Discrete Surface Ricci Flow

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Thanks for the invitation.
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Klein’s Program

Klein’s Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion
Conformal geometry lays down the theoretic foundation for
- Surface mapping
- Geometry classification
- Shape analysis

Applied in computer graphics, computer vision, geometric modeling, wireless sensor networking and medical imaging, and many other engineering, medical fields.
In pure mathematics, conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, differential geometry, partial differential equation.

In applied mathematics, computational complex function theory has been developed, which focuses on the conformal mapping between planar domains.

Recently, computational conformal geometry has been developed, which focuses on the conformal mapping between surfaces.
Conventional conformal geometric method can only handle the mappings among planar domains.

- Applied in thin plate deformation (biharmonic equation)
- Membrane vibration
- Electro-magnetic field design (Laplace equation)
- Fluid dynamics
- Aerospace design
Reasons for Booming

Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.
Reasons for Booming

Computational Power

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE’s much easier.
Fundamental Problems

1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmüller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics.
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.
Complete Tools

Computational Conformal Geometry Library

1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation
The theory, algorithms and sample code can be found in the following books.

You can find them in the book store.
Please email me gu@cmsa.fas.harvard.edu for updated code library on computational conformal geometry.
Conformal Mapping
Suppose $f : \mathbb{C} \to \mathbb{C}$ is invertible, both $f$ and $f^{-1}$ are holomorphic, then $f$ is a biholomorphic function.
The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.
Definition (Conformal Map)

Let $\phi : (S_1, g_1) \rightarrow (S_2, g_2)$ is a homeomorphism, $\phi$ is conformal if and only if

$$\phi^* g_2 = e^{2u} g_1.$$ 

Conformal Mapping preserves angles.
Conformal maps Properties
Map a circle field on the surface to a circle field on the plane.
Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.
Uniformization
Theorem (Poincaré Uniformization Theorem)

Let \((\Sigma, g)\) be a compact 2-dimensional Riemannian manifold. Then there is a metric \(\tilde{g} = e^{2\lambda}g\) conformal to \(g\) which has constant Gauss curvature.
Definition (Circle Domain)
A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem
Any domain $\Omega$ in $\hat{\mathbb{C}}$, whose boundary $\partial \Omega$ has at most countably many components, is conformally homeomorphic to a circle domain $\Omega^*$ in $\hat{\mathbb{C}}$. Moreover $\Omega^*$ is unique upto Möbius transformations, and every conformal automorphism of $\Omega^*$ is the restriction of a Möbius transformation.
Uniformization of Open Surfaces

- Spherical
- Euclidean
- Hyperbolic
Smooth Surface Ricci Flow
Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface $M$ with a Riemannian metric $g$, a local coordinate system $(u, v)$ is an isothermal coordinate system, if

$$g = e^{2\lambda(u, v)}(du^2 + dv^2).$$
Suppose $\bar{g} = e^{2\lambda} g$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta g \lambda = -\frac{1}{e^{2\lambda}} \Delta \lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}.$$
Conformal Metric Deformation

Definition

Suppose $M$ is a surface with a Riemannian metric,

$$ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} $$

Suppose $\lambda : \Sigma \rightarrow \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda} g$ is also a Riemannian metric on $\Sigma$ and called a conformal metric. $\lambda$ is called the conformal factor.

$$ g \rightarrow e^{2\lambda} g $$

Conformal metric deformation.

Angles are invariant measured by conformal metrics.
Yamabi Equation

Suppose $\bar{g} = e^{2\lambda}g$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(K - \Delta_g \lambda),$$

geodesic curvature on the boundary

$$\bar{k}_g = e^{-\lambda}(k_g - \partial_g, n \lambda).$$
Theorem (Poincaré Uniformization Theorem)

Let \((\Sigma, g)\) be a compact 2-dimensional Riemannian manifold. Then there is a metric \(\tilde{g} = e^{2\lambda} g\) conformal to \(g\) which has constant Gauss curvature.
Uniformization of Open Surfaces

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Discrete Surface Ricci Flow
Key ideas:

- Conformal metric deformation
  \[ g \rightarrow e^{2\lambda} g \]

- Ricci flow
  \[ \frac{d\lambda}{dt} = -K, \]

- Gaussian curvature \( K = -\Delta_g \lambda \), evolution equation
  \[ \frac{dK}{dt} = \Delta_g K + 2K^2 \]
  diffusion-reaction equation.
Definition (Normalized Hamilton’s Surface Ricci Flow)

A closed surface $S$ with a Riemannian metric $g$, the Ricci flow on it is defined as

$$
\frac{dg_{ij}}{dt} = \left( \frac{4\pi \chi(S)}{A(0)} - 2K \right) g_{ij}.
$$

where $\chi(S)$ is the Euler characteristic number of $S$, $A(0)$ is the initial total area.

The ricci flow preserves the total area during the flow, converge to a metric with constant Gaussian curvature $\frac{4\pi \chi(S)}{A(0)}$. 
Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.
Discrete Surface
Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 

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Discrete Surface Ricci Flow
- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 

(Images of generic surface models with triangular meshes)
- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 
Discrete Generalization

Concepts

1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation
A Discrete Metric on a triangular mesh is a function defined on the vertices, \( l : E = \{ \text{all edges} \} \rightarrow \mathbb{R}^+ \), satisfies triangular inequality.

A mesh has infinite metrics.
**Definition (Discrete Curvature)**

Discrete curvature: \( K : V = \{ \text{vertices} \} \rightarrow \mathbb{R}^1 \).

\[
K(v) = 2\pi - \sum \alpha_i, \ v \notin \partial M; \ K(v) = \pi - \sum \alpha_i, \ v \in \partial M
\]

**Theorem (Discrete Gauss-Bonnet theorem)**

\[
\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi \chi(M).
\]
Discrete Metrics Determines the Curvatures

\[ \cos l_i = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} \] (1)

\[ \cosh l_i = \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k} \] (2)

\[ 1 = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} \] (3)
Lemma (Derivative Cosine Law)

Suppose corner angles are the functions of edge lengths, then

\[
\frac{\partial \theta_i}{\partial l_i} = \frac{l_i}{A} \quad \frac{\partial \theta_j}{\partial l_j} = -\frac{\partial \theta_i}{\partial l_i} \cos \theta_k
\]

where \( A = l_j l_k \sin \theta_i \).
Discrete Conformal Structure
Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.
Discrete Conformal Metric Deformation vs CP

Discrete Surface Ricci Flow
Discrete Conformal Metric Deformation vs CP
We associate each vertex $v_i$ with a circle with radius $\gamma_i$. On edge $e_{ij}$, the two circles intersect at the angle of $\Phi_{ij}$. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos \Phi_{ij}$$

CP Metric $(T, \Gamma, \Phi)$, $T$ triangulation,

$$\Gamma = \{\gamma_i|\forall v_i\}, \Phi = \{\phi_{ij}|\forall e_{ij}\}$$
Conformal Equivalence

Two CP metrics \((T_1, \Gamma_1, \Phi_1)\) and \((T_2, \Gamma_2, \Phi_2)\) are conformal equivalent, if they satisfy the following conditions:

\[ T_1 = T_2 \text{ and } \Phi_1 = \Phi_2. \]
Definition (Power Circle)

The unit circle orthogonal to three circles at the vertices $(v_i, \gamma_i)$, $(v_j, \gamma_j)$ and $(v_k, \gamma_k)$ is called the power circle. The center is called the power center. The distance from the power center to three edges are denoted as $h_i, h_j, h_k$ respectively.
Derivative cosine law

Theorem (Symmetry)

\[
\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k} \\
\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i} \\
\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}
\]

Therefore the differential 1-form \(\omega = \theta_i du_i + \theta_j du_j + \theta_k du_k\) is closed.
Discrete Ricci Energy

Definition (Discrete Ricci Energy)

The functional associated with a CP metric on a triangle is

$$E(u) = \int_{(0,0,0)} (u_i, u_j, u_k) \theta_i(u) du_i + \theta_j(u) du_j + \theta_k(u) du_k.$$

Geometrical interpretation: the volume of a truncated hyperbolic hyper-ideal tetrahedron.
Definition (Tangential Circle Packing)

\[ l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j. \]
Generalized Circle Packing/Pattern

Definition (Inversive Distance Circle Packing)

\[ l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\eta_{ij}. \]

where \( \eta_{ij} > 1. \)
Generalized Circle Packing/Pattern

**Definition (Discrete Yamabe Flow)**

\[ l_{ij}^2 = 2 \gamma_i \gamma_j \eta_{ij}. \]

where \( \eta_{ij} > 0. \)
Definition (Voronoi Diagram)

Given $p_1, \ldots, p_k$ in $\mathbb{R}^n$, the Voronoi cell $W_i$ at $p_i$ is

$$W_i = \{ x | |x - p_i|^2 \leq |x - p_j|^2, \forall j \}.$$ 

The dual triangulation to the Voronoi diagram is called the Delaunay triangulation.
Power Distance

Given \( p_i \) associated with a sphere \((p_i, r_i)\), the power distance from \( q \in \mathbb{R}^n \) to \( p_i \) is

\[
pow(p_i, q) = |p_i - q|^2 - r_i^2.
\]
Definition (Power Diagram)

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$ and sphere radii $\gamma_1, \cdots, \gamma_k$, the power Voronoi cell $W_i$ at $p_i$ is

$$W_i = \{ x | \text{Pow}(x, p_i) \leq \text{Pow}(x, p_j), \forall j \}.$$  

The dual triangulation to Power diagram is called the Power Delaunay triangulation.
Definition (Voronoi Diagram)

Let \((S, V)\) be a punctured surface, \(V\) is the vertex set. \(d\) is a flat cone metric, where the cone singularities are at the vertices. The Voronoi diagram is a cell decomposition of the surface, Voronoi cell \(W_i\) at \(v_i\) is

\[
W_i = \{ p \in S | d(p, v_i) \leq d(p, v_j), \forall j \}.
\]

The dual triangulation to the voronoi diagram is called the Delaunay triangulation.
Definition (Power Diagram)

Let \((S, V)\) be a punctured surface, with a generalized circle packing metric. The Power diagram is a cell decomposition of the surface, a Power cell \(W_i\) at \(v_i\) is

\[ W_i = \{ p \in S | \text{Pow}(p, v_i) \leq \text{Pow}(p, v_j), \forall j \} \]

The dual triangulation to the power diagram is called the power Delaunay triangulation.
Definition (Edge Weight)

\((S, V, d)\), \(d\) a generalized CP metric. \(D\) the Power diagram, \(T\) the Power Delaunay triangulation. \(\forall e \in D\), the dual edge \(\bar{e} \in T\), the weight

\[
w(e) = \frac{|e|}{|\bar{e}|}.
\]
Discrete Surface Ricci Flow
Discrete Conformal Factor

Conformal Factor

Defined on each vertex $u : V \to \mathbb{R}$,

$$u_i = \begin{cases} 
\log \gamma_i & \mathbb{R}^2 \\
\log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\
\log \tan \frac{\gamma_i}{2} & \mathbb{S}^2
\end{cases}$$
Definition (Discrete Surface Ricci Flow with Surgery)

Suppose \((S, V, d)\) is a triangle mesh with a generalized CP metric, the discrete surface Ricci flow is given by

\[
\frac{du_i}{dt} = \bar{K}_i - K_i,
\]

where \(\bar{K}_i\) is the target curvature. Furthermore, during the flow, the Triangulation preserves to be Power Delaunay.

Theorem (Exponential Convergence)

The flow converges to the target curvature \(K_i(\infty) = \bar{K}_i\).
Furthermore, there exists \(c_1, c_2 > 0\), such that

\[
|K_i(t) - K_i(\infty)| < c_1 e^{-c_2 t}, |u_i(t) - u_i(\infty)| < c_1 e^{-c_2 t},
\]
Discrete Conformal Metric Deformation

Properties

- Symmetry

\[ \frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i} = -w_{ij} \]

- Discrete Laplace Equation

\[ dK_i = \sum_{[v_i, v_j] \in E} w_{ij}(du_i - du_j) \]

namely

\[ dK = \Delta du, \]
Discrete Laplace-Beltrami operator

Definition (Laplace-Beltrami operator)

\( \Delta \) is the discrete Laplace-Beltrami operator, \( \Delta = (d_{ij}) \), where

\[
d_{ij} = \begin{cases} 
\sum_k w_{ik} & i = j \\
-w_{ij} & i \neq j, [v_i, v_j] \in E \\
0 & \text{otherwise}
\end{cases}
\]

Lemma

Given \((S, V, d)\) with generalized CP metric, if \(T\) is the Power Delaunay triangulation, then \(\Delta\) is positive definite on the linear space \(\sum_i u_i = 0\).

Because \(\Delta\) is diagonal dominant.

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Discrete Surface Ricci Flow
Definition (Discrete Surface Ricci Energy)

Suppose \((S, V, d)\) is a triangle mesh with a generalized CP metric, the discrete surface energy is defined as

\[
E(u) = \int_0^u \sum_{i=1}^k (\bar{K}_i - K_i) du_i.
\]

1. **Gradient**
   \(\nabla E = \bar{K} - K\),

2. **Hessian**
   \[
   \left( \frac{\partial^2 E}{\partial u_i \partial u_j} \right) = \Delta,
   \]

3. Ricci flow is the gradient flow of the Ricci energy,

4. Ricci energy is concave, the solution is the unique global maximal point, which can be obtained by Newton’s method.
Input: a closed triangle mesh $M$, target curvature $\bar{K}$, step length $\delta$, threshold $\epsilon$
Output: a PL metric conformal to the original metric, realizing $\bar{K}$.

1. Initialize $u_i = 0, \forall v_i \in V$.
2. compute edge length, corner angle, discrete curvature $K_i$
3. update to Delaunay triangulation by edge swap
4. compute edge weight $w_{ij}$.
5. $u_+ = \delta \Delta^{-1} (\bar{K} - K)$
6. normalize $u$ such that the mean of $u_i$’s is 0.
7. repeat step 2 through 6, until the max $|\bar{K}_i - K_i| < \epsilon$. 
Genus One Example

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Discrete Surface Ricci Flow
Hyperbolic Discrete Surface Yamabe Flow

Discrete conformal metric deformation:

\[ y_{k} = e^{u_i \frac{l_k}{2}} e^{u_j}, \quad \mathbb{R}^2 \]
\[ \sinh y_{k} = e^{u_i \sinh \frac{l_k}{2}} e^{u_j}, \quad \mathbb{H}^2 \]
\[ \sin y_{k} = e^{u_i \sin \frac{l_k}{2}} e^{u_j}, \quad \mathbb{S}^2 \]

Properties: \[ \frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i} \text{ and } dK = \Delta du. \]
Curvature flow

\[ \frac{du}{dt} = \bar{K} - K, \]

Energy

\[ E(u) = \int \sum_i (\bar{K}_i - K_i) du_i, \]

Hessian of $E$ denoted as $\Delta$,

\[ dK = \Delta du. \]
Genus Two Example
Genus Three Example
Existence Theorem
Definition (Delaunay Triangulation)

Each PL metric $d$ on $(S, V)$ has a Delaunay triangulation $T$, such that for each edge $e$ of $T$,

$$a + a' \leq \pi,$$

It is the dual of Voronoi decomposition of $(S, V, d)$

$$R(v_i) = \{x|d(x, v_j) \leq d(x, v_i) \text{ for all } v_j\}$$
Conformal factor $u : V \rightarrow \mathbb{R}$. Discrete conformal change is vertex scaling.

proposed by physicists Rocek and Williams in 1984 in the Lorenzian setting. Luo discovered a variational principle associated to it in 2004.
Definition (Discrete Yamabe Flow)

The discrete conformal factor deforms proportional to the difference between the target curvature and the current curvature

\[
\frac{du(v_i)}{dt} = \bar{K}(v_i) - K(v_i),
\]

the triangulation is updated to be Delaunay during the flow.
Definition (Discrete Conformal Equivalence)

PL metrics \( d, d' \) on \((S, V)\) are discrete conformal,

\[
d \sim d'
\]

if there is a sequence \( d = d_1, d_2, \cdots, d_k = d' \) and \( T_1, T_2, \cdots, T_k \) on \((S, V)\), such that

1. \( T_i \) is Delaunay in \( d_i \)
2. if \( T_i \neq T_{i+1} \), then \((S, d_i) \cong (S, d_{i+1})\) by an isometry homotopic to \( id \)
3. if \( T_i = T_{i+1} \), \( \exists u : V \rightarrow \mathbb{R} \), such that \( \forall \) edge \( e = [v_i, v_j] \),

\[
l_{d_{i+1}}(e) = e^{u(v_i)}l_{d_i}e^{u(v_j)}
\]
Discrete Conformality

Discrete conformal metrics

Discrete Surface Ricci Flow
Main Theorem

Theorem (Gu-Luo-Sun-Wu (2013))

∀ PL metrics $d$ on closed $(S, V)$ and $\forall \bar{K} : V \rightarrow (-\infty, 2\pi)$, such that $\sum \bar{K}(v) = 2\pi \chi(S)$, $\exists$ a PL metric $\bar{d}$, unique up to scaling on $(S, V)$, such that

1. $\bar{d}$ is discrete conformal to $d$
2. The discrete curvature of $\bar{d}$ is $\bar{K}$.

Furthermore, $\bar{d}$ can be found from $d$ from a discrete curvature flow.

Remark

$\bar{K} = \frac{2\pi \chi(S)}{|V|}$, discrete uniformization.
Main Theorem

1. The uniqueness of the solution is obtained by the convexity of discrete surface Ricci energy and the convexity of the admissible conformal factor space (\(u\)-space).

2. The existence is given by the equivalence between PL metrics on \((S, V)\) and the decorated hyperbolic metrics on \((S, V)\) and the Ptolemy identity.

PL Metric Teichmüller Space
Definition (Marked Surface)
Suppose $\Sigma$ is a closed topological surface, $V = \{v_1, v_2, \ldots, v_n\} \subset \Sigma$ is a set of disjoint points on $\Sigma$, satisfying $\chi(\Sigma - V) < 0$.

Definition (Metric Equivalence)
Two polyhedral metrics $d$ and $d'$ are equivalent, if there is an isometric transformation $h : (\Sigma, V, d) \rightarrow (\Sigma, V, d')$, $h$ is homotopic to the identity of the marked surface $(\Sigma, V)$.

Definition (PL Teichmüller Space)
All the equivalence classes of the PL metrics on the marked surface $(\Sigma, V)$ consist the Teichmüller space

$$T_{PL}(\Sigma, V) := \{d|\text{polyhedralmetric}(\Sigma, V)\}/\{\text{isometry} \sim id(\Sigma, V)\}.$$
Definition (Local Chart for PL Teichmüller Space)

Assume $\mathcal{T}$ is a triangulation of $(\Sigma, V)$, the edge length function determines a unique PL metric, 

$\Phi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})}_\Delta \to T_{PL}(\Sigma, V),$

this gives a local coordinates of the PL Teichmüller space, where the domain

$\mathbb{R}^{E(\mathcal{T})}_\Delta = \left\{ x \in \mathbb{R}^{E(\mathcal{T})}_\Delta \mid \forall \Delta = \{e_i, e_j, e_k\}, x(e_i) + x(e_j) > x(e_k) \right\}$

is a convex set. We use $\mathcal{P}_{\mathcal{T}}$ to represent the image of $\Phi_{\mathcal{T}}$, then $(\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1})$ form a local chart of $T_{PL}(\Sigma, V)$.
Definition (Atlas of PL metric Teichmüller Space)

Given a closed marked surface \((\Sigma, V)\), the atlas of \(T_{PL}(\Sigma, V)\) consists of all local charts \((P_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1})\), where \(\mathcal{T}\) exhaust all possible triangulations,

\[
\mathcal{A}(T_{pl}(S, V)) = \bigcup_{\mathcal{T}}(P_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1}).
\]

From \(|V| + |F| - |E| = 2 - 2g\) and \(3|F| = 2|E|\), we obtain \(|E| = 6g - 6 + 3|V|\).

Theorem (Troyanov)

Given a closed marked surface \((\Sigma, V)\), the PL metric Teichmüller space \(T_{PL}(\Sigma, V)\) and the Euclidean space \(\mathbb{R}^{6g-6+3|V|}\) is diffeomorphic.
Complete Hyperbolic Metric Teichmüller Space
The unit disk is with hyperbolic Riemannian metric

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2},$$

**Figure:** Hyperbolic geodesics in the Poincare model.
The upper half plane is with hyperbolic Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

Figure: All hyperbolic ideal triangles are isometric.
Definition (Thurston’s Shear Coordinates)

Given an ideal quadrilateral, Thurston’s shear coordinates equal to the oriented distance from $L$ to $R$ along the diagonal.

Figure: Hyperbolic Ideal Quadrilateral.
Hyperbolic Ideal Quadrilateral

Definition (Thurston’s Shear Coordinates)

Given an ideal quadrilateral, Thurston’s shear coordinates equal to the oriented distance from $L$ to $R$ along the diagonal.

Figure: Hyperbolic Ideal Quadrilateral.
Assume a genus $g$ surface with $n$ vertices removed, $\Sigma = \Sigma_g - \{v_1, v_2, \ldots, v_n\}, n \geq 1, \chi(\Sigma) < 0, (\Sigma, \mathcal{T})$ is a triangulation. Given a function defined on edges, $x : E(\mathcal{T}) \to \mathbb{R}$, construct a hyperbolic structure $\pi(X)$

1. for every triangle $\Delta \in \mathcal{T}$, construct a hyperbolic ideal triangle, $\Delta \to \Delta^*$;

2. for every edge $e \in E(\mathcal{T})$, adjacent to two faces $\Delta_1 \cap \Delta_2 = e$, glue two ideal triangles $\Delta_1^* \cap \Delta_2^*$ along $e$ isometrically, the shear coordinates on $e$ equals to $x(e)$.

**Figure:** Construction of a complete metric.
Lemma

If $\pi(x)$ is a complete metric with finite area, namely each vertex becomes a cusp, then for each $v \in \{v_1, v_2, \ldots, v_n\}$,

$$\sum_{e \sim v} x(e) = 0.$$
Define linear space:

\[ \mathbb{R}^E_P = \left\{ x \in \mathbb{R}^E \big| \forall v \in V, \sum_{v \sim e} x(e) = 0 \right\} \]

**Theorem (Thurston)**

The mapping

\[ \Phi_T : \mathbb{R}^E_P \rightarrow T(\Sigma), x \mapsto [\pi(x)] \]

is injective and surjective, \( \Phi_T(x) \) under \( T \) has shear coordinates \( x(e) \).
Hyperbolic Teichmüller Space

**Definition (Complete Hyperbolic Metric Teichmüller Space)**

Given a closed marked surface \((\Sigma, V)\) with genus \(g\), \(\chi(\Sigma - V) < 0\), all the complete hyperbolic metrics defined on \(\Sigma - V\) with finite area, and each \(v \in V\) being a cusp, form the hyperbolic metric Teichmüller space of \(\Sigma - V\), denoted as \(T_H(\Sigma, V)\).

From \(|V| + |F| - |E| = 2 - 2g\) and \(3|F| = 2|E|\), we obtain \(|E| = 6g - 6 + 3|V|\). The cusp condition removes \(|V|\) freedoms.

**Corollary**

The hyperbolic metric Teichmüller Space \(T(\Sigma, V)\) is a real analytic manifold, diffeomorphic to \(\mathbb{R}^{6g-6+2|V|}\), where \(g\) is the genus of the closed surface \(\Sigma\).
Definition (Complete Hyperbolic Teichmüller Space)

Given a closed marked surface \((\Sigma, V)\), \(\chi(\Sigma - V) < 0\), all the equivalence classes of the complete hyperbolic metrics with finite area on \((\Sigma, V)\) form the Teichmüller space:

\[
T_H(\Sigma - V) = \{h| hcompelete, finitearea\}/\{isometry \sim idof(\Sigma - V)\}
\]
Complete Hyperbolic Metric Teichmüller Space

**Definition (Local Chart of** \( T_H(\Sigma - V) \)**)

Assume \( \mathcal{T} \) is a triangulation of \((\Sigma, V)\), its shear coordinates determines a unique complete hyperbolic metric with finite area,

\[
\Theta_\mathcal{T} : \Omega_\mathcal{T} \rightarrow T_H(\Sigma - V)
\]  

(5)

this gives a local chart of the Teichmüller space, where the domain \( \Omega_\mathcal{T} \) is a sublinear space in \( \mathbb{R}^{E(\mathcal{T})} \), satisfying the cusp conditions. Then \( (\Omega_\mathcal{T}, \Theta_\mathcal{T}^{-1}) \) form a local chart of \( T_H(\Sigma - V) \).

**Definition (Atlas of** \( T_H(\Sigma - V) \)**)

Each triangulation \( \mathcal{T} \) of \((\Sigma, V)\) corresponds to a local chart \( (\Omega_\mathcal{T}, \Theta_\mathcal{T}^{-1}) \). By exhausting all possible triangulations, the union of all local charts gives the atlas of \( T_H(\Sigma - V) \):

\[
\mathcal{A}(T_H(\Sigma - V)) = \bigcup_{\mathcal{T}} \left( \Omega_\mathcal{T}, \Theta_\mathcal{T}^{-1} \right).
\]
Decorated Hyperbolic Metric Teichmüller Space
\( \tau \) is a decorated ideal hyperbolic triangle, three infinite vertices are \( v_1, v_2, v_3 \in \partial \mathbb{H}^2 \). Each \( v_i \) is associated with a horoball \( H_i \), the length of \( \partial H_i \cap \tau \) is \( \alpha_i \); the oriented length of \( e_i \) is \( l_i \): if \( H_j \cap H_k = \emptyset \) then \( l_i > 0 \), otherwise \( l_i < 0 \). Penner’s \( \lambda \)-length \( L_i \) is defined as

\[
L_i := e^{\frac{1}{2}l_i}.
\]

\[\text{Figure: Decorated ideal hyperbolic triangle, left frame } l_i > 0, \text{ right frame } l_i < 0.\]
Definition (Decorated Hyperbolic Metric)

A decorated hyperbolic metric on a marked closed surface $(\Sigma, V)$ is represented as $(d, w)$:

1. $d$ is a complete, with finite area hyperbolic metric;
2. each cusp $v_i$ is associated with a horoball $H_i$. The center of $H_i$ is $v_i$, the length of $\partial H_i$ is $w_i$. $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_{>0}$

Figure: Decorated hyperbolic metric.
Definition (Decorated Hyperbolic Metric Equivalence)

Two decorated hyperbolic metric $(d_1, w_1)$ and $(d_2, w_2)$ on $(\Sigma, V)$ are equivalent, if there is an isometric transformation $h$ between them, $h$ preserves all the horoballs and is isotopic to the identity map of $\Sigma - V$.

Definition (Decorated Hyperbolic Metric Teichmüller Space)

Given a closed marked surface $(\Sigma, V)$, $\chi(S - V) < 0$, the decorated hyperbolic metric Teichmüller space of $(\Sigma, V)$ is defined as

$$T_D(\Sigma, V) := \{(d, w) | \text{decorated hyperbolic metric} \over \text{isometry homotopic to id, preserves horoballs}\}$$
Mappings Among Teichmüller Spaces
Relation between Teichmüller Spaces

**Theorem**

Given a closed marked surface \((\Sigma, V)\), \(\chi(\Sigma - V) < 0\), the decorated hyperbolic metric Teichmüller space and the complete hyperbolic metric Teichmüller space has the relation:

\[
T_D(\Sigma, V) = T_H(\Sigma, V) \times \mathbb{R}|V|_{>0}.
\]

where \(\mathbb{R}|V|_{>0}\) represents the length of the decoration \(\partial H_i\).
Fix a triangulation $\mathcal{T}$ of $(\Sigma, V)$, construct a mapping between the local charts determined by $\mathcal{T}$,

$$\Phi_\mathcal{T} : T_{PL}(\Sigma, V) \rightarrow T_{D}(\Sigma, V), x(e) \mapsto 2 \ln x(e).$$

Figure: Euclidean metric to decorated hyperbolic metric.
Definition (Cross Ratio)

Given a marked surface with a PL metric and a triangulation $(\Sigma, d, \mathcal{T})$, for a pair of adjacent faces $\{A, C, B\} \circlearrowleft \{A, B, D\}$ sharing the edge $\{A, B\}$, the cross ratio on the common edge is defined as:

$$\text{Cr}(\{A, B\}) := \frac{aa'}{bb'},$$

where $a, a', b, b'$ are the lengths of the edges $\{A, C\}, \{B, D\}, \{B, C\}, \{A, D\}$ under the PL metric $d$. 

![Length cross ratio diagram](image)
Length cross ratio of \((\Sigma, V, d, \mathcal{T})\) satisfies the cusp condition, hence we can construct a mapping \(\psi_{\mathcal{T}} : T_{PL}(\Sigma, V) \rightarrow T_H(\Sigma, V)\), such that the shear coordinates of the complete hyperbolic metric equals to the length cross ratio of the PL metric.

\[x(\Delta_1^{*}, \Delta_2^{*}) = \Delta_1^{*} \Delta_2^{*}\]

**Figure:** Euclidean metric to complete hyperbolic metric.
Consistency among the transformations

\[ T_{PL}(\Sigma, V) \xrightarrow{Cr} T_{PL}(\Sigma, V) \]
\[ \Phi_{\mathcal{T}} \downarrow \quad \downarrow \Psi_{\mathcal{T}} \]
\[ T_{D}(\Sigma, V) \xrightarrow{Sh} T_{H}(\Sigma, V) \]

The above diagram commutes.

Figure: Cross ratio, Penner’s $\lambda$ length, shear coordinates.

Fix a triangulation $\mathcal{T}$,

David Gu Discrete Surface Ricci Flow
Definition (Euclidean Delaunay Triangulation)

Given a marked surface with a PL metric \((\Sigma, V, d)\), Delaunay triangulation \(\mathcal{T}\) satisfies condition, for all edges \(\alpha + \alpha' \leq \pi\). Equivalently \(\cos \alpha + \cos \alpha' \geq 0\),

\[
\frac{x_1^2 + x_2^2 - x_0^2}{2x_1x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3x_4} \geq 0.
\] (6)
Lemma

The transformation \( \Phi_T : T_{PL}(\Sigma, V) \rightarrow T_D(\Sigma, V) \) preserves Delaunay triangulations.

Since both situations:

\[
\frac{x_1^2 + x_2^2 - x_0^2}{2x_1 x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3 x_4} \geq 0. \tag{7}
\]
Ptolemy Conditions

Let $A, A', B, B', C, C'$ are edge lengths of the Euclidean quadrilateral and the Penner's $\lambda$-length of the decorated hyperbolic ideal quadrilateral, then both of them satisfy the Ptolemy conditions:

$$CC' = AA' + BB'.$$
The mapping $\Phi_T : T_{PL}(\Sigma, V) \rightarrow T_{D}(\Sigma, V)$ is defined on each local chart, by Tolemy condition. By Ptolemy condition, all the locally defined mappings $\Phi_T$ can be glued together to form a global map

$$\Phi : T_{PL}(\Sigma, V) \rightarrow T_{D}(\Sigma, V),$$

Ptolemy condition shows that the global mapping is continuous. Further computation shows that $\Phi$ is globally $C^1$. 
Define the cell decomposition of the Teichmüller spaces

\[ T_{PL}(\Sigma, V) = \bigcup_{\mathcal{T}} C_{PL}(\mathcal{T}) \]

where

\[ C_{PL}(\mathcal{T}) := \{ [d] \in T_{PL} | \mathcal{T} \text{ is Delaunay under } d \} \].

Similarly

\[ T_{D}(\Sigma, V) = \bigcup_{\mathcal{T}} C_{D}(\mathcal{T}) \]

where

\[ C_{D}(\mathcal{T}) := \{ [d] \in T_{D} | \mathcal{T} \text{ is Delaunay under } d \} \].

Inside the cells, the mapping \( \Phi_{\mathcal{T}} : C_{PL}(\mathcal{T}) \to C_{D}(\mathcal{T}) \) is a diffeomorphism.
On the boundary of the cells, restricted on $C_{PL}(\mathcal{T}) \cap C_{PL}(\mathcal{T}')$, where four points are cocircle,

$$
\begin{align*}
C_{PL}(\mathcal{T}) & \xrightarrow{\text{Euclidean Ptolemy}} C_{PL}(\mathcal{T}') \\
\downarrow \phi_{\mathcal{T}} & \quad & \downarrow \phi_{\mathcal{T}'} \\
C_{D}(\mathcal{T}) & \xrightarrow{\text{Hyperbolic Ptolemy}} C_{D}(\mathcal{T}')
\end{align*}
$$

Furthermore,

$$
\begin{align*}
C_{PL}(\mathcal{T}) & \xrightarrow{\text{Euclidean Ptolemy}} C_{PL}(\mathcal{T}') \\
\downarrow \nabla \phi_{\mathcal{T}} & \quad & \downarrow \nabla \phi_{\mathcal{T}'} \\
C_{D}(\mathcal{T}) & \xrightarrow{\text{Hyperbolic Ptolemy}} C_{D}(\mathcal{T}')
\end{align*}
$$

the diagram commutes. So the piecewise diffeomorphism $\phi_{\mathcal{T}}$ can be glued together to form a global $C^1$ map:

$$\Phi : T_{PL}(\Sigma, V) \rightarrow T_{D}(\Sigma, V).$$
Existence of Solution to Discrete Surface Ricci Flow
Domain $\Omega_u$ is the space of discrete conformal factor,

$$\Omega_u = \mathbb{R}^n \cap \left\{ u \mid \sum_{i=1}^{n} u_i = 0 \right\}.$$

The range $\Omega_K$ is the space of discrete curvatures,

$$\Omega_K = \left\{ K \in (-\infty, 2\pi)^n \mid \sum_{i=1}^{n} K_i = 2\pi \chi(S) \right\}$$

both of them are open sets in $\mathbb{R}^{|V|^{-1}}$. The global mapping is

$$F : \Omega_u \xrightarrow{\exp} \{ p \} \times \mathbb{R}_{>0} \rightarrow T_D(\Sigma, V) \xrightarrow{\Phi^{-1}} T_{PL}(\Sigma, V) \xrightarrow{K} \Omega_K$$
Existence Proof

The global mapping is \( C^1 \),

\[
F : \Omega_u \xrightarrow{\exp} \{ p \} \times \mathbb{R}_{>0} |V| \rightarrow T_D(\Sigma, V) \xrightarrow{\phi^{-1}} T_{PL}(\Sigma, V) \xrightarrow{K} \Omega_K
\]

During the flow, the triangulation is always Delaunay, the cotangent edge weight is non-negative, the discrete Laplace-Beltrami matrix is strictly positive definite. Hence the Hessian matrix of the energy

\[
E(u) = \int_{u}^{n} \sum_{i=1}^{n} K_i du_i
\]

is strictly convex. \( F \) is the gradient map of the energy,

\[
F(u) = \nabla E(u),
\]

because \( \Omega_u \) is convex, the mapping is a diffeomorphism.
Convergence of Solutions to Discrete Surface Ricci Flow
Convergence Proof

**Definition ($\delta$ triangulation)**

Given a compact polyhedral surface $(\Sigma, V, d)$, a triangulation $T$ is a $\delta$-triangulation, $\delta > 0$, if all the inner angles are in the interval $(\delta, \pi/2 - \delta)$.

**Definition ($($($\delta$, $c$)$)$-triangulation)**

Given a compact triangulated polyhedral surface $(S, T, l^*)$, a geometric subdivision sequence $(T_n, l^*_n)$ is a $(\delta, c)$ subdivision sequence, $\delta > 0$, $c > 0$, if each $(T_n, l^*_n)$ is a $\delta$ triangulation, and the edge lengths satisfy

\[
l^*_n e \in \left( \frac{1}{cn}, \frac{c}{n} \right), \forall e \in E(T_n)
\]

Polyhedral surface can be replaced by a surface with a Riemannian metric, triangulation can be replaced by geodesic triangulation, then we obtain $(\delta, c)$ geodesic subdivision sequence.
Theorem (Discrete Surface Ricci Flow Convergence)

Given a simply connected Riemannian surface \((S, g)\) with a single boundary, the inner angles at the three corners are \(\frac{\pi}{3}\). Given a \((\delta, c)\) geodesic subdivision sequence \((\mathcal{T}_n, L_n)\), for any edge \(e \in E(T_n)\), \(L_n(e)\) is the geodesic length under the metric \(g\). There exists discrete conformal factor \(w_n \in \mathbb{R}^{V(T_n)}\), such that for large enough \(n\), \(C_n = (S, \mathcal{T}_n, w_n \ast L_n)\) satisfies

a. \(C_n\) is isometric to a planar equilateral triangle \(\triangle\), and \(C_n\) is a \(\delta_S/2\)-triangulation

b. discrete uniformizations map \(\phi_n: C_n \rightarrow \triangle\) converge to the smooth uniformization map \(\phi: (S, g) \rightarrow (\triangle, dzd\bar{z})\) uniformly, such that

\[
\lim_{n \to \infty} \| \phi_n|_{V(\mathcal{T}_n)} - \phi|_{V(\mathcal{T}_n)} \|_{\infty} = 0.
\]
References


Topological Quadrilateral
Figure: Topological quadrilateral.
Definition (Topological Quadrilateral)

Suppose $S$ is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wise. Then $S$ is called a topological quadrilateral, and denoted as $Q(p_1, p_2, p_3, p_4)$.

Theorem

Suppose $Q(p_1, p_2, p_3, p_4)$ is a topological quadrilateral with a Riemannian metric $g$, then there exists a unique conformal map $\phi : S \rightarrow \mathbb{C}$, such that $\phi$ maps $Q$ to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.
Algorithm: Topological Quadrilateral

Input: A topological quadrilateral \( M \)
Output: Conformal mapping from \( M \) to a planar rectangle \( \phi : M \rightarrow \mathbb{D} \)

1. Set the target curvatures at corners \( \{p_0, p_1, p_2, p_3\} \) to be \( \frac{\pi}{2} \),
2. Set the target curvatures to be 0 everywhere else,
3. Run ricci flow to get the target conformal metric \( \bar{u} \),
4. Isometrically embed the surface using the target metric.
Topological Annulus
Figure: Topological annulus.
Definition (Topological Annulus)

Suppose $S$ is a surface of genus zero with two boundaries, the $S$ is called a topological annulus.

Theorem

Suppose $S$ is a topological annulus with a Riemannian metric $g$, the boundary of $S$ are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \rightarrow \mathbb{C}$, which maps $S$ to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius $\gamma$. Then $-\log \gamma$ is the conformal module of $S$. The mapping $\phi$ is unique up to a planar rotation.
Algorithm: Topological Annulus

Input: A topological annulus $M$, $\partial M = \gamma_0 - \gamma_1$
Output: a conformal mapping from the surface to a planar annulus $\phi : M \to \mathbb{A}$

1. Set the target curvature to be 0 every where,
2. Run Ricci flow to get the target metric,
3. Find the shortest path $\gamma_2$ connecting $\gamma_0$ and $\gamma_1$, slice $M$ along $\gamma_2$ to obtain $\bar{M}$,
4. Isometrically embed $\bar{M}$ to the plane, further transform it to a flat annulus

$$\{z | r \leq \text{Re}(z) \leq 0\}/\{z \to z + 2k\sqrt{-1}\pi\}$$

by planar translation and scaling,
5. Compute the exponential map $z \to \exp(z)$, maps the flat annulus to a canonical annulus.
Riemann Mapping
Conformal Module

Simply Connected Domains
Definition (Topological Disk)

Suppose $S$ is a surface of genus zero with one boundary, the $S$ is called a topological disk.

Theorem

Suppose $S$ is a topological disk with a Riemannian metric $g$, then there exists a conformal mapping $\phi : S \rightarrow \mathbb{C}$, which maps $S$ to the canonical disk. The mapping $\phi$ is unique up to a Möbius transformation,

$$z \rightarrow e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$
Algorithm: Topological Disk

Input: A topological disk $M$, an interior point $p \in M$
Output: Riemann mapping $\phi : M \rightarrow \mathbb{D}$, maps $M$ to the unit disk and $p$ to the origin

1. Punch a small hole at $p$ in the mesh $M$,
2. Use the algorithm for topological annulus to compute the conformal mapping.
Multiply connected domains
Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)
Suppose $S$ is a surface of genus zero with multiple boundaries, then $S$ is called a multiply connected annulus.

Theorem
Suppose $S$ is a multiply connected annulus with a Riemannian metric $g$, then there exists a conformal mapping $\phi : S \rightarrow \mathbb{C}$, which maps $S$ to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of $S$. Such kind of conformal mapping are unique up to Möbius transformations.
Input: A multiply-connected annulus $M$, 

$$\partial M = \gamma_0 - \gamma_1, \cdots, \gamma_n.$$ 

Output: a conformal mapping $\phi : M \rightarrow \mathbb{A}$, $\mathbb{A}$ is a circle domain.

1. Fill all the interior holes $\gamma_1$ to $\gamma_n$
2. Punch a hole at $\gamma_k$, $1 \leq k \leq n$
3. Conformally map the annulus to a planar canonical annulus
4. Fill the inner circular hole of the canonical annulus
5. Repeat step 2 through 4, each round choose different interior boundary $\gamma_k$. The holes become rounder and rounder, and converge to canonical circles.
Figure: Koebe’s iteration for computing conformal maps for multiply connected domains.
Koebe’s iteration for computing conformal maps for multiply connected domains.
Figure: Koebe’s iteration for computing conformal maps for multiply connected domains.
Theorem (Gu and Luo 2009)

Suppose genus zero surface has \( n \) boundaries, then there exists constants \( C_1 > 0 \) and \( 0 < C_2 < 1 \), for step \( k \), for all \( z \in \mathbb{C} \),

\[
|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2 \left[ \frac{k}{n} \right]},
\]

where \( f \) is the desired conformal mapping.

Topological Torus
Figure: Genus one closed surface.
Algorithm: Topological Torus

Input: A topological torus $M$
Output: A conformal mapping which maps $M$ to a flat torus $\mathbb{C}/\{m + n\alpha | m, n \mathbb{Z}\}$

1. Compute a basis for the fundamental group $\pi_1(M)$, $\{\gamma_1, \gamma_2\}$.
2. Slice the surface along $\gamma_1, \gamma_2$ to get a fundamental domain $\bar{M}$,
3. Set the target curvature to be 0 everywhere
4. Run Ricci flow to get the flat metric
5. Isometrically embed $\tilde{S}$ to the plane
Hyperbolic Ricci Flow

Computational results for genus 2 and genus 3 surfaces.
Thanks

For more information, please email to gu@cmsa.fas.harvard.edu.

Thank you!