Approximation Algorithms for Geometric Proximity Problems: Part II: Approximating Convex Bodies

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Geometric Queries with Convex Bodies

Preprocess a geometric set to answer queries efficiently

Focus on convex bodies: closed, bounded convex sets:

- Convex hull of a set of n points in $\mathbb{R}^d$
- Intersection of a set of n closed halfspaces in $\mathbb{R}^d$ (within an enclosure)

Sample queries:

- Membership/Containment: $q \in P$?, $Q \subseteq P$?
- Intersection: $Q \cap P \neq \emptyset$?
- Extrema: Ray shooting, directional extrema (linear-programming queries)
- Distance: Directional width, longest parallel segment, separation distance

Assumptions

- Bodies reside in $\mathbb{R}^d$, where d is a constant. Bodies are full dimensional.
- Euclidean distance

Geometric Queries with Convex Bodies

Gold Standard for exact queries: $O(n)$ space and $O(\log n)$ query time

Good exact solutions exist in $\mathbb{R}^2$ and $\mathbb{R}^3$, but not in higher dimensions:

- The worst-case combinatorial complexity grows as $O(n^{\lfloor \frac{d}{2} \rfloor})$
- Point membership, halfspace emptiness, ray shooting:
  $\mathbb{R}^2, \mathbb{R}^3$: $O(n)$ space, $O(\log n)$ query time
  $\mathbb{R}^d$: $O(n)$ space, $\widetilde{O}(n^{1-\frac{2}{d}})$ query time [Matoušek 92]
- Intersection detection of preprocessed convex polytopes:
  $\mathbb{R}^2$: $O(n)$ space, $O(\log n)$ query time [Dobkin and Kirkpatrick 83]
  $\mathbb{R}^3$: $O(n)$ space, $O(\log^2 n)$ query time [Dobkin and Kirkpatrick 90]
  $O(n)$ space, $O(\log n)$ query time [Barba and Langerman 15]
  $\mathbb{R}^d$: $O(\log n)$ query time but space $O(N^{\lfloor \frac{d}{2} \rfloor})$
  where $N =$ total combinatorial complexity [Barba and Langerman 15]

Approximating Convex Bodies

Given a convex body K, and $\varepsilon > 0$:

- Inner ε-approximation: Any set $K_\varepsilon^- \subseteq K$ within Hausdorff distance $\varepsilon \cdot \mathrm{diam}(K)$ of K
- Outer ε-approximation: Any set $K_\varepsilon^+ \supseteq K$ within Hausdorff distance $\varepsilon \cdot \mathrm{diam}(K)$ of K

The representation often suggests which. Let P be point set, and $\mathcal{H}$ a set of halfspaces

- $K = \mathrm{conv}(P)$: Inner approximation $K_\varepsilon^- = \mathrm{conv}(P')$ for some $P' \subseteq P$
- $K = \bigcap(\mathcal{H})$: Outer approximation $K_\varepsilon^+ = \bigcap(\mathcal{H}')$, for some $\mathcal{H}' \subseteq \mathcal{H}$

- Many queries are equivalent through point-hyperplane duality
- Most results can be adapted to any combination inner/outer, point/halfspace

ε-Approximate Query

An answer is valid if it is consistent with any ε-approximation to K

It is often useful to have a directionally sensitive notion of approximation

- Given a vector v, define $\text{width}_v(K)$ to be the minimum distance between two hyperplanes orthogonal to v that enclose K.
- Width-sensitive (outer) ε-approximation: Any set $K_\varepsilon^+ \supseteq K$ such that $\text{width}_v(K^+) \leq (1+\varepsilon) \cdot \text{width}_v(K)$, for all v.

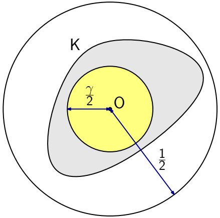
Width-Sensitive Approximation

An answer is valid if it is consistent with any width-sensitive ε-approximation to K

Preconditioning - Canonical Form

γ-Canonical Form

K is nested between two origin-centered balls of radii $\gamma/2$ and $1/2$



- Can convert to $\frac{1}{d}$-canonical form in $O(n)$ time — John's Theorem + fast minimum enclosing/enclosed ellipsoid [Chazelle and Matoušek 1996]
- Since diameter ≤ 1, can use absolute error of ε
- Uniform approximation to TK induces a width-sensitive approximation to K

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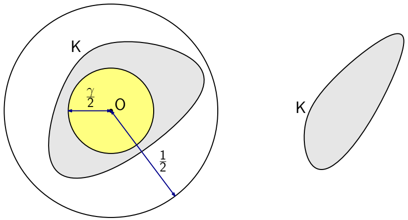

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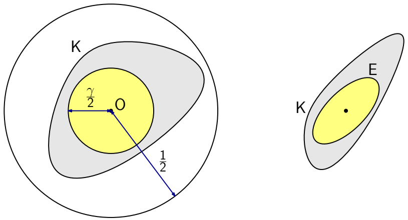

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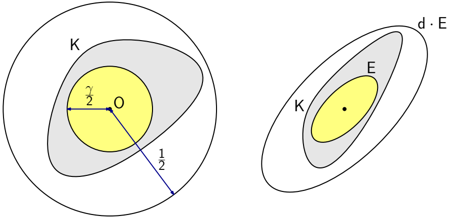

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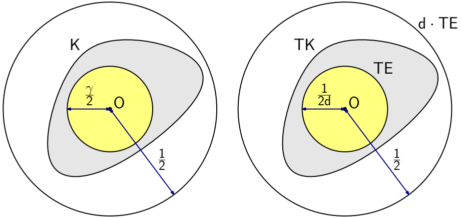

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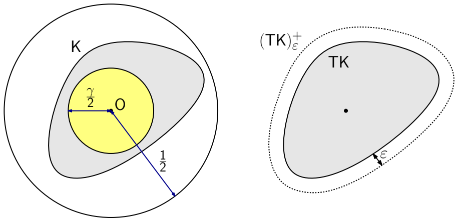

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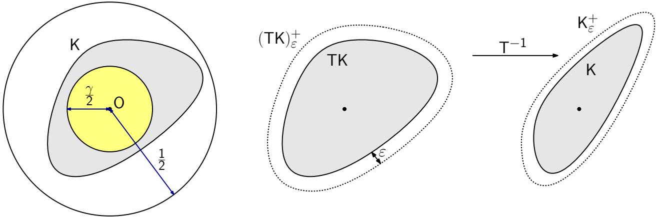

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First Stab - Quadtree-based Approximation

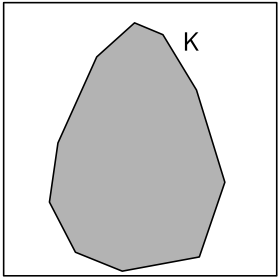
Query: ε-Approximate Polytope Membership (ε-APM)

- Preprocessing: Build a quadtree, subdividing each node that cannot be resolved as being inside or outside
- Stop at diameter ε
- Query: Find the leaf node containing q and return its label

Performance:

Query time: $O(\log \frac{1}{\varepsilon})$ — Quadtree descent

Storage: $O(1/\varepsilon^{d-1})$ — No. of leaves $\longleftarrow$ independent of n



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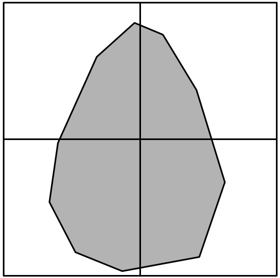

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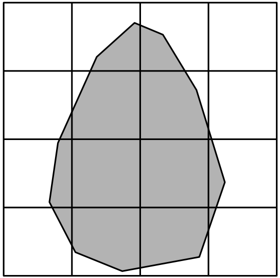

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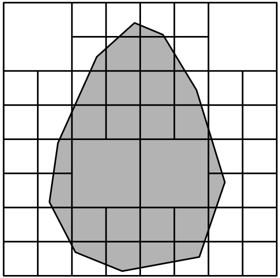

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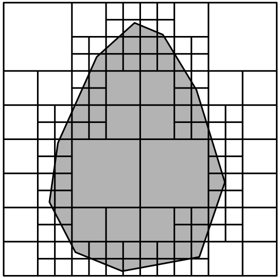

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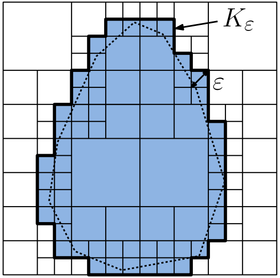

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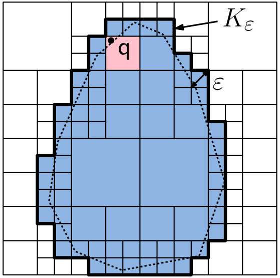

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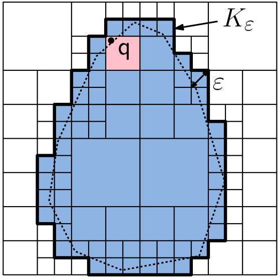

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Updating the Gold Standard

(Dudley 1974), (Bronshteyn and Ivanov 1976)

A convex body of unit diameter can be inner (outer) ε-approximated by a polytope with $O(1/\varepsilon^{\frac{d-1}{2}})$ vertices (facets)

- Transform K into canonical form
- $B \leftarrow$ ball of radius 2
- $N \leftarrow \sqrt{\varepsilon}$-net on B
- $N' \leftarrow$ closest points on K to each point of N
- $K_\varepsilon^- \leftarrow \text{conv}(N')$
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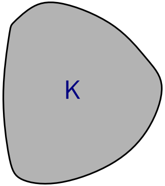

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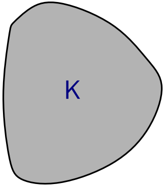

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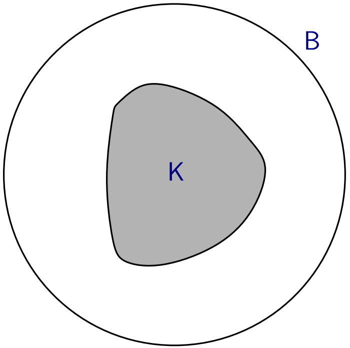

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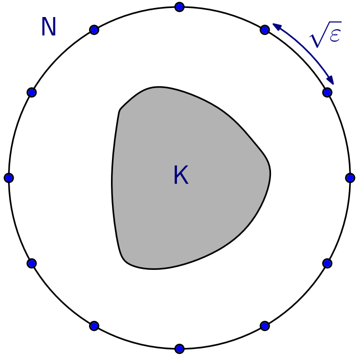

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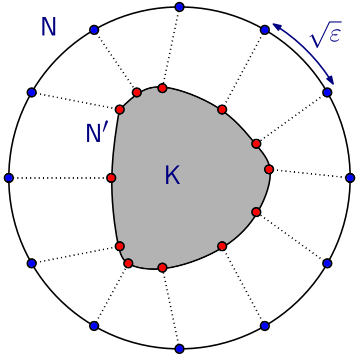

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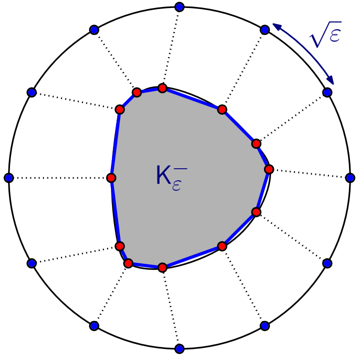

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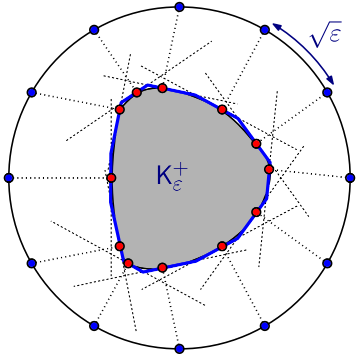
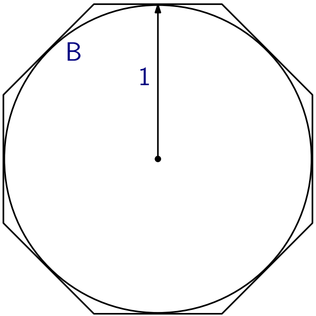


Updating the Gold Standard

Worst-Case Optimality

Any inner (outer) ε-approximation of a Euclidean unit ball requires $\Omega(1/\varepsilon^{\frac{d-1}{2}})$ vertices (facets)

- Consider a unit ball B and it ε-expansion B^+
- Any approximating facet cannot extend beyond B^+
- Extension beyond distance $\sqrt{3\varepsilon}$ goes too far
- Facet normals must be $O(\sqrt{\varepsilon})$-dense
- Need $\Omega((\frac{1}{\sqrt{\varepsilon}})^{d-1}) = \Omega(1/\varepsilon^{\frac{d-1}{2}})$ facets



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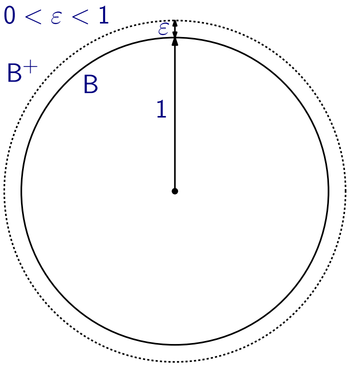

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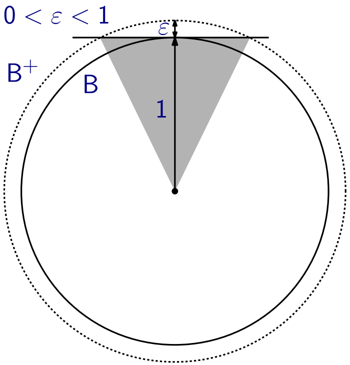

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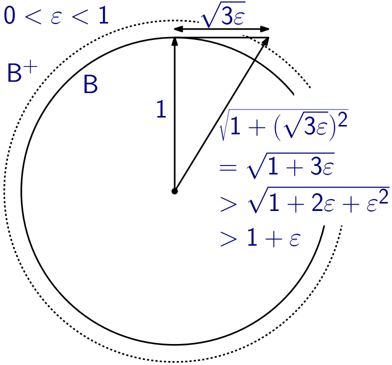

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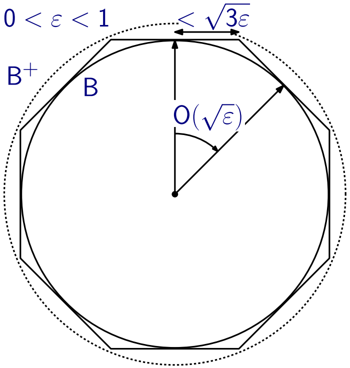

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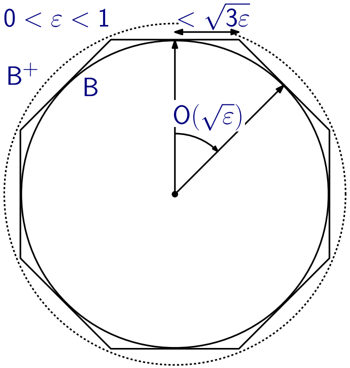


Achieving the Gold Standard

An approach to convex approximation that is space and time optimal:

ε-Approximate Polytope Membership (ε-APM):

Query time: $O(\log \frac{1}{\varepsilon})$

Storage: $O(1/\varepsilon^{\frac{d-1}{2}})$

Numerous applications:

- Best known space-time tradeoffs for ε-approximate Euclidean nearest neighbor searching (Arya et al. 2011, Arya, et al. 2012)
- Near worst-case optimal computation of ε-kernels and approximating the diameter of a convex body (Arya et al. 2017)
- Best known algorithms for approximating bichromatic closest pairs and bottleneck Euclidean minimum spanning trees (Arya et al. 2017)
- Best known algorithm for computing an ε-approximation to the width of a convex body (Arya et al. 2018)

Intuition - Hierarchy of Covers by Balls

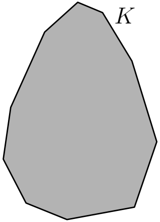
Hierarchy of covering balls:

- Preprocessing: Cover K by balls of diameter $1, \frac{1}{2}, \frac{1}{4}, \ldots, \varepsilon$
- DAG Structure: Each ball stores pointers to overlapping balls at next level
- Query: Find any ball at each level that contains q. If none $\Rightarrow$ "outside".
- Need only check $O(1)$ balls that overlap previous

Analysis:

Query: $O(\log \frac{1}{\varepsilon})$ (Log depth, constant degree)

Storage: $O(1/\varepsilon^d)$ (Number of leaves)



Intuition - Hierarchy of Covers by Balls

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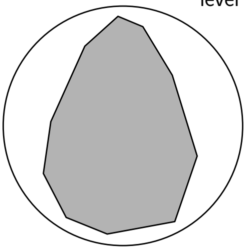

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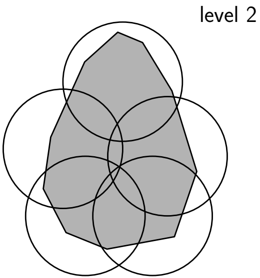

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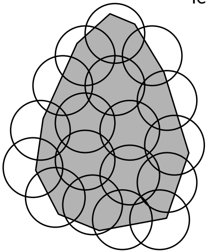

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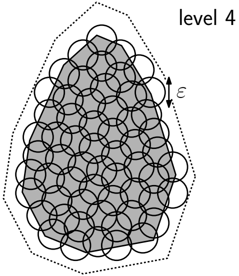

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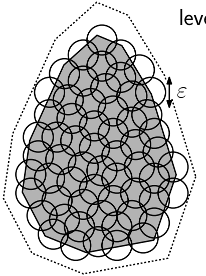

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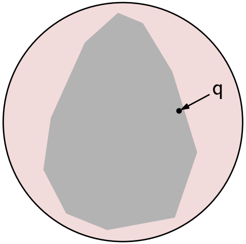

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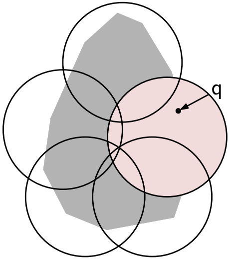

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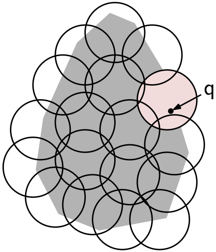

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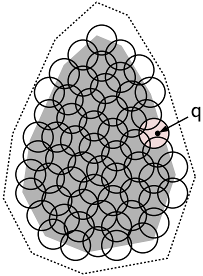

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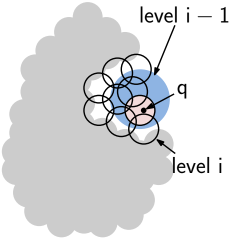

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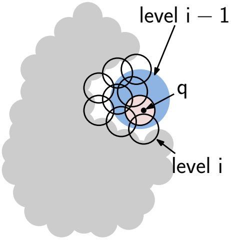

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Metric Space Perspective

The balls yield hierarchical covering of K by balls, but are their more economical solutions?

Yes! But first we need to recall a bit about ...

Metric Space: A set $\mathbb{X}$ and distance measure $f : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ that satisfies:

- Nonnegativity: $f(x, y) \geq 0$, and $f(x, y) = 0$ if and only if $x = y$
- Symmetry: $f(x, y) = f(y, x)$
- Triangle Inequality: $f(x, z) \leq f(x, y) + f(y, z)$.

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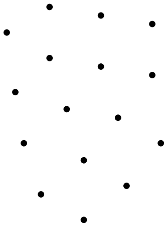
Delone Sets



A subset $X \subseteq \mathbb{X}$ is an:

- ε-packing: If the balls of radius $\varepsilon/2$ centered at every point of X are disjoint
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- $(\varepsilon_p, \varepsilon_c)$-Delone Set: If X is an ε_p-packing and an ε_c-covering
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We seek economical Delone sets for K, that fit within K's δ-expansion for $\delta = 1, \frac{1}{2}, \frac{1}{4}, \ldots, \varepsilon$

[a]Warning: Different from ε-nets for range spaces!

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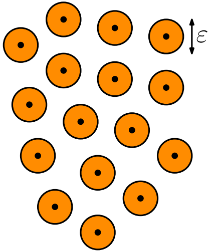

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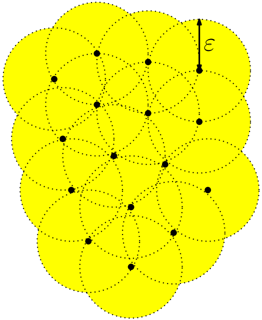

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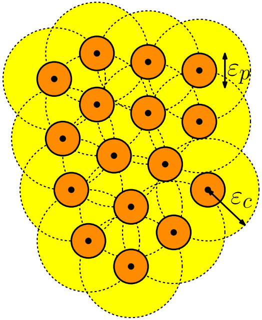

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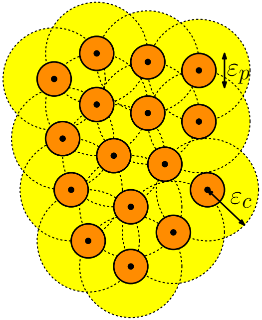

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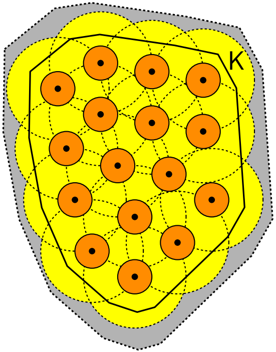

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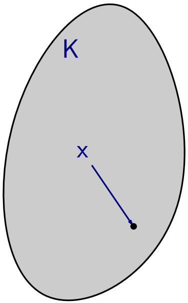
Macbeath Regions

- Euclidean balls are not sensitive to K's shape
- Want metric balls that conform locally



Macbeath Region [Macbeath (1952)]

Given convex body K, $x \in K$, and $\lambda > 0$:

- $M_K^\lambda(x) = x + \lambda((K - x) \cap (x - K))$
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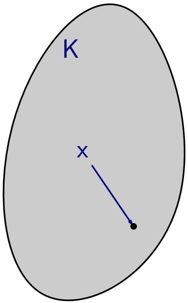

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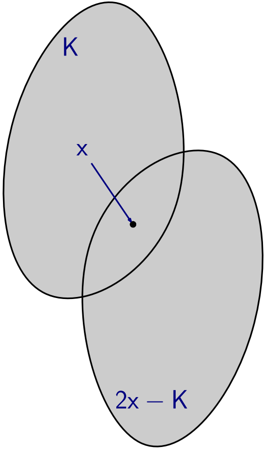

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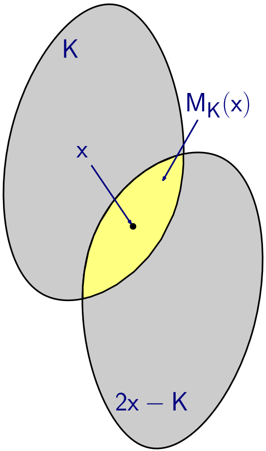

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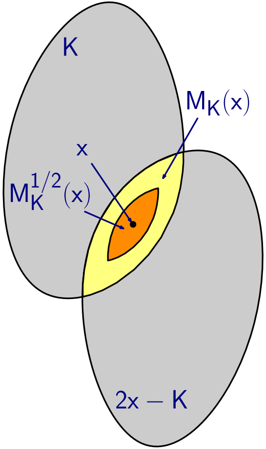

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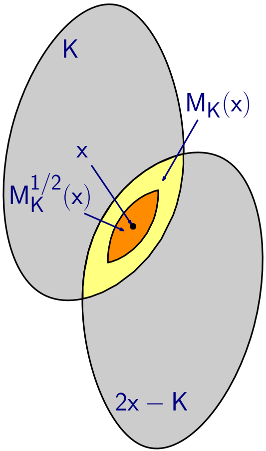


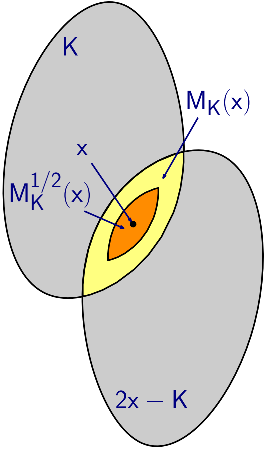
Properties of Macbeath Regions

Properties:

- Symmetry: $M^{\lambda}(x)$ is convex and centrally symmetric about x
- Expansion-Containment: [Ewald et al (1970)] If for $\lambda < 1$, $M^{\lambda}(x)$ and $M^{\lambda}(y)$ intersect, then

$$M^{\lambda}(y) \subseteq M^{c\lambda}(x), \quad \text{where } c = \frac{3+\lambda}{1-\lambda}.$$

Upshot: By expansion-containment, shrunken Macbeath regions behave "like" Euclidean balls, but they conform locally to K's boundary



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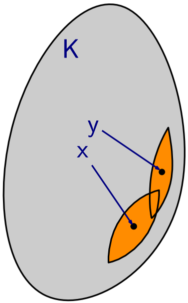

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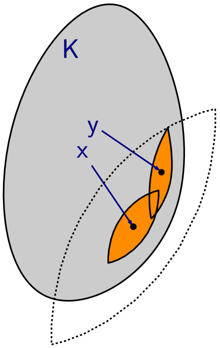

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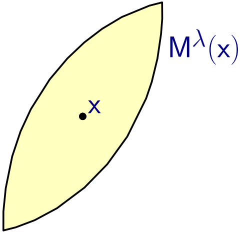
Macbeath Ellipsoids



Macbeath regions can be combinatorially complex.
Want a coarse approximation of low-complexity.

John ellipsoid [John (1948)]

Given a centrally symmetric convex body M in $\mathbb{R}^d$, there exist ellipsoids E_1, E_2 such that $E_1 \subseteq M \subseteq E_2$ and E_2 is a $\sqrt{d}$-scaling of E_1

Macbeath ellipsoid:

- $E(x)$: maximum volume ellipsoid in $M(x)$
- $E^{\lambda}(x)$: scaling by factor λ
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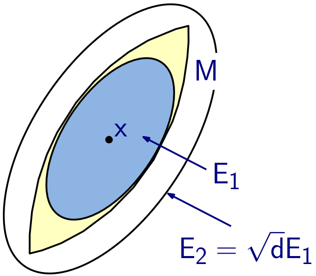

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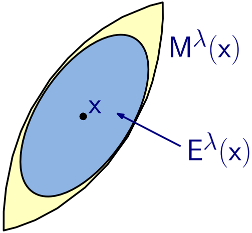

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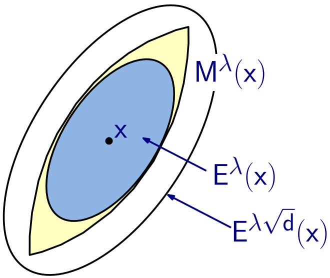
Macbeath Ellipsoids



Macbeath regions can be combinatorially complex.
Want a coarse approximation of low-complexity.

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Macbeath Ellipsoids and Delone Sets

Delone sets from Macbeath ellipsoids:

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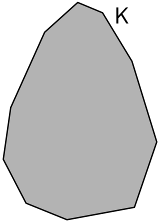
X_δ is essentially a $(\frac{1}{2}, 2\lambda_0)$-Delone set for K



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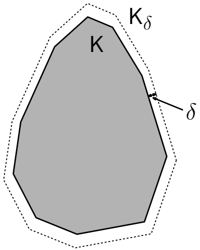

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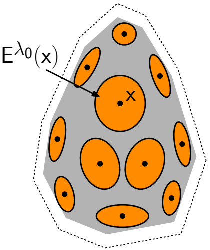

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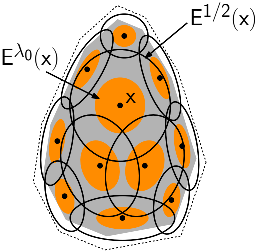

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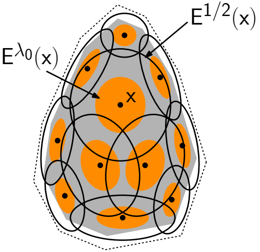

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Macbeath Regions and Hilbert Balls

For all $x \in K$ and $0 \leq \lambda < 1$:

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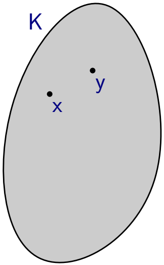

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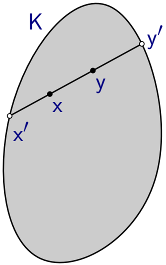

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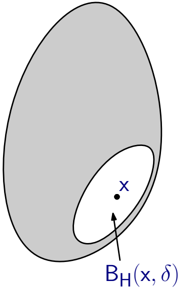

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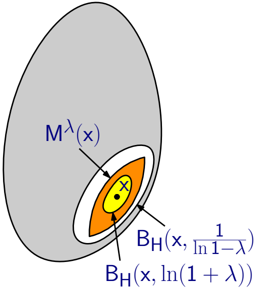

Approximate Polytope Membership (APM) Data Structure

Preprocessing:

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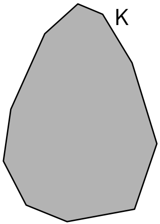

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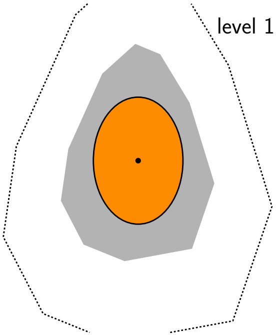

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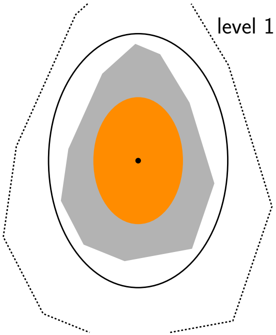

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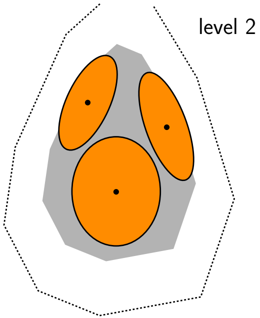

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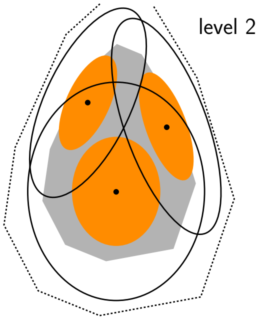

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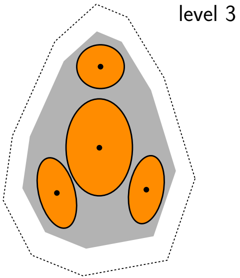

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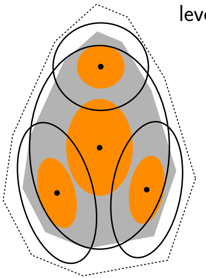

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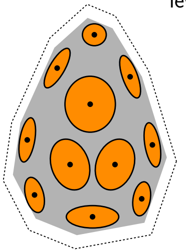

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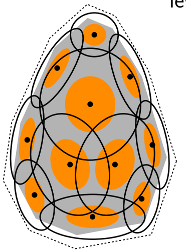

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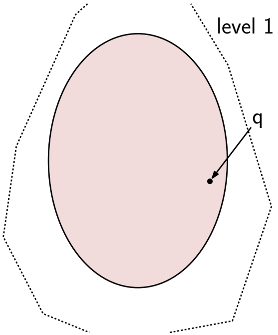

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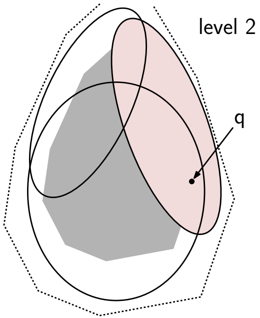

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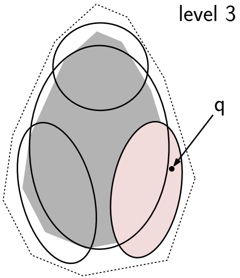

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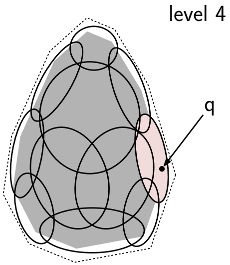

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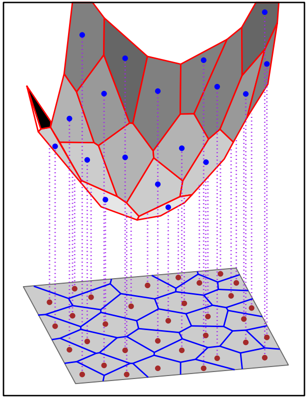
Lifting and Voronoi Diagrams

Lift the points of P to Ψ, take the upper envelope of the tangent hyperplanes, and project the skeleton back onto the plane. The result is the Voronoi diagram of P.



Intuition: Improvements to APM queries leads to improvements in ANN queries

Concluding Remarks

- Optimal solution to ε-APM queries:
  - Query time: $O(\log \frac{1}{\varepsilon})$
  - Storage: $O(1/\varepsilon^{(d-1)/2})$
  - Many applications!
- Still, several problems remain open:
  - Some polytopes can be approximated by much fewer than $O(1/\varepsilon^{(d-1)/2})$ elements. Instance-optimal approximation?
  - Generalizations to other nearest-neighbor problems: Bregman divergence? kth-nearest neighbor? Mahalanobis distance?

Concluding Remarks

- Optimal solution to ε-APM queries:
  - Query time: $O(\log \frac{1}{\varepsilon})$
  - Storage: $O(1/\varepsilon^{(d-1)/2})$
  - Many applications!

- Still, several problems remain open:
  - Some polytopes can be approximated by much fewer than $O(1/\varepsilon^{(d-1)/2})$ elements. Instance-optimal approximation?
  - Generalizations to other nearest-neighbor problems: Bregman divergence? kth-nearest neighbor? Mahalanobis distance?

Thank you for your attention!

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