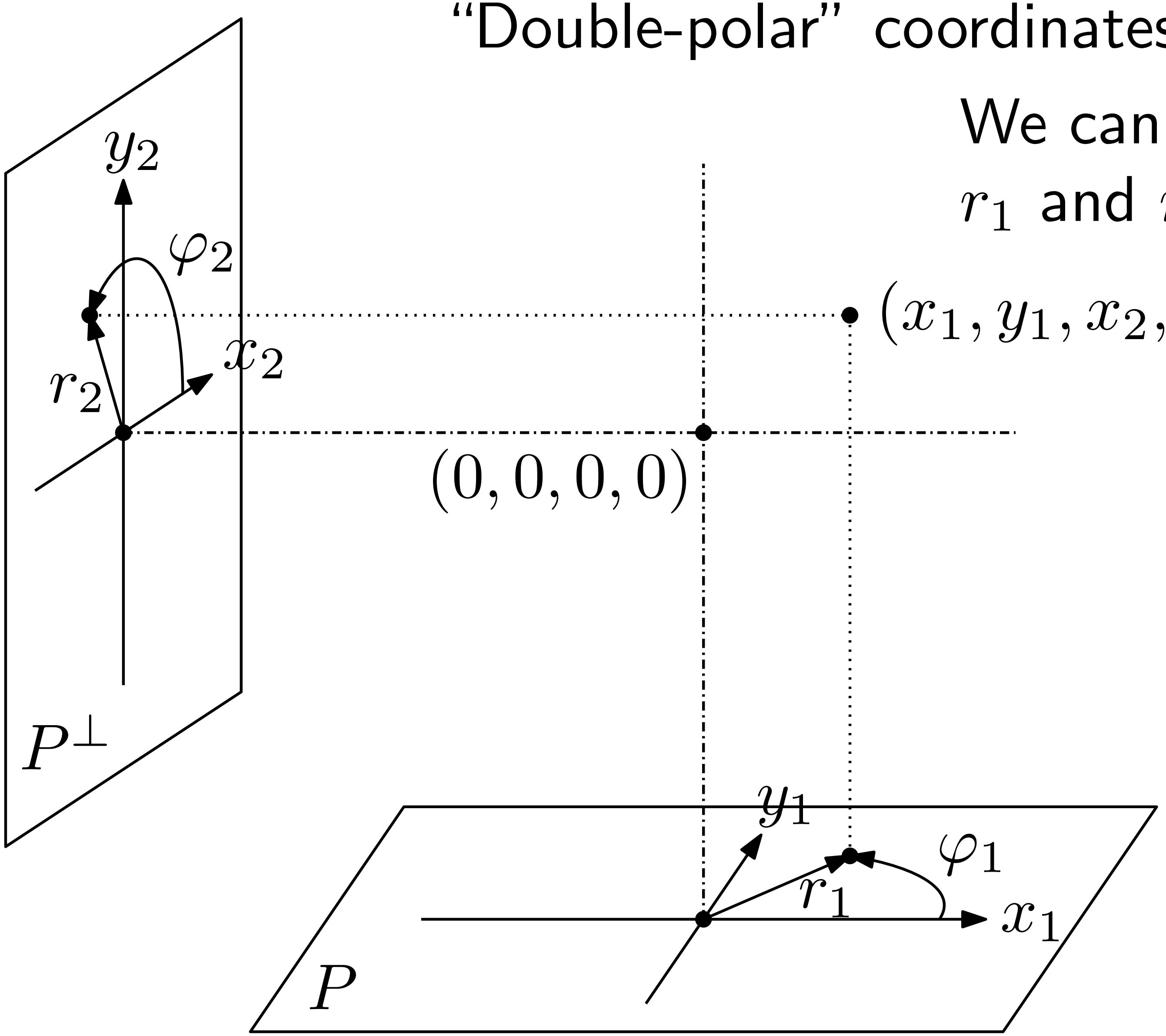
We have a plane P and we know its image in B.

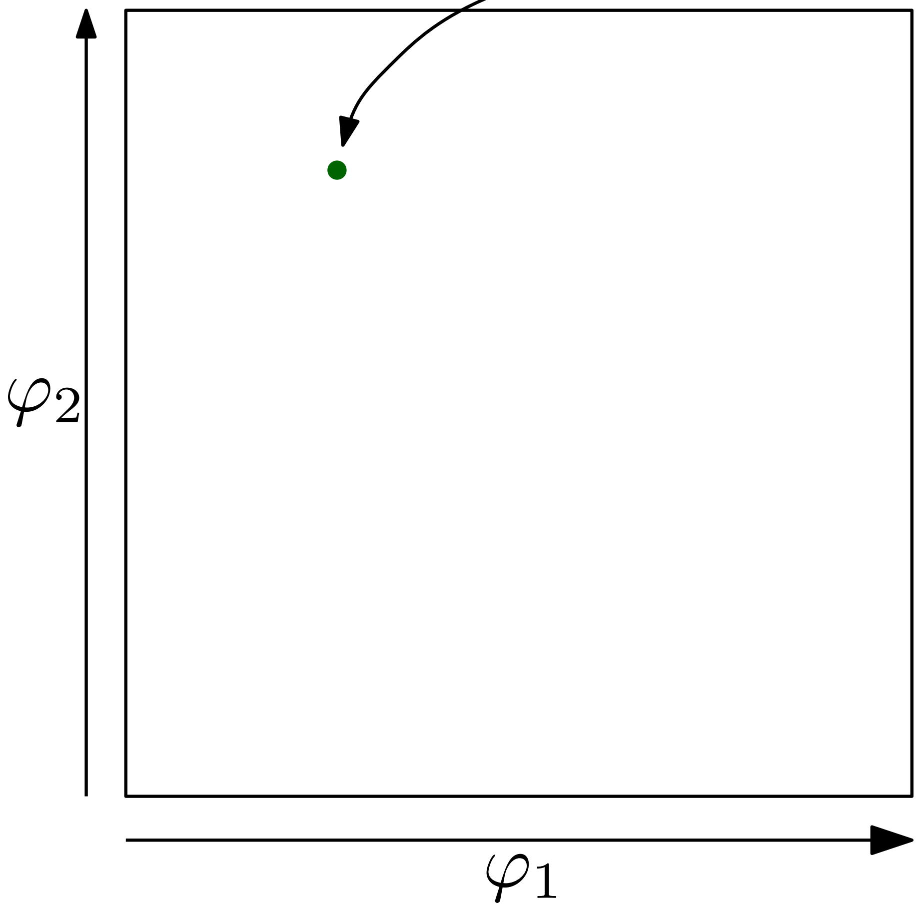
"Double-polar" coordinates $(r_1, \varphi_1, r_2, \varphi_2)$

We can change φ_1 and φ_2.
r_1 and r_2 are fixed.



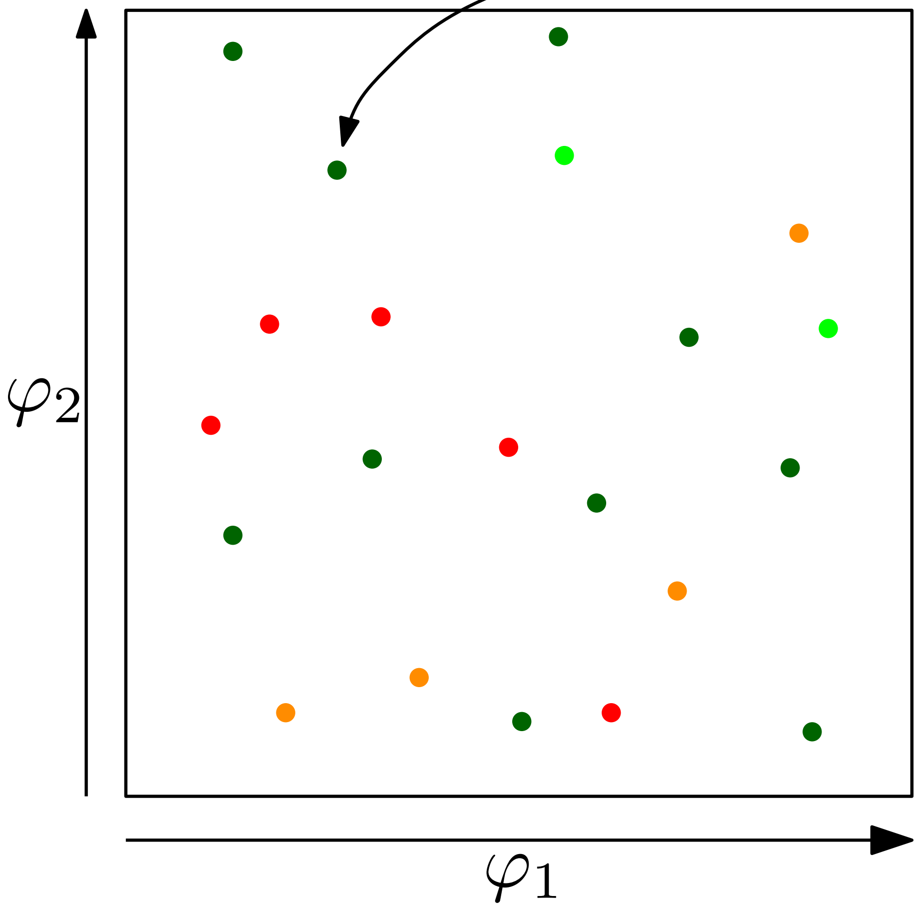
2+2 Dimension Reduction

attach (r_1, r_2) as a label (color)

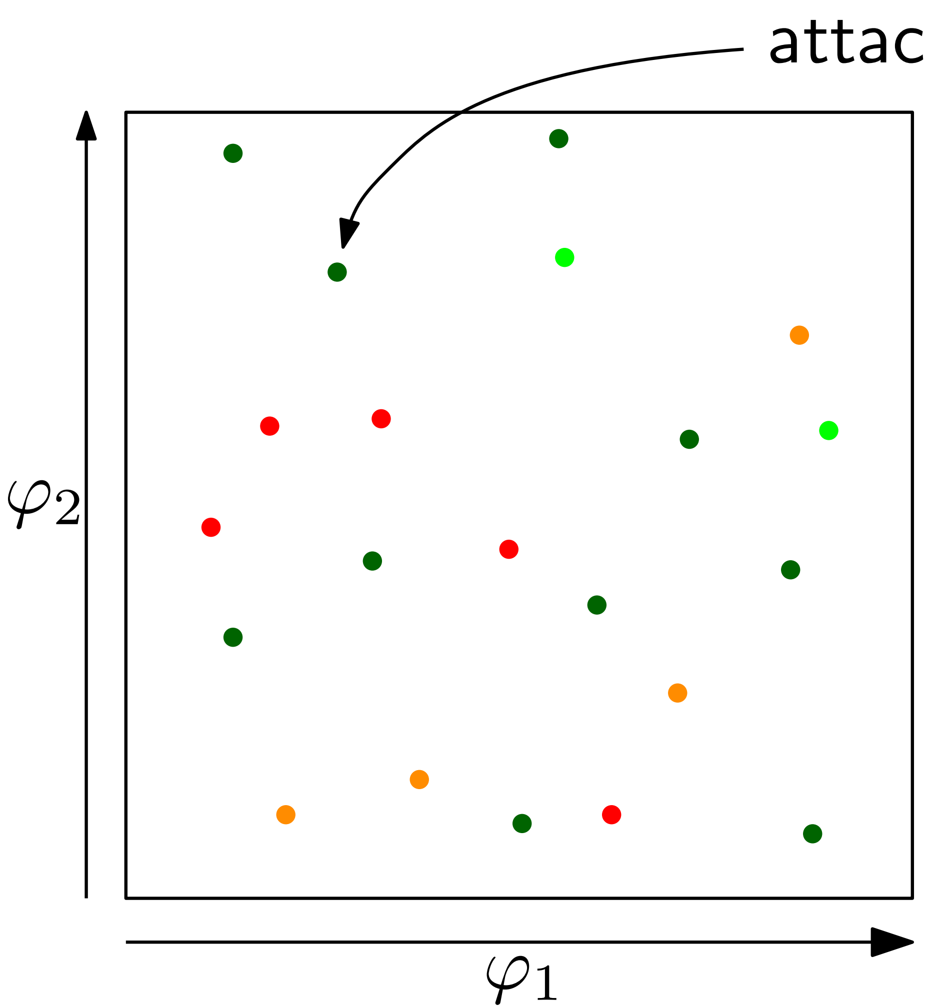


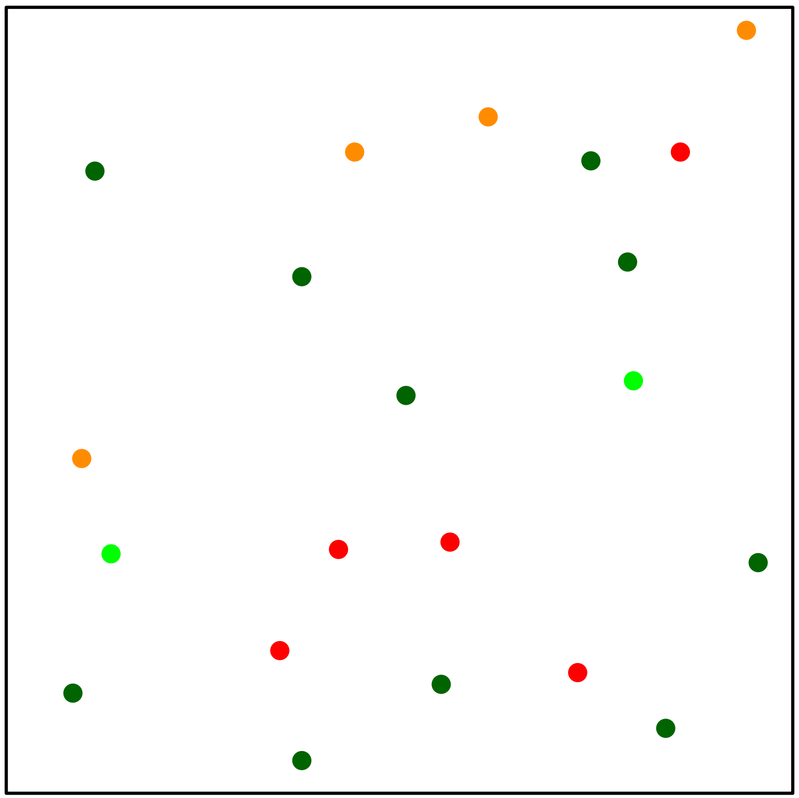
2+2 Dimension Reduction

attach (r_1, r_2) as a label (color)



2+2 Dimension Reduction

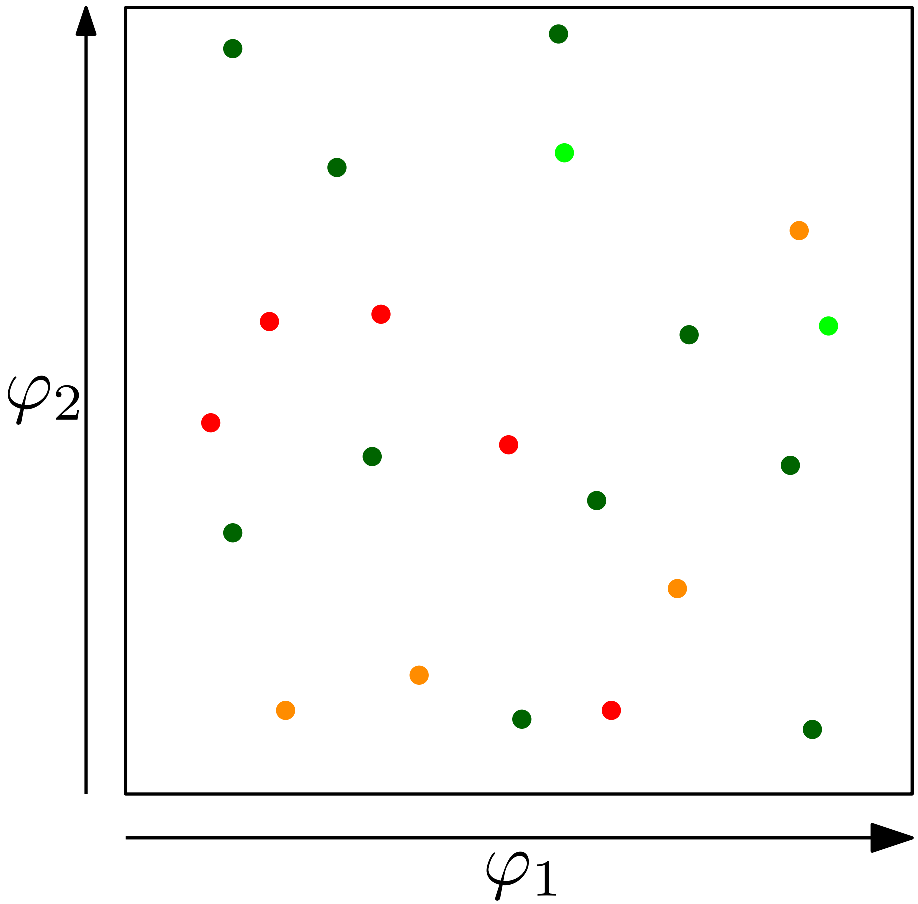




the picture for set B

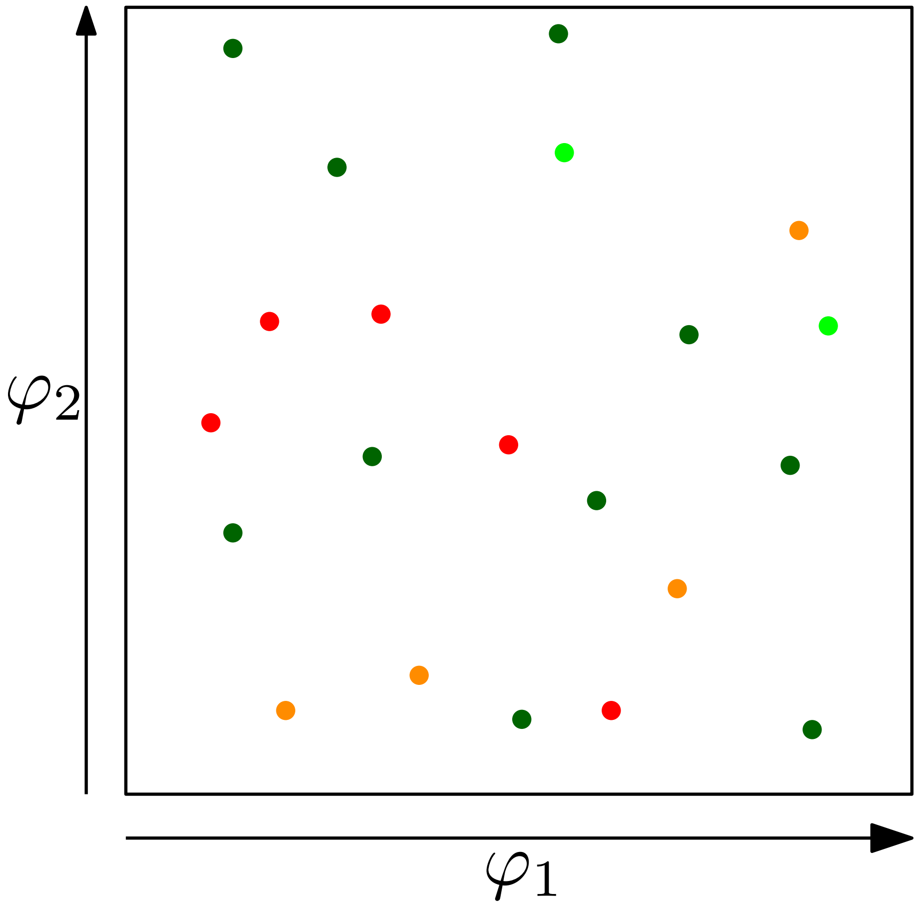
Are they the same up to translation on the φ_1, φ_2-torus?

2+2 Dimension Reduction



Prune without losing information:
(CANONICAL SET)

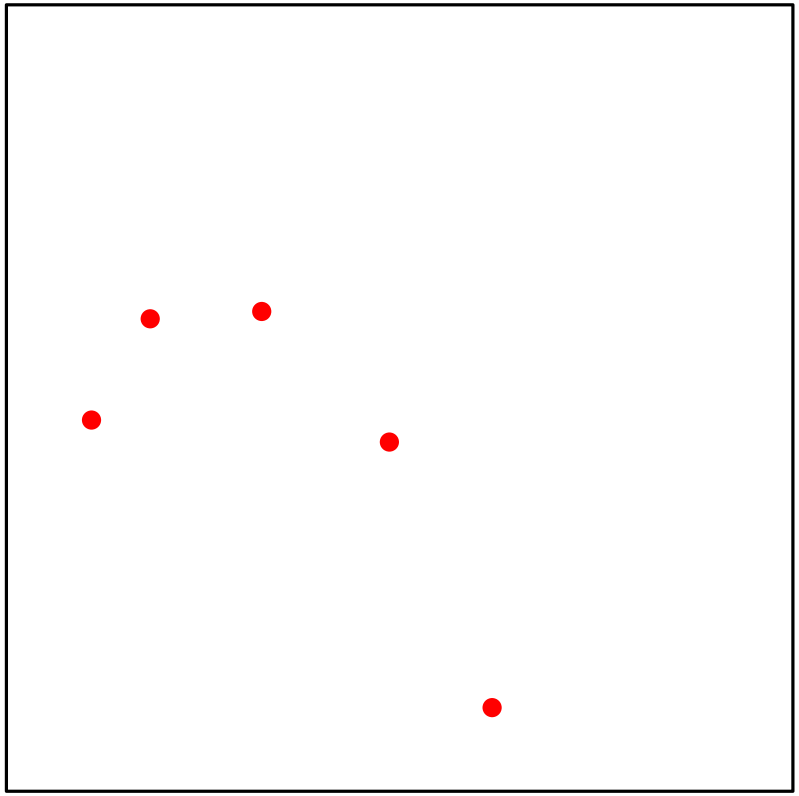
2+2 Dimension Reduction



Prune without losing information:
(CANONICAL SET)

Pick a color class

2+2 Dimension Reduction



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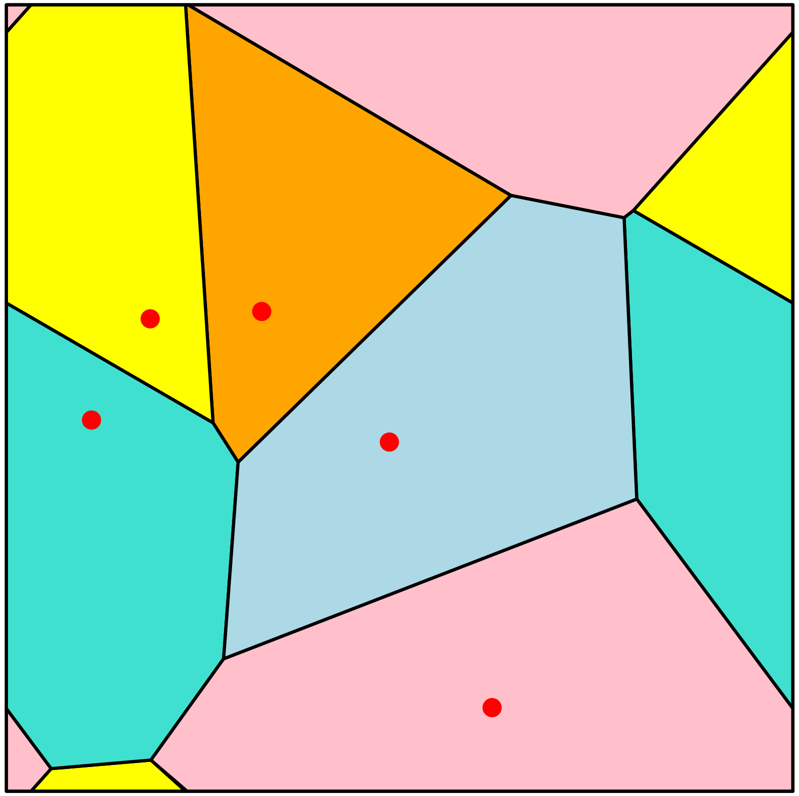
2+2 Dimension Reduction



Prune without losing information: (CANONICAL SET)

Pick a color class

Compute the Voronoi diagram

2+2 Dimension Reduction

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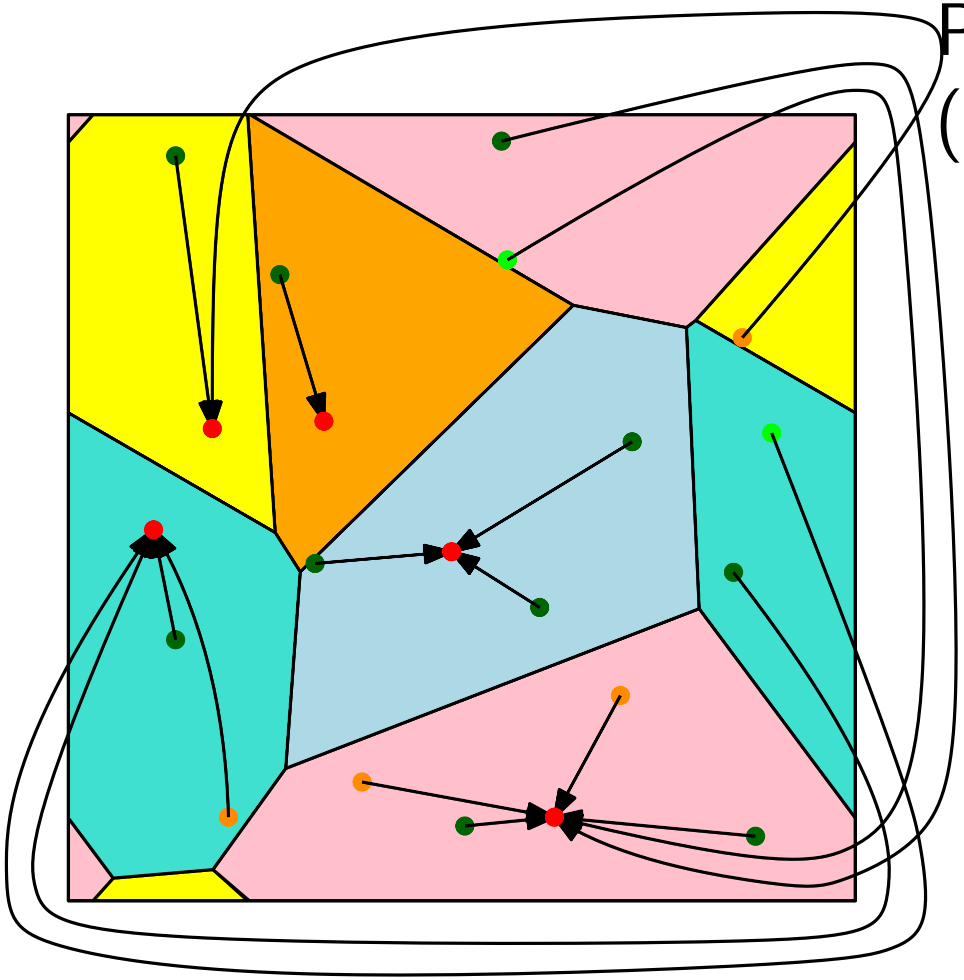


Prune without losing information:
(CANONICAL SET)

Pick a color class

Compute the Voronoi diagram

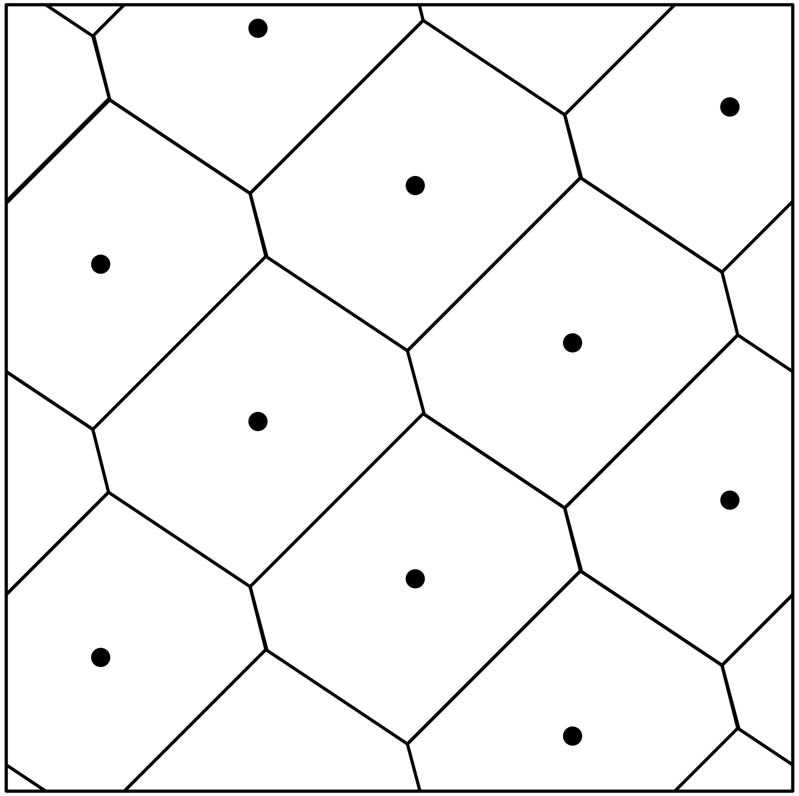
Assign other points to cells.

Refine the coloring, based on color and relative position of assigned points, shape of Voronoi cell.

Repeat.

After recoloring, the reduced set has THE SAME translational symmetries as the old set.

2+2 Dimension Reduction



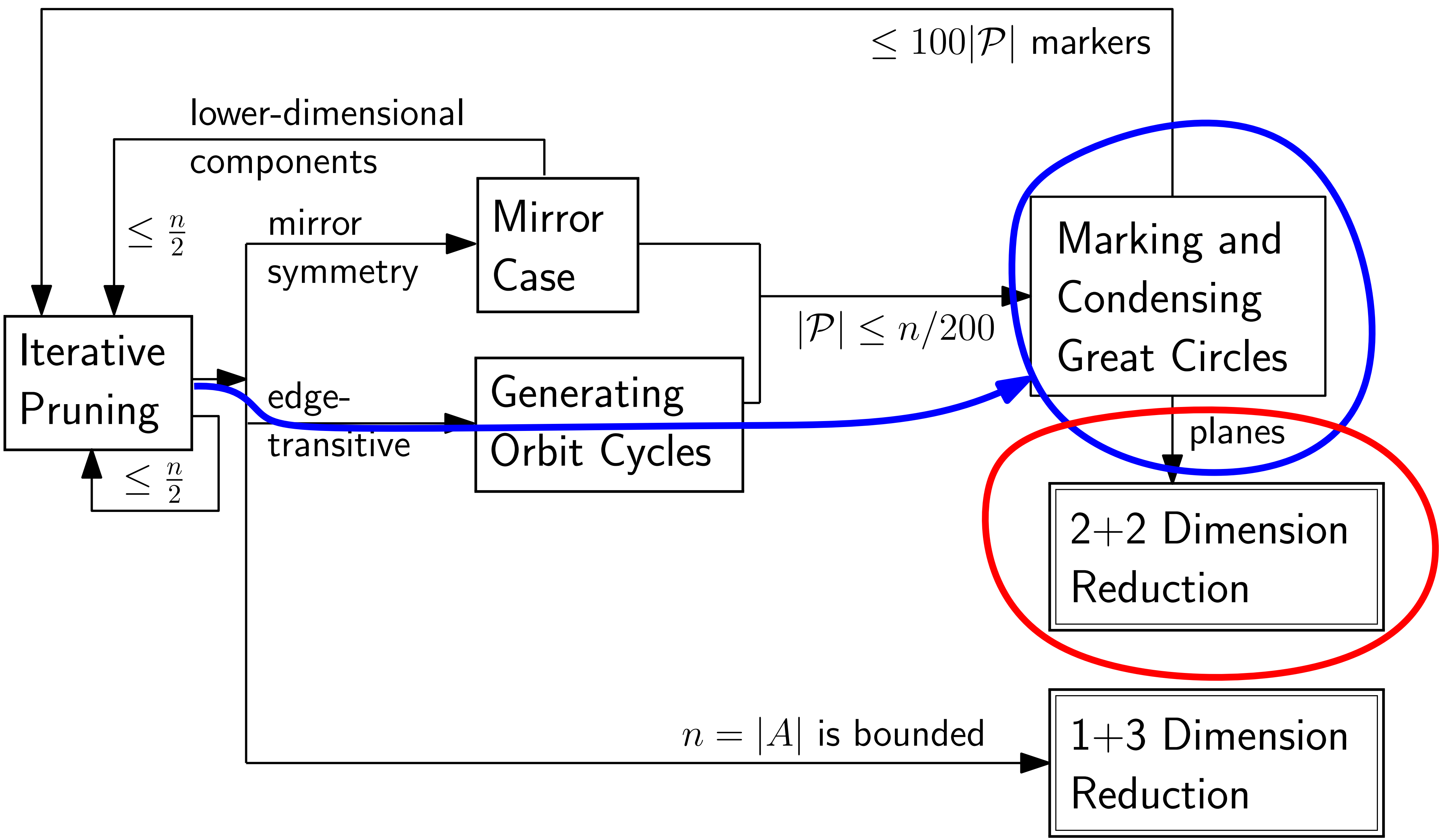
Termination:
All points have the same color and the same cell shape (a modular *lattice*)

ANY point is as good a representative as any other.

CANONICAL SET $c(A)$:
move (any) representative point to $(\varphi_1, \varphi_2) = (0, 0)$, or to $(x_1, 0, x_3, 0)$.

$$\exists T \text{ with } TP = P \text{ and } TA = B \iff c(A) = c(B)$$

Algorithm Overview



Algorithm Overview

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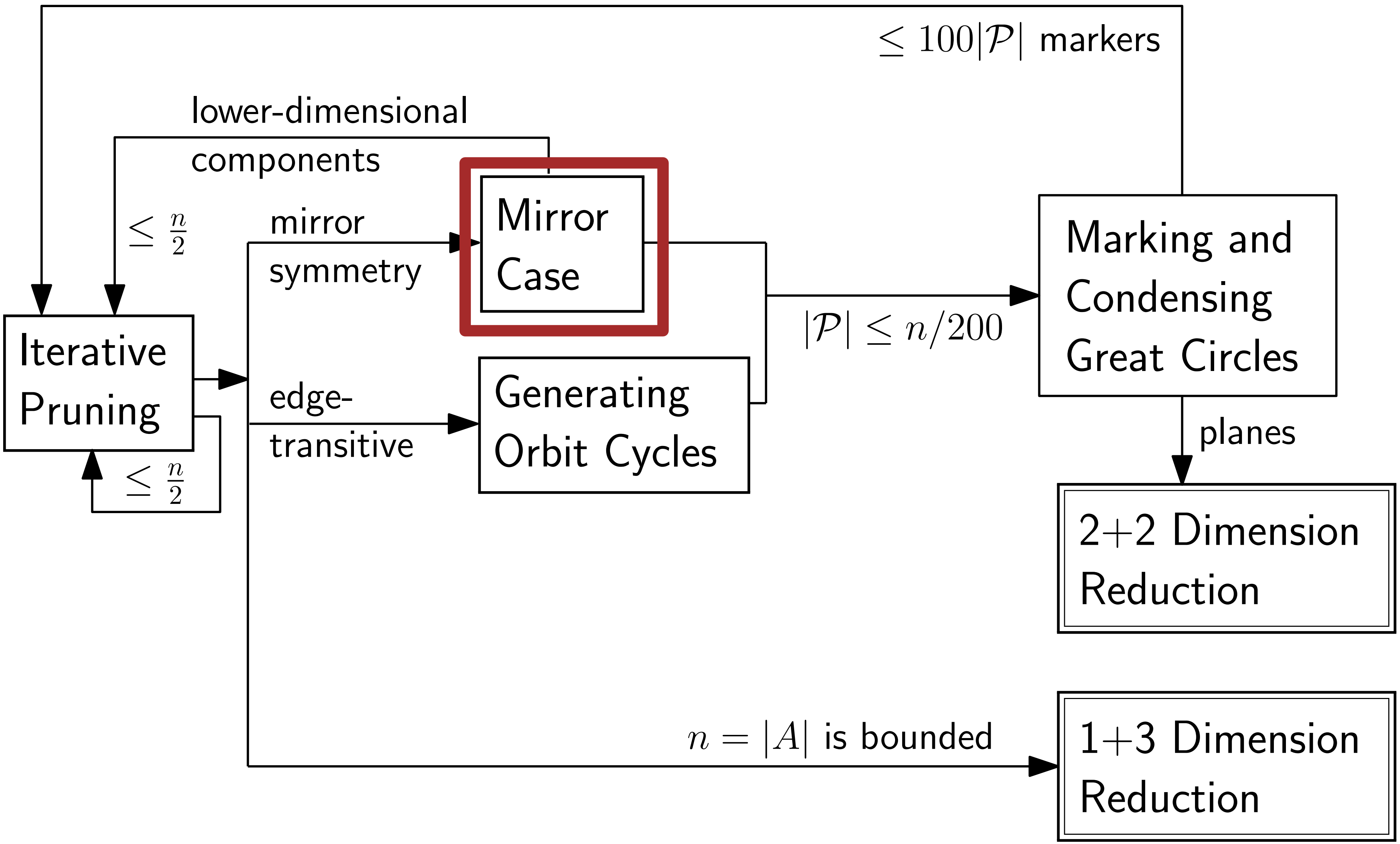
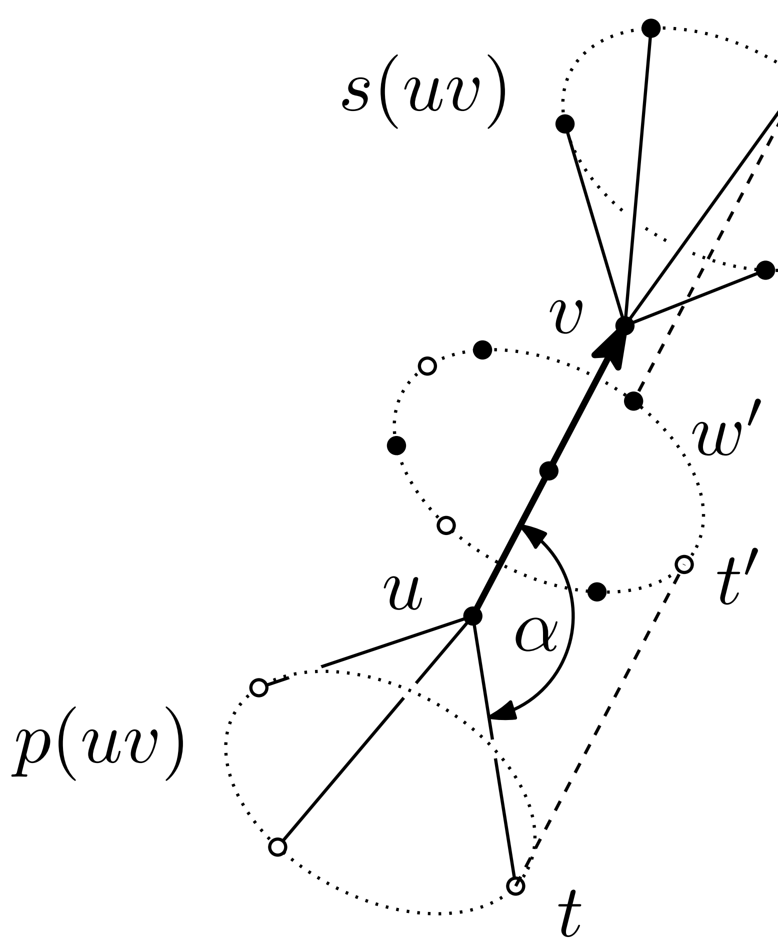
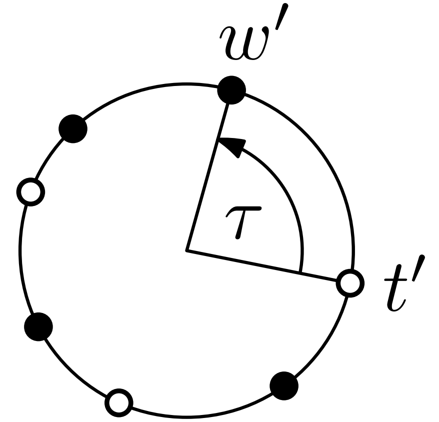


The Mirror Case

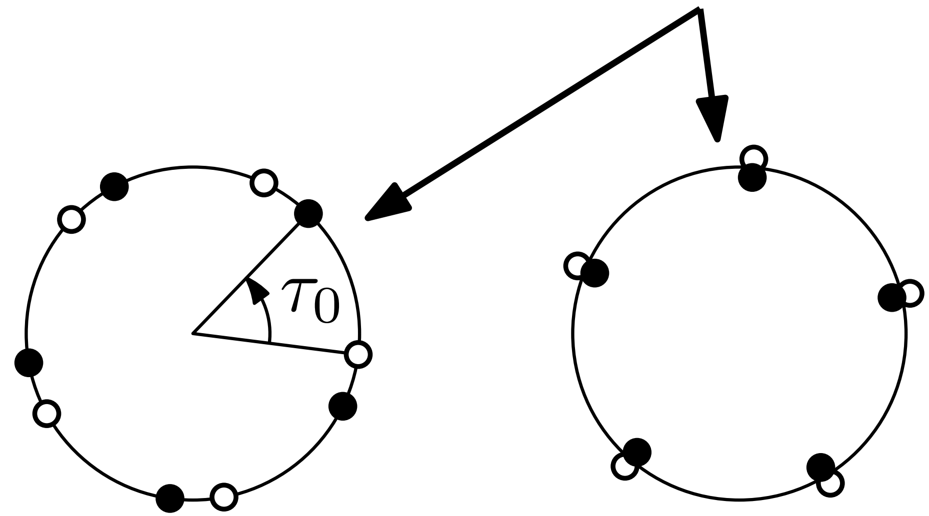
Pick some α. $\quad s(uv) := \{vw : vw \in E, \angle uvw = \alpha\}$





torsion angle τ

can PRUNE arcs from $s(u,v)$



The Mirror Case

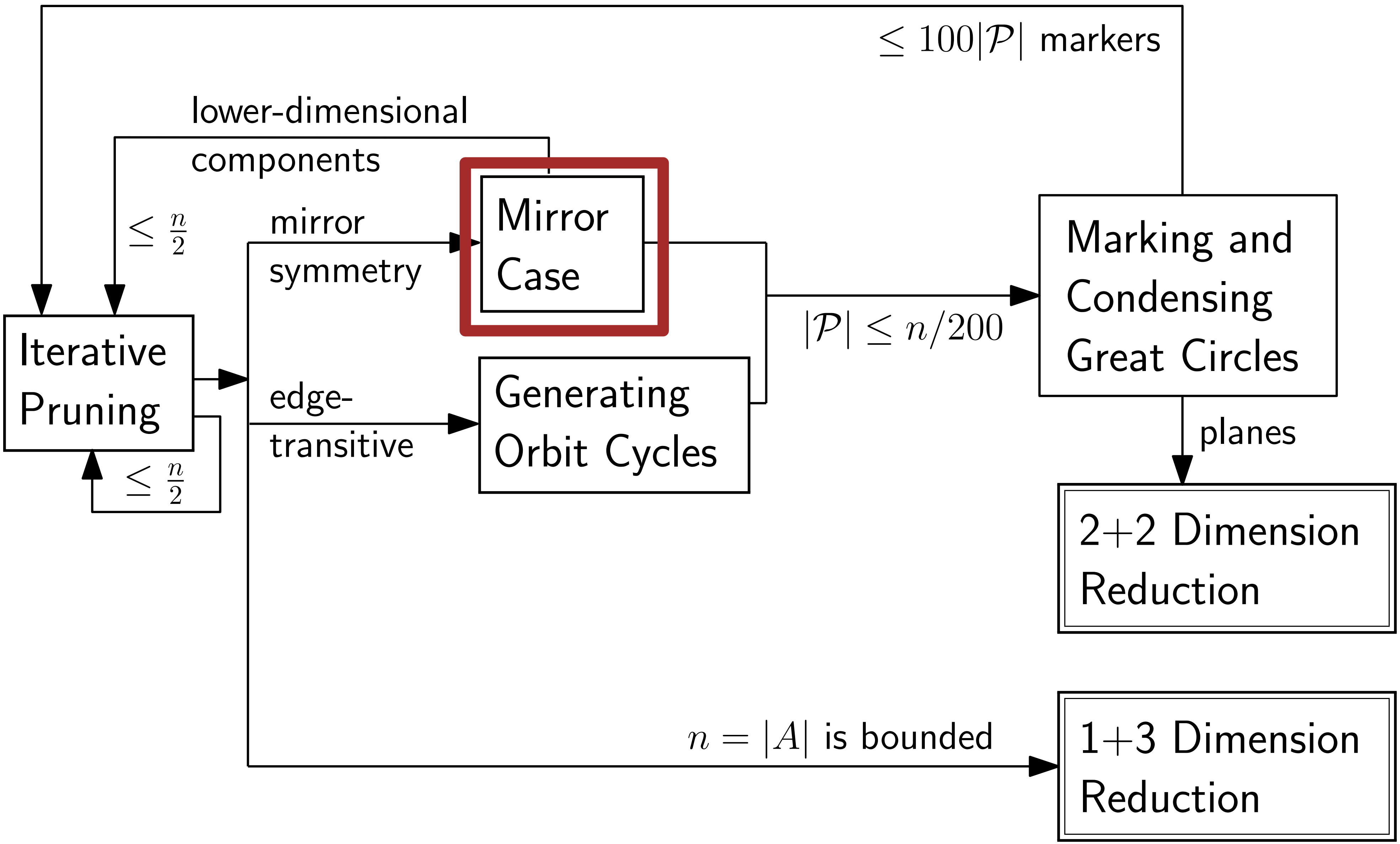
Every edge acts like a perfect mirror of the neighborhood.

$\rightarrow$ Every connnected component is the orbit of a point under a group generated by reflections.

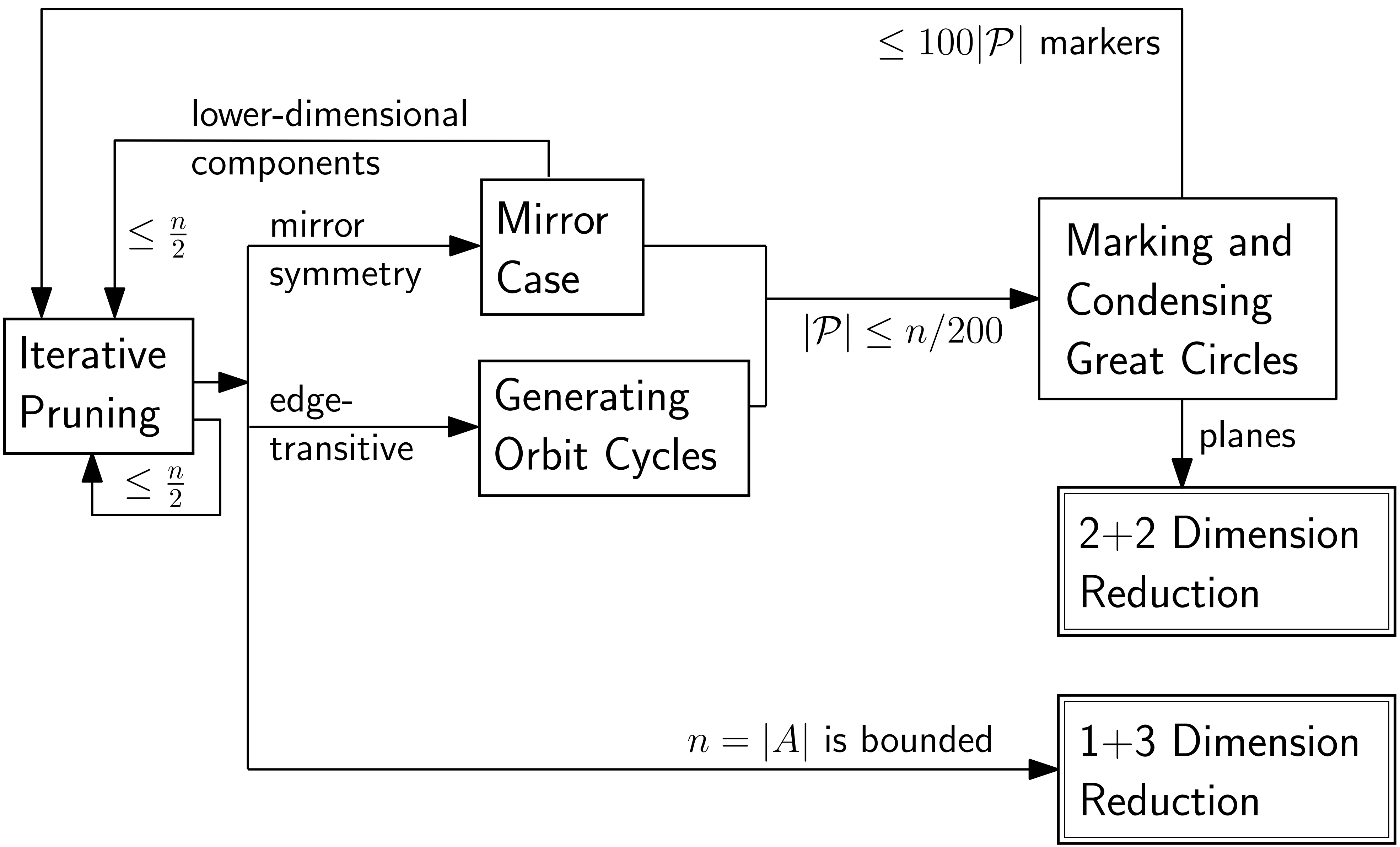
These groups have been classified. (Coxeter groups)

- "small" components
  $\rightarrow$ condensing
- Cartesian product of 2-dimensional groups (infinite family)
  $\rightarrow$ 2+2 dimension reduction
- "large" components (finite family)
  $\rightarrow |A| \le n_0$

Algorithm Overview



Algorithm Overview



Afterthoughts

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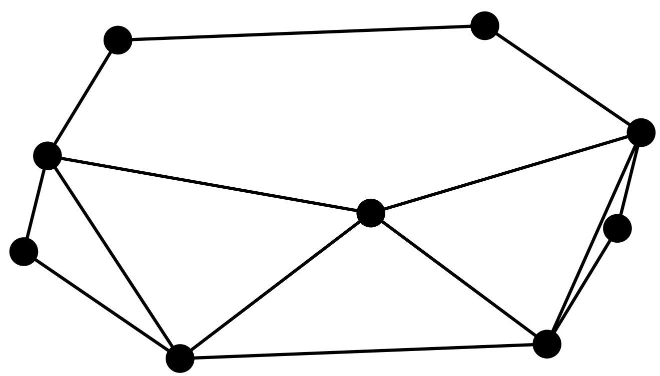
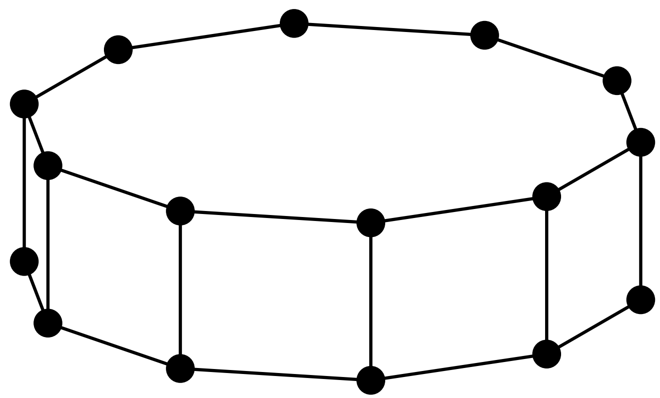
- 5 dimensions and higher
- terrible constants
- chimeras
- tolerances, $\leq \varepsilon$ versus $\geq 10\varepsilon$
- depth of construction ($\rightarrow$ degree of predicates)
- Plücker space
- point groups in 4 dimensions

Symmetry groups

COROLLARY. The symmetry group of a finite full-dimensional point set in 3-space ($=$ a discrete subgroup of $O(3)$) is

- the symmetry group of a Platonic solid,
- the symmetry group of a regular prism,
- or a subgroup of such a group.





The *point groups* (discrete subgroups of $O(3)$) are classified (Hessel's Theorem).

[F. Hessel 1830, M. L. Frankenheim 1826]

Point groups in higher dimensions

Bold and naive CONJECTURE:

¿The symmetry group of a finite full-dimensional point set in d-space ($=$ a discrete subgroup of $O(d)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - * a hypercube (or its dual, the crosspolytope)
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

Point groups in higher dimensions

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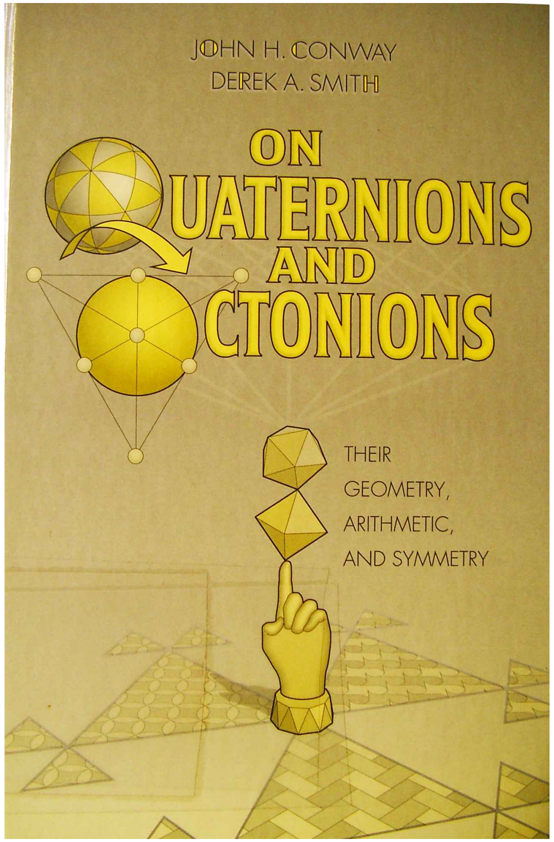
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

Counterexample (Paco Santos, by divisibility). The symmetry groups of the root systems E_6, E_7, E_8 in 6, 7, 8 dimensions.

The four-dimensional point groups

- [W. Threlfall and H. Seifert, Math. Annalen, 1931, 1933] enumerated discrete subgroups of $SO(4)$ (determinant $+1$)
- [J. Conway and D. Smith 2003] complete enumeration of point groups



4d-rotation $T \leftrightarrow$ pair (R, S) of 3d-rotations.
(for example, via quaternions)

Goursat's Lemma: [É. Goursat 1890]
Pairs of 3d point groups
$+$ additional information
$\rightarrow$ 4d point groups

- The groups generated by reflections (Coxeter groups) have been enumerated up to 8 dimensions.

[Norman Johnson, unpublished book manuscript]

The four-dimensional point groups

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umerate the groups. (see section 3

Group	Generators
$\pm[I\times O]$	$[i_I,1],[\omega,1],[1,i_O],[1,\omega];$
$\pm[I\times T]$	$[i_I,1],[\omega,1],[1,i],[1,\omega];$
$\pm[I\times D_{2n}]$	$[i_I,1],[\omega,1],[1,e_n],[1,j];$
$\pm[I\times C_n]$	$[i_I,1],[\omega,1],[1,e_n];$
$\pm[O\times T]$	$[i_O,1],[\omega,1],[1,i],[1,\omega];$
$\pm[O\times D_{2n}]$	$[i_O,1],[\omega,1],[1,e_n],[1,j];$
$\pm\frac{1}{2}[O\times D_{2n}]$	$[i,1],[\omega,1],[1,e_n];[i_O,j]$
$\pm\frac{1}{2}[O\times \overline{D}_{4n}]$	$[i,1],[\omega,1],[1,e_n],[1,j];[i_O,e_{2n}]$
$\pm\frac{1}{6}[O\times D_{6n}]$	$[i,1],[j,1],[1,e_n];[i_O,j],[\omega,e_{3n}]$
$\pm[O\times C_n]$	$[i_O,1],[\omega,1],[1,e_n];$
$\pm\frac{1}{2}[O\times C_{2n}]$	$[i,1],[\omega,1],[1,e_n];[i_O,e_{2n}]$
$\pm[T\times D_{2n}]$	$[i,1],[\omega,1],[1,e_n],[1,j];$
$\pm[T\times C_n]$	$[i,1],[\omega,1],[1,e_n];$
$\pm\frac{1}{3}[T\times C_{3n}]$	$[i,1],[1,e_n];[\omega,e_{3n}]$
$\pm\frac{1}{2}[D_{2m}\times \overline{D}_{4n}]$	$[e_m,1],[1,e_n],[1,j];[j,e_{2n}]$
$\pm[D_{2m}\times C_n]$	$[e_m,1],[j,1],[1,e_n];$
$\pm\frac{1}{2}[D_{2m}\times C_{2n}]$	$[e_m,1],[1,e_n];[j,e_{2n}]$
$+\frac{1}{2}[D_{2m}\times C_{2n}]$	$-\ ,\ -\ ;\ +$
$\pm\frac{1}{2}[\overline{D}_{4m}\times C_{2n}]$	$[e_m,1],[j,1],[1,e_n];[e_{2m},e_{2n}]$

Table 4.1. Chiral groups, I. These are most of the "metachiral" groups—s 4.6—some others appear in the last few lines of Table 4.2.

Table 4.1. The *chiral* groups (groups of orientation-preserving orthogonal transformations)

[Conway and Smith 2003]

both m and n must be odd.

The four-dimensional point groups

Table 4.2.
The *chiral* groups
(continued)

Group	Generators	Coxeter Name
$\pm[I\times I]$	$[i_I,1],[\omega,1],[1,i_I],[1,\omega];$	$[3,3,5]^+$
$\pm\frac{1}{60}[I\times I]$	$;[\omega,\omega],[i_I,i_I]$	$2.[3,5]^+$
$+\frac{1}{60}[I\times I]$	$;\ +\ ,\ +$	$[3,5]^+$
$\pm\frac{1}{60}[I\times\overline{I}]$	$;[\omega,\omega],[i_I,i'_I]$	$2.[3,3,3]^+$
$+\frac{1}{60}[I\times\overline{I}]$	$;\ +\ ,\ +$	$[3,3,3]^+$
$\pm[O\times O]$	$[i_O,1],[\omega,1],[1,i_O],[1,\omega];$	$[3,4,3]^+:2$
$\pm\frac{1}{2}[O\times O]$	$[i,1],[\omega,1],[1,i],[1,\omega];[i_O,i_O]$	$[3,4,3]^+$
$\pm\frac{1}{6}[O\times O]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\omega],[i_O,i_O]$	$[3,3,4]^+$
$\pm\frac{1}{24}[O\times O]$	$;[\omega,\omega],[i_O,i_O]$	$2.[3,4]^+$
$+\frac{1}{24}[O\times O]$	$;\ +\ ,\ +$	$[3,4]^+$
$+\frac{1}{24}[O\times\overline{O}]$	$;\ +\ ,\ -$	$[2,3,3]^+$
$\pm[T\times T]$	$[i,1],[\omega,1],[1,i],[1,\omega];$	$[^+3,4,3^+]$
$\pm\frac{1}{3}[T\times T]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\omega]$	$[^+3,3,4^+]$
$\cong\pm\frac{1}{3}[T\times\overline{T}]$	$[i,1],[j,1],[1,i],[1,j];[\omega,\overline{\omega}]$	”
$\pm\frac{1}{12}[T\times T]$	$;[\omega,\omega],[i,i]$	$2.[3,3]^+$
$\cong\pm\frac{1}{12}[T\times\overline{T}]$	$;[\omega,\overline{\omega}],[i,-i]$	”
$+\frac{1}{12}[T\times T]$	$;\ +\ ,\ +$	$[3,3]^+$
$\cong+\frac{1}{12}[T\times\overline{T}]$	$;\ +\ ,\ +$	”
$\pm[D_{2m}\times D_{2n}]$	$[e_m,1],[j,1],[1,e_n],[1,j];$	
$\pm\frac{1}{2}[\overline{D}_{4m}\times\overline{D}_{4n}]$	$[e_m,1],[j,1],[1,e_n],[1,j];[e_{2m},e_{2n}]$	
$\pm\frac{1}{4}[D_{4m}\times\overline{D}_{4n}]$	$[e_m,1],[1,e_n];[e_{2m},j],[j,e_{2n}]$	<u>Conditions</u>
$+\frac{1}{4}[D_{4m}\times\overline{D}_{4n}]$	$-\ ,\ -\ ;\ +\ ,\ +$	m,n odd
$\pm\frac{1}{2f}[D_{2mf}\times D^{(s)}_{2nf}]$	$[e_m,1],[1,e_n];[e_{mf},e^s_{nf}],[j,j]$	$(s,f)=1$
$+\frac{1}{2f}[D_{2mf}\times D^{(s)}_{2nf}]$	$-\ ,\ -\ ;\ +\ ,\ +$	m,n odd, $(s,2f)=1$
$\pm\frac{1}{f}[C_{mf}\times C^{(s)}_{nf}]$	$[e_m,1],[1,e_n];[e_{mf},e^s_{nf}]$	$(s,f)=1$
$+\frac{1}{f}[C_{mf}\times C^{(s)}_{nf}]$	$-\ ,\ -\ ;\ +$	m,n odd, $(s,2f)=1$

Table 4.2. Chiral groups, II. These groups are mostly “orthochiral,” with a few “parachiral”

The four-dimensional point groups

Freie Universität Berlin

Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \overline{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I\times \overline{I}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O\times O]\cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O\times O]\cdot 2$ or $\overline{2}$	$*$ or $*[1,i_O]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
$\pm\frac{1}{6}[O\times O]\cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O\times O]\cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O\times O]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O\times \overline{O}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T\times T]\cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T\times T]\cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T\times \overline{T}]\cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T\times T]\cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T\times \overline{T}]\cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T\times T]\cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T\times \overline{T}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n}\times D_{2n}]\cdot 2$	$*$	
$\pm\frac{1}{2}[\overline{D}_{4n}\times \overline{D}_{4n}]\cdot 2$ or $\overline{2}$	$*$ or $*[1,e_{2n}]$	
$\pm\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2$	$*$	<u>Conditions</u>
$+\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1, e^{\gamma(f,s+1)}_{2nf}]$	*in*
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1, e^{\gamma(f,s+1)}_{2nf}]$	*Appendix*

Table 4.3. Achiral groups.

Table 4.3.
The *achiral* groups

The four-dimensional point groups

Group	Extending element	Coxeter Name
$\pm[I\times I]\cdot 2$	$*$	$[3,3,5]$
$\pm\frac{1}{60}[I\times I]\cdot 2$	$*$	$2.[3,5]$
$+\frac{1}{60}[I\times I]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,5]$ or $[3,5]^\circ$
$\pm\frac{1}{60}[I\times \overline{I}]\cdot 2$	$*$	$2.[3,3,3]$
$+\frac{1}{60}[I\times \overline{I}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3,3]^\circ$ or $[3,3,3]$
$\pm[O\times O]\cdot 2$	$*$	$[3,4,3]:2$
$\pm\frac{1}{2}[O\times O]\cdot 2$ or $\overline{2}$	$*$ or $*[1,i_O]$	$[3,4,3]$ or $[3,4,3]^{+}\cdot 2$
$\pm\frac{1}{6}[O\times O]\cdot 2$	$*$	$[3,3,4]$
$\pm\frac{1}{24}[O\times O]\cdot 2$	$*$	$2.[3,4]$
$+\frac{1}{24}[O\times O]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,4]$ or $[3,4]^\circ$
$+\frac{1}{24}[O\times \overline{O}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[2,3,3]^\circ$ or $[2,3,3]$
$\pm[T\times T]\cdot 2$	$*$	$[3,4,3^+]$
$\pm\frac{1}{3}[T\times T]\cdot 2$	$*$	$[^+3,3,4]$
$\pm\frac{1}{3}[T\times \overline{T}]\cdot 2$	$*$	$[3,3,4^+]$
$\pm\frac{1}{12}[T\times T]\cdot 2$	$*$	$2.[^+3,4]$
$\pm\frac{1}{12}[T\times \overline{T}]\cdot 2$	$*$	$2.[3,3]$
$+\frac{1}{12}[T\times T]\cdot 2_3$ or 2_1	$*$ or $-*$	$[^+3,4]$ or $[^+3,4]^\circ$
$+\frac{1}{12}[T\times \overline{T}]\cdot 2_3$ or 2_1	$*$ or $-*$	$[3,3]^\circ$ or $[3,3]$
$\pm[D_{2n}\times D_{2n}]\cdot 2$	$*$	
$\pm\frac{1}{2}[\overline{D}_{4n}\times \overline{D}_{4n}]\cdot 2$ or $\overline{2}$	$*$ or $*[1,e_{2n}]$	
$\pm\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2$	$*$	<u>Conditions</u>
$+\frac{1}{4}[D_{4n}\times \overline{D}_{4n}]\cdot 2_3$ or 2_1	$*$ or $-*$	n odd
$\pm\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*See*
$+\frac{1}{2f}[D_{2nf}\times D^{(s)}_{2nf}]\cdot 2^{(\alpha,\beta)}$ or $\overline{2}$	$*[e^{\alpha}_{2nf}, e^{\alpha s+\beta f}_{2nf}]$ or $*[1,j]$	*Text*
$\pm\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e^{\gamma(f,s+1)}_{2nf}]$	*in*
$+\frac{1}{f}[C_{nf}\times C^{(s)}_{nf}]\cdot 2^{(\gamma)}$	$*[1,e^{\gamma(f,s+1)}_{2nf}]$	*Appendix*

Table 4.3. Achiral groups.

Table 4.3.
The *achiral* groups

- Project: Visualize these groups: Schlegel diagram of a 4-polytope which has these symmetries.

Point groups in four dimensions

Bold and naive CONJECTURE:

¿The symmetry group of a finite full-dimensional point set in 4-space ($=$ a discrete subgroup of $O(4)$) is

- the symmetry group of a regular d-dimensional polytope:
  - a regular simplex
  - a regular n-gon in two dimensions
  - a dodecahedron (or its dual, the icosahedron) in 3 d.
  - a 24-cell, or a 120-cell (or its dual, the 600-cell) in 4 d.
- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

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- the symmetry group of the Cartesian product of lower-dimensional regular polytopes,
- or a subgroup of such a group? ?

Counterexample: $I \times C_n$ (group-theoretic product, but not geometric Cartesian product)

Icosahedron on $\mathbb{S}^2 \Rightarrow 12$ great circles with regular n-gons in $\mathbb{S}^3$

Point groups in four dimensions

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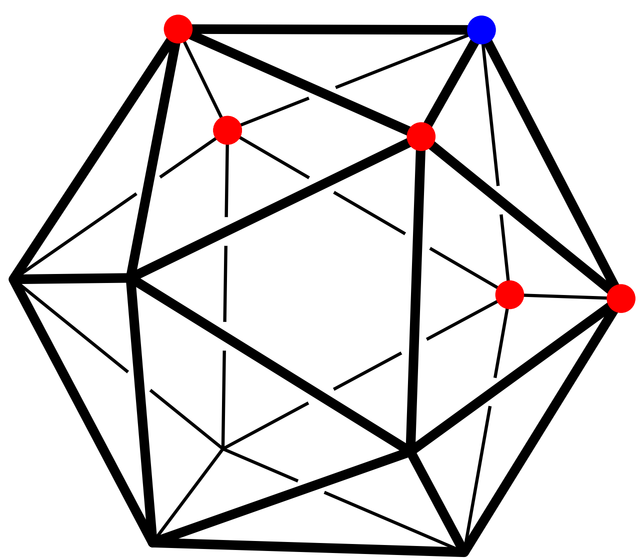
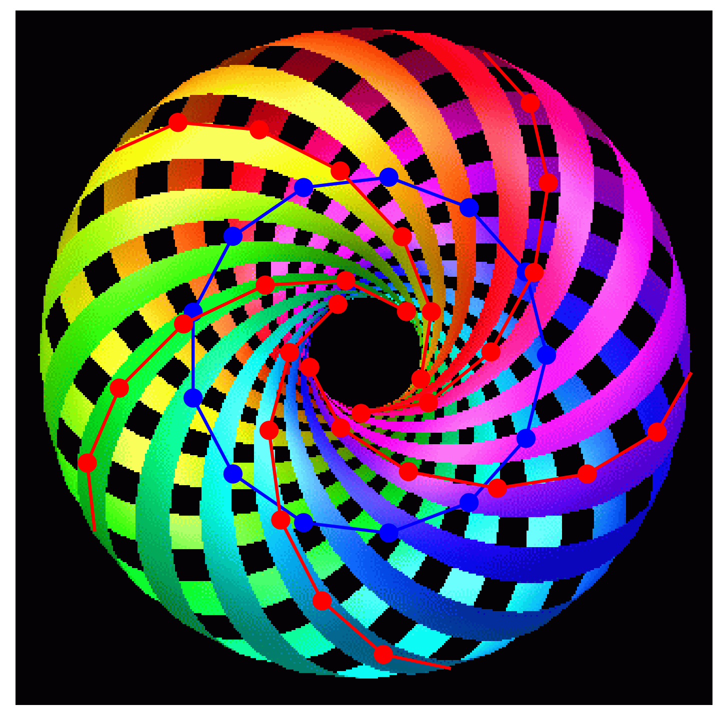


http://www.geom.uiuc.edu/~banchoff/script/b3d/hypertorus.html



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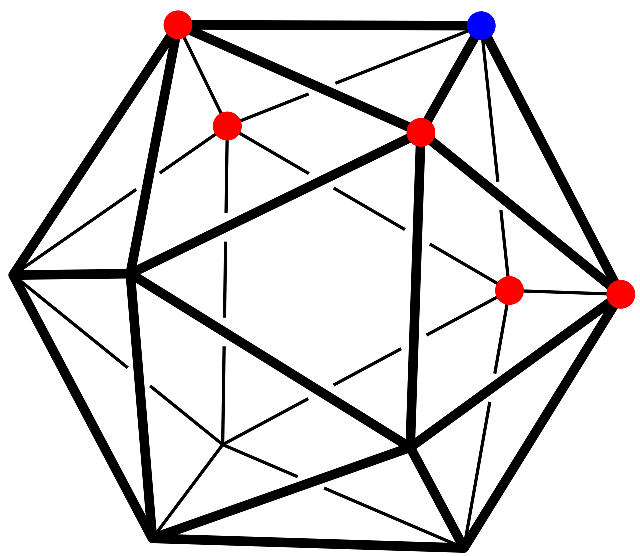

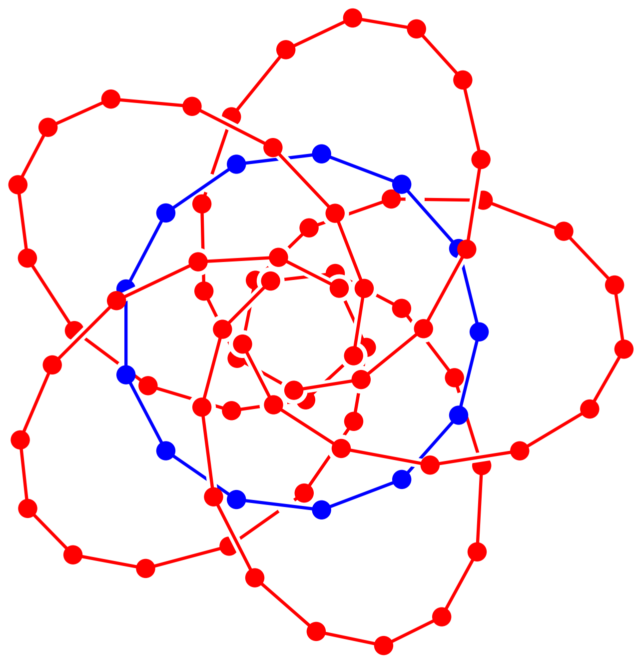

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