Discrete Surface Ricci Flow

David Gu[1]

[1]Computer Science Department
Stony Brook University, USA

Center of Mathematical Sciences and Applications
Harvard University

Geometric Computation and Applications
Trinity College, Dublin, Ireland

Thanks

Thanks for the invitation.

Collaborators

The work is collaborated with Shing-Tung Yau, Yalin Wang, Feng Luo, Ronald Lok Ming Lui, Paul M. Thompson, Tony F. Chan, Arie Kaufman, Hong Qin, Dimitris Samaras, Jie Gao and many other mathematicians, computer scientists and doctors.

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Motivation

Conformal geometry lays down the theoretic foundation for

- Surface mapping
- Geometry classification
- Shape analysis

Applied in computer graphics, computer vision, geometric modeling, wireless sensor networking and medical imaging, and many other engineering, medical fields.

History

History

- In pure mathematics, conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, differential geometry, partial differential equation.
- In applied mathematics, computational complex function theory has been developed, which focuses on the conformal mapping between planar domains.
- Recently, computational conformal geometry has been developed, which focuses on the conformal mapping between surfaces.

History

History

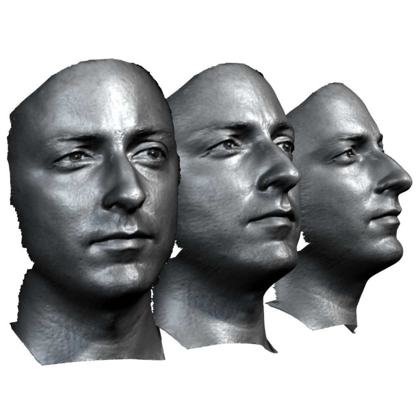
Conventional conformal geometric method can only handle the mappings among planar domains.

- Applied in thin plate deformation (biharmonic equation)
- Membrane vibration
- Electro-magnetic field design (Laplace equation)
- Fluid dynamics
- Aerospace design

Reasons for Booming

Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.



Reasons for Booming

Computational Power

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE's much easier.

Fundamental Problems

1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmuller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.

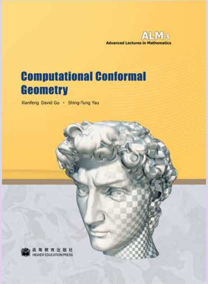
Complete Tools

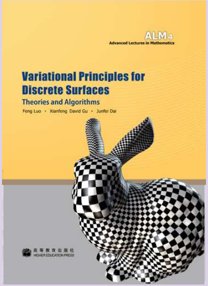
Computational Conformal Geometry Library

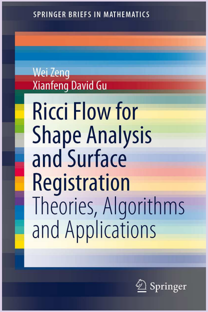
1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation

Books

The theory, algorithms and sample code can be found in the following books.



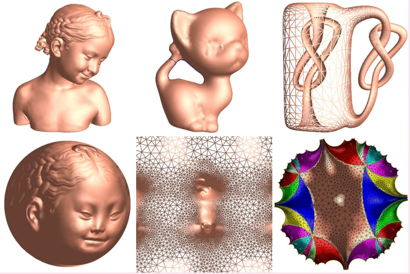


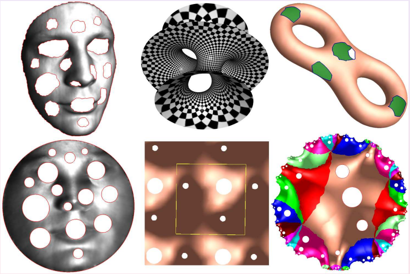


You can find them in the book store.

Source Code Library

Please email me gu@cmsa.fas.harvard.edu for updated code library on computational conformal geometry.





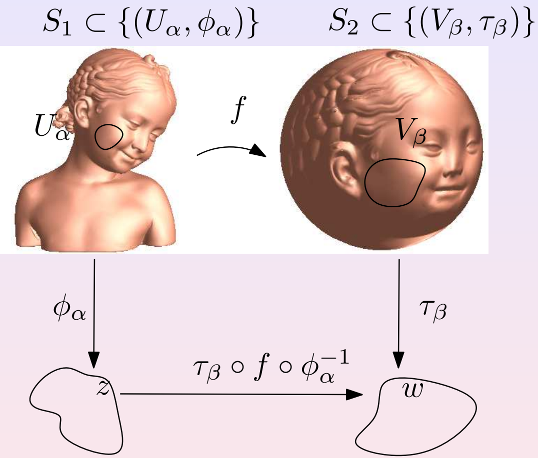
Conformal Mapping

biholomorphic Function

Definition (biholomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

Conformal Map



The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.

Conformal Mapping

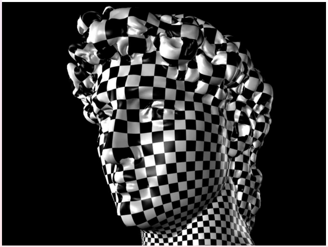
Conformal Geometry

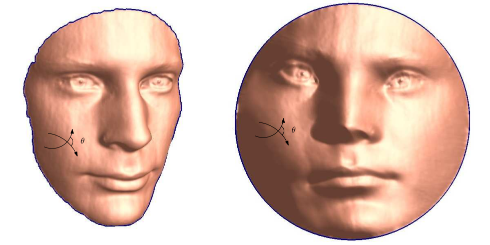
Definition (Conformal Map)

Let $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is a homeomorphism, ϕ is conformal if and only if

$$\phi^* \mathbf{g}_2 = e^{2u} \mathbf{g}_1.$$

Conformal Mapping preserves angles.

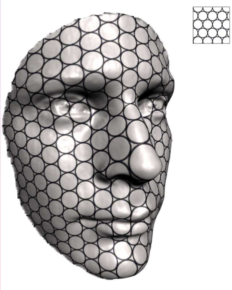




Conformal Mapping

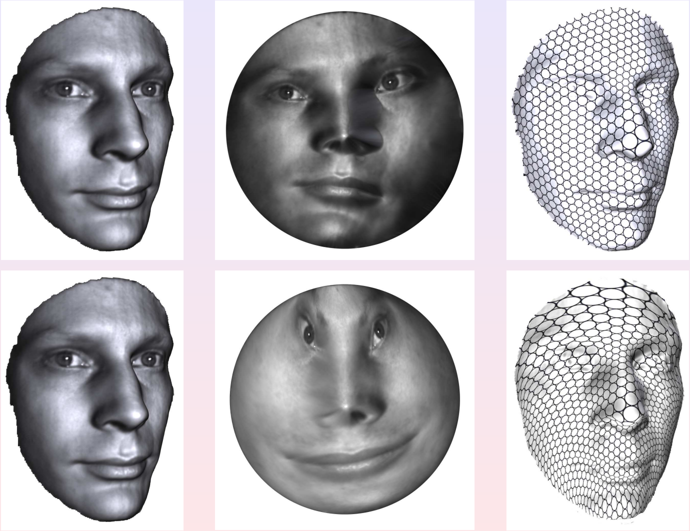
Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.



Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.

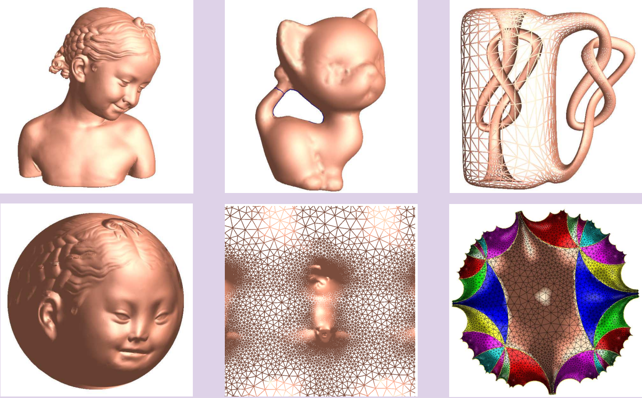


Uniformization

Conformal Canonical Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



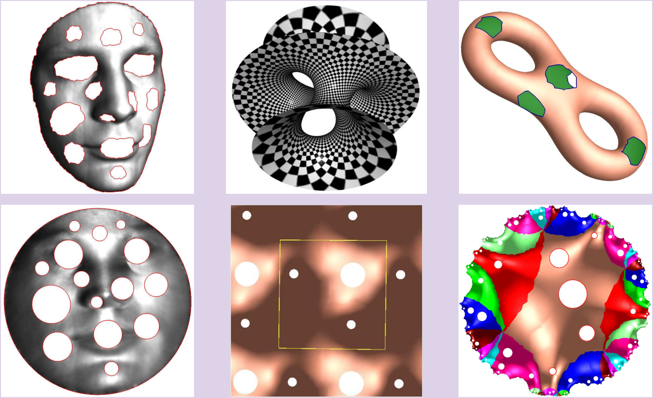
Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Uniformization of Open Surfaces



Spherical Euclidean Hyperbolic

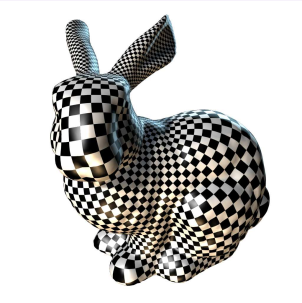
Smooth Surface Ricci Flow

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface M with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

Gaussian Curvature

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda}}\Delta\lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

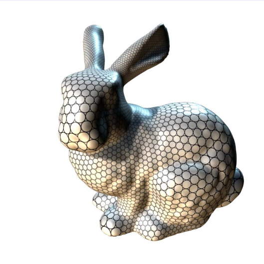
Suppose M is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(K - \Delta_{\mathbf{g}}\lambda),$$

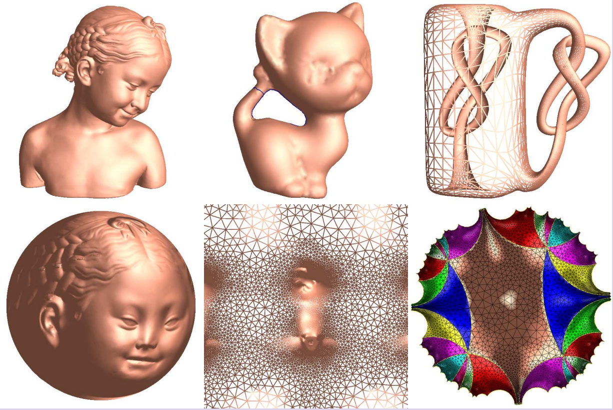
geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(k_g - \partial_{\mathbf{g},n}\lambda).$$

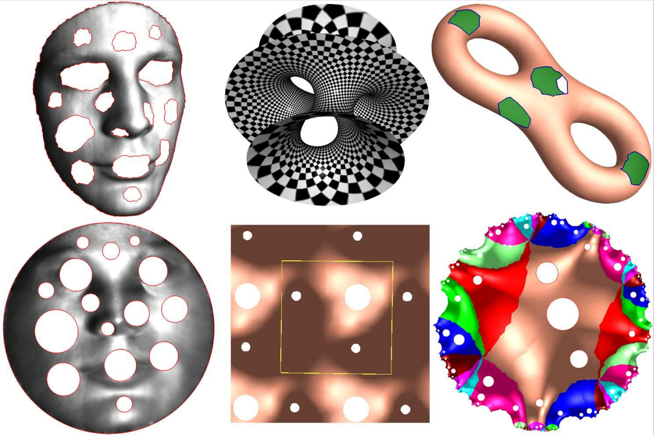
Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Uniformization of Open Surfaces



Key ideas:

- Conformal metric deformation

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

- Ricci flow

$$\frac{d\lambda}{dt} = -K,$$

- Gaussian curvature $K = -\Delta_{\mathbf{g}}\lambda$, evolution equation

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + 2K^2$$

diffusion-reaction equation.

Definition (Normalized Hamilton's Surface Ricci Flow)

A closed surface S with a Riemannian metric $\mathbf{g}$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = \left(\frac{4\pi\chi(S)}{A(0)} - 2K\right) g_{ij}.$$

where $\chi(S)$ is the Euler characteristic number of S, $A(0)$ is the initial total area.

The ricci flow preserves the total area during the flow, converge to a metric with constant Gaussian curvature $\frac{4\pi\chi(S)}{A(0)}$.

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

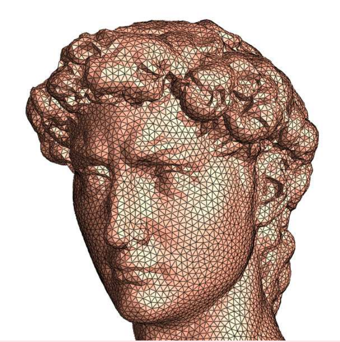
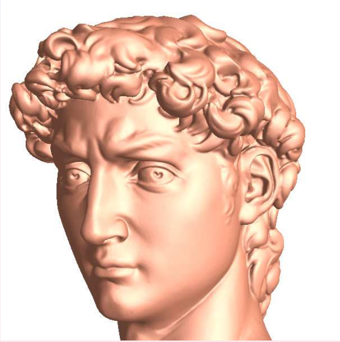
Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Discrete Surface

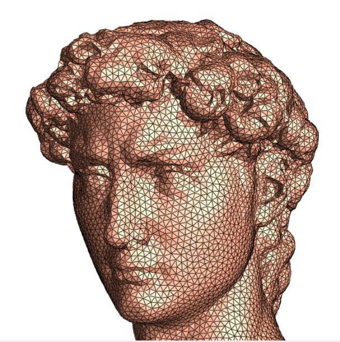
Generic Surface Model - Triangular Mesh

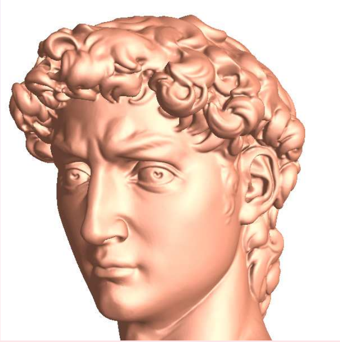
- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

Generic Surface Model - Triangular Mesh

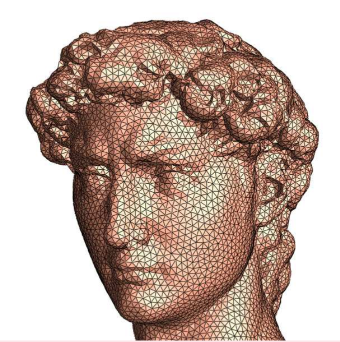
- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

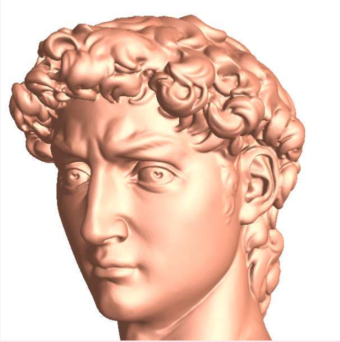




Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





Discrete Generalization

Concepts

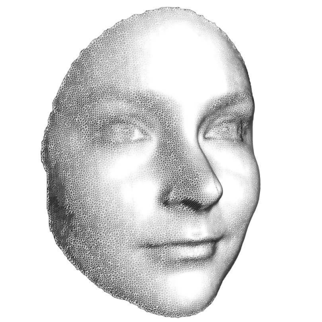
1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation

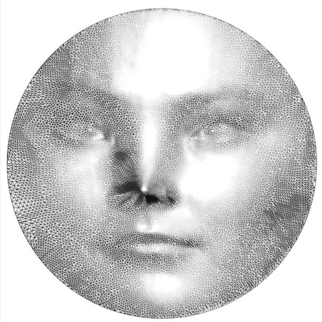
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





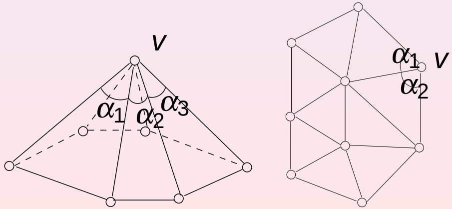
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

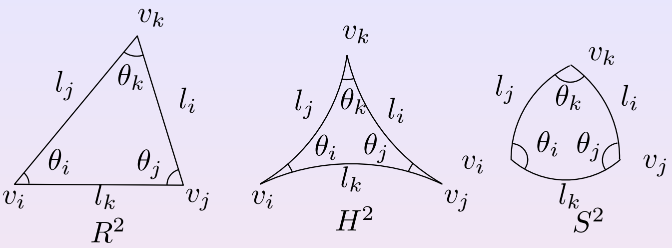
$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



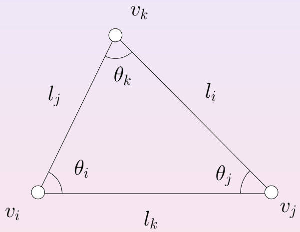
cosine laws

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \quad (1)$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \quad (2)$$

$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \quad (3)$$

Derivative cosine law



Lemma (Derivative Cosine Law)

Suppose corner angles are the functions of edge lengths, then

$$\frac{\partial \theta_i}{\partial l_i} = \frac{l_i}{A}$$

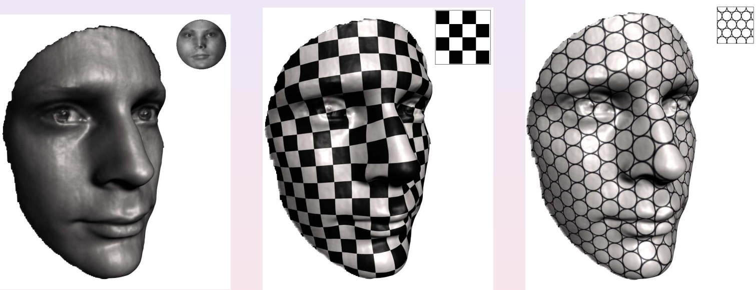
$$\frac{\partial \theta_i}{\partial l_j} = -\frac{\partial \theta_i}{\partial l_i} \cos \theta_k$$

where $A = l_j l_k \sin \theta_i$.

Discrete Conformal Structure

Conformal maps Properties

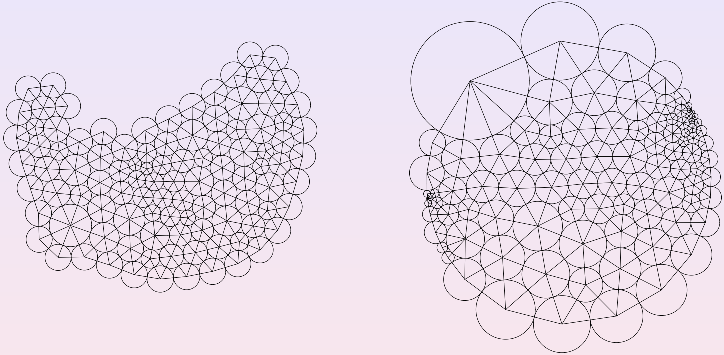
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.



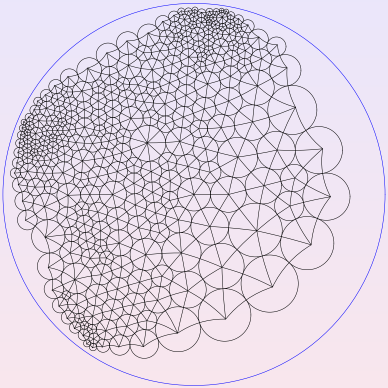
Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

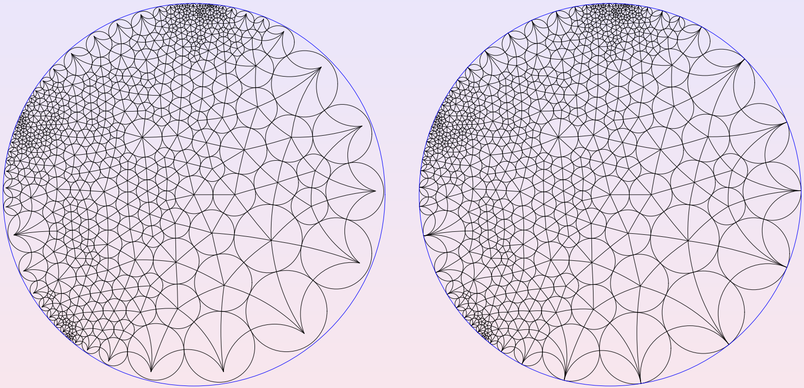
Discrete Conformal Metric Deformation vs CP



Discrete Conformal Metric Deformation vs CP



Discrete Conformal Metric Deformation vs CP



Thurston's Circle Packing Metric

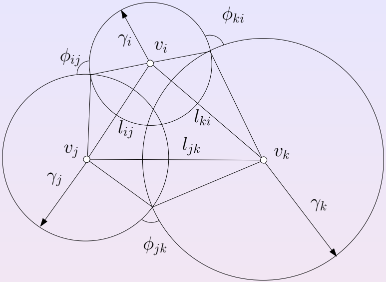
Thurston's CP Metric

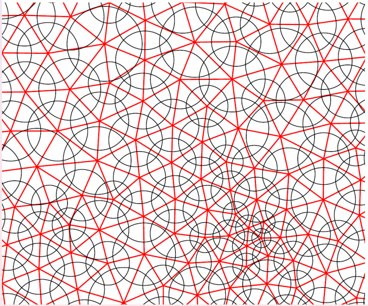
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (T, Γ, Φ), T triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$

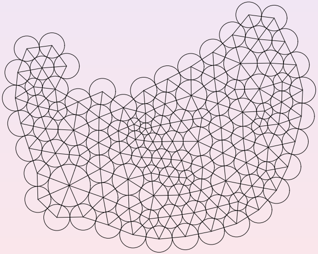


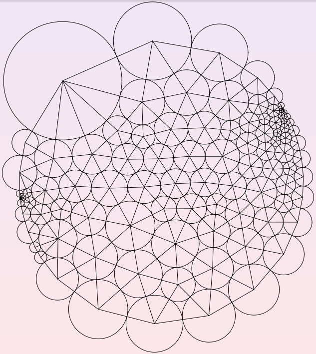


Definition

Conformal Equivalence Two CP metrics (T_1,Γ_1,Φ_1) and (T_2,Γ_2,Φ_2) are conformal equivalent, if they satisfy the following conditions

$$T_1 = T_2 \textit{ and } \Phi_1 = \Phi_2.$$

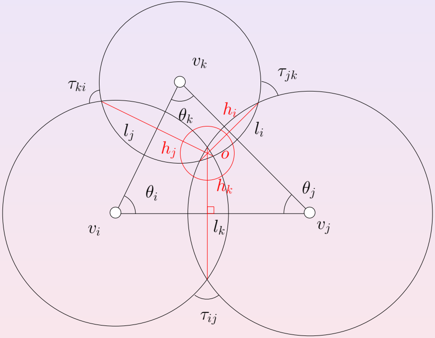




Power Circle

Definition (Power Circle)

The unit circle orthogonal to three circles at the vertices (v_i,γ_i), (v_j,γ_j) and (v_k,γ_k) is called the power circle. The center is called the power center. The distance from the power center to three edges are denoted as h_i, h_j, h_k respectively.

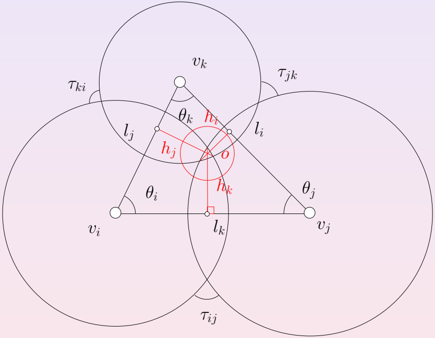


Derivative cosine law

Theorem (Symmetry)

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$
$$\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i}$$
$$\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}$$

Therefore the differential 1-form $\omega = \theta_i du_i + \theta_j du_j + \theta_k du_k$ is closed.

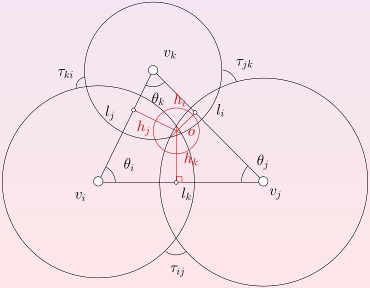


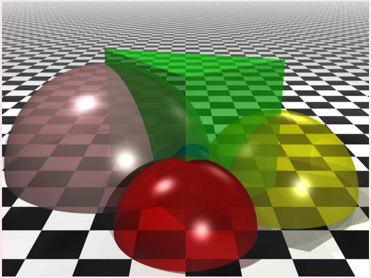
Definition (Discrete Ricci Energy)

The functional associated with a CP metric on a triangle is

$$E(\mathbf{u}) = \int_{(0,0,0)}^{(u_i,u_j,u_k)} \theta_i(\mathbf{u})du_i + \theta_j(\mathbf{u})du_j + \theta_k(\mathbf{u})du_k.$$

Geometrical interpretation: the volume of a truncated hyperbolic hyper-ideal tetrahedron.

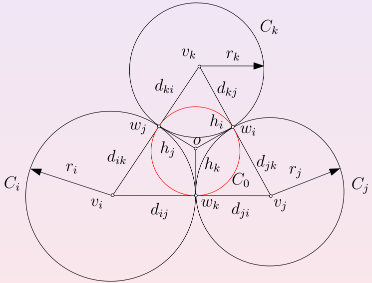


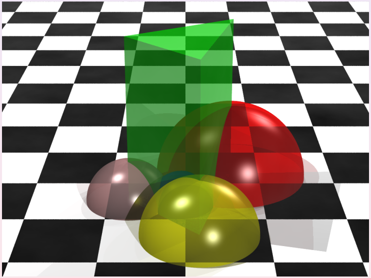


Generalized Circle Packing/Pattern

Definition (Tangential Circle Packing)

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j.$$



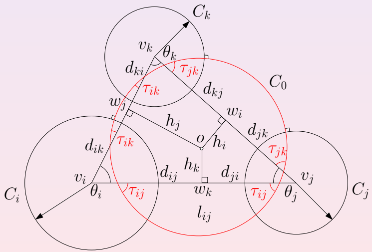


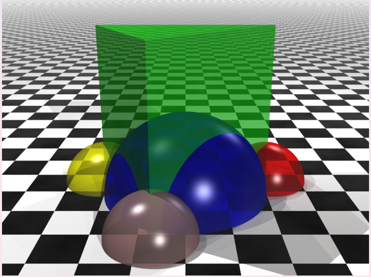
Generalized Circle Packing/Pattern

Definition (Inversive Distance Circle Packing)

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\eta_{ij}.$$

where $\eta_{ij} > 1$.



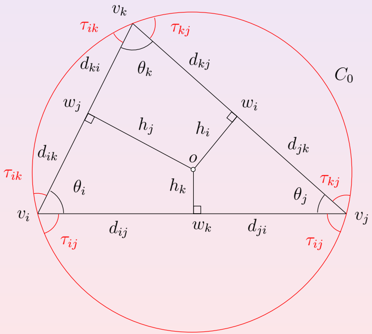


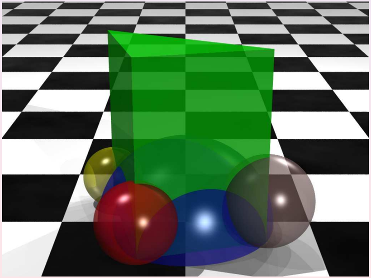
Generalized Circle Packing/Pattern

Definition (Discrete Yamabe Flow)

$$l_{ij}^2 = 2\gamma_i \gamma_j \eta_{ij}.$$

where $\eta_{ij} > 0$.





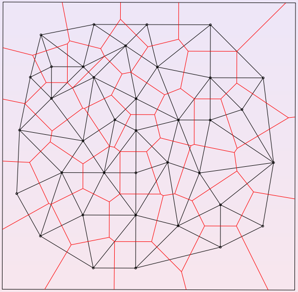
Voronoi Diagram

Definition (Voronoi Diagram)

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$, the Voronoi cell W_i at p_i is

$$W_i = \{\mathbf{x} | |\mathbf{x} - p_i|^2 \leq |\mathbf{x} - p_j|^2, \forall j\}.$$

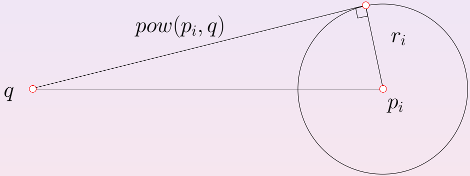
The dual triangulation to the Voronoi diagram is called the Delaunay triangulation.



Power Distance

Given $\mathbf{p}_i$ associated with a sphere $(\mathbf{p}_i, r_i)$ the power distance from $\mathbf{q} \in \mathbb{R}^n$ to $\mathbf{p}_i$ is

$$pow(\mathbf{p}_i, \mathbf{q}) = |\mathbf{p}_i - \mathbf{q}|^2 - r_i^2.$$



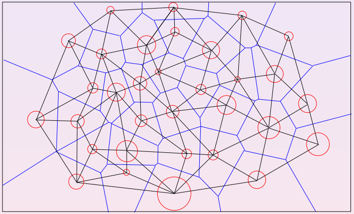
Power Diagram

Definition (Power Diagram)

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$ and sphere radii $\gamma_1, \cdots, \gamma_k$, the power Voronoi cell W_i at p_i is

$$W_i = \{\mathbf{x} | Pow(\mathbf{x}, p_i) \leq Pow(\mathbf{x}, p_j), \forall j\}.$$

The dual triangulation to Power diagram is called the Power Delaunay triangulation.



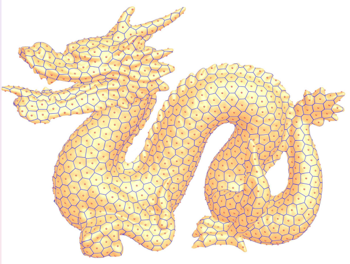
Voronoi Diagram Delaunay Triangulation

Definition (Voronoi Diagram)

Let (S,V) be a punctured surface, V is the vertex set. d is a flat cone metric, where the cone singularities are at the vertices. The Voronoi diagram is a cell decomposition of the surface, Voronoi cell W_i at v_i is

$$W_i = \{\mathbf{p} \in S | d(\mathbf{p}, v_i) \leq d(\mathbf{p}, v_j), \forall j\}.$$

The dual triangulation to the voronoi diagram is called the Delaunay triangulation.

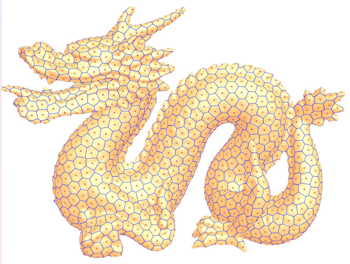


Definition (Power Diagram)

Let (S,V) be a punctured surface, with a generalized circle packing metric. The Power diagram is a cell decomposition of the surface, a Power cell W_i at v_i is

$$W_i = \{\mathbf{p} \in S | Pow(\mathbf{p}, v_i) \leq Pow(\mathbf{p}, v_j), \forall j\}.$$

The dual triangulation to the power diagram is called the power Delaunay triangulation.

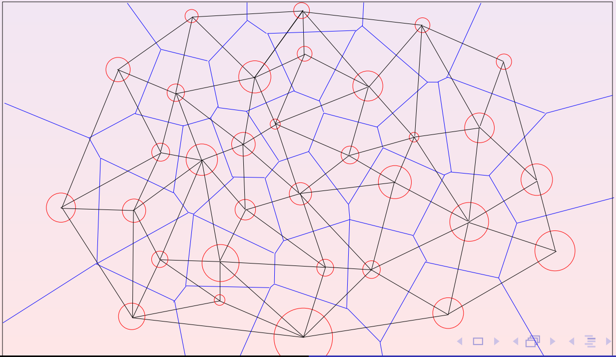


Edge Weight

Definition (Edge Weight)

(S, V, d), d a generalized CP metric. D the Power diagram, T the Power Delaunay triangulation. $\forall e \in D$, the dual edge $\bar{e} \in T$, the weight

$$w(e) = \frac{|e|}{|\bar{e}|}.$$



Discrete Surface Ricci Flow

Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u} : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Definition (Discrete Surface Ricci Flow with Surgery)

Suppose (S,V,d) is a triangle mesh with a generalized CP metric, the discrete surface Ricci flow is given by

$$\frac{du_i}{dt} = \bar{K}_i - K_i,$$

where $\bar{K}_i$ is the target curvature. Furthermore, during the flow, the Triangulation preserves to be Power Delaunay.

Theorem (Exponential Convergence)

The flow converges to the target curvature $K_i(\infty) = \bar{K}_i$*. Furthermore, there exists* $c_1, c_2 > 0$*, such that*

$$|K_i(t) - K_i(\infty)| < c_1 e^{-c_2 t}, |u_i(t) - u_i(\infty)| < c_1 e^{-c_2 t},$$

Discrete Conformal Metric Deformation

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i} = -w_{ij}$$

- Discrete Laplace Equation

$$dK_i = \sum_{[v_i,v_j]\in E} w_{ij}(du_i - du_j)$$

namely

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Discrete Laplace-Beltrami operator

Definition (Laplace-Beltrami operator)

Δ is the discrete Lapalce-Beltrami operator, $\Delta = (d_{ij})$, where

$$d_{ij} = \begin{cases} \sum_k w_{ik} & i = j \\ -w_{ij} & i \neq j, [v_i, v_j] \in E \\ 0 & otherwise \end{cases}$$

Lemma

Given (S, V, d) with generalized CP metric, if T is the Power Delaunay triangulation, then Δ is positive definite on the linear space $\sum_i u_i = 0$.

Because Δ is diagonal dominant.

Discrete Surface Ricci Energy

Definition (Discrete Surface Ricci Energy)

Suppose (S,V,d) is a triangle mesh with a generalized CP metric, the discrete surface energy is defined as

$$E(\mathbf{u}) = \int_{\mathbf{0}}^{\mathbf{u}} \sum_{i=1}^{k} (\bar{K}_i - K_i) du_i.$$

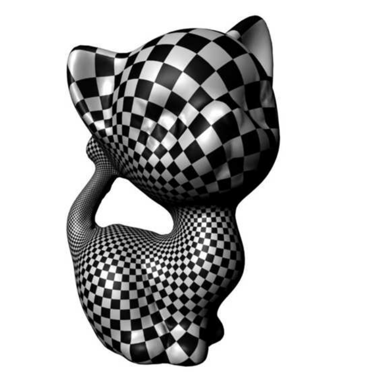
1. gradient $\nabla E = \bar{\mathbf{K}} - \mathbf{K}$,
2. Hessian
$$\left(\frac{\partial^2 E}{\partial u_i \partial u_j}\right) = \Delta,$$
3. Ricci flow is the gradient flow of the Ricci energy,
4. Ricci energy is concave, the solution is the unique global maximal point, which can be obtained by Newton's method.

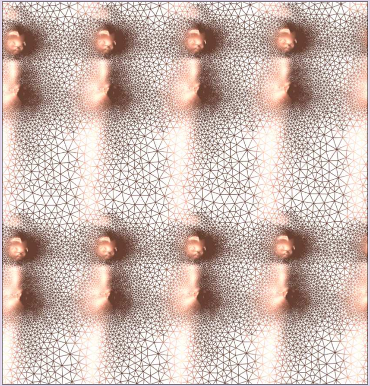
Algorithm

Input: a closed triangle mesh M, target curvature $\bar{K}$, step length δ, threshold ε
Output:a PL metric conformal to the original metric, realizing $\bar{K}$.

1. Initialize $u_i = 0, \forall v_i \in V$.
2. compute edge length, corner angle, discrete curvature K_i
3. update to Delaunay triangulation by edge swap
4. compute edge weight w_{ij}.
5. $\mathbf{u} += \delta\Delta^{-1}(\bar{\mathbf{K}} - \mathbf{K})$
6. normalize $\mathbf{u}$ such that the mean of u_i's is 0.
7. repeat step 2 through 6, until the max $|\bar{K}_i - K_i| < \varepsilon$.

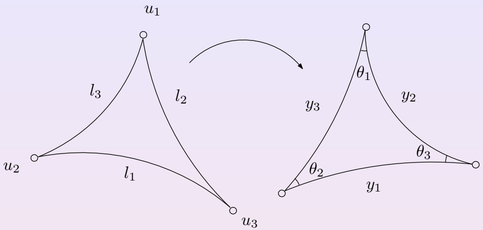
Genus One Example





Hyperbolic Discrete Surface Yamabe Flow

Discrete conformal metric deformation:



conformal factor

$$
\begin{aligned}
\frac{y_k}{2} &= e^{u_i}\frac{l_k}{2}e^{u_j} && \mathbb{R}^2\\
\sinh\frac{y_k}{2} &= e^{u_i}\sinh\frac{l_k}{2}e^{u_j} && \mathbb{H}^2\\
\sin\frac{y_k}{2} &= e^{u_i}\sin\frac{l_k}{2}e^{u_j} && \mathbb{S}^2
\end{aligned}
$$

Properties: $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and $d\mathbf{K} = \Delta d\mathbf{u}$.

Unified framework for both Discrete Ricci flow and Yamabe flow

- Curvature flow

$$\frac{du}{dt} = \bar{K} - K,$$

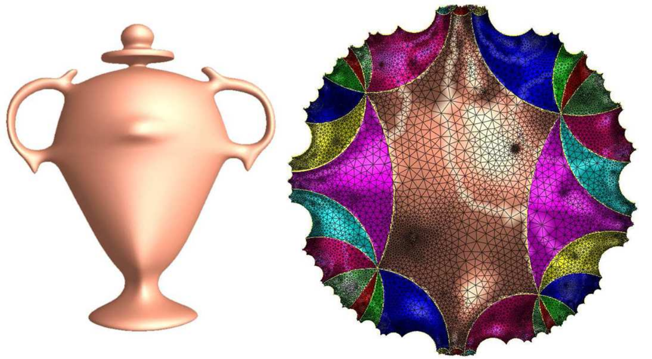
- Energy

$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$

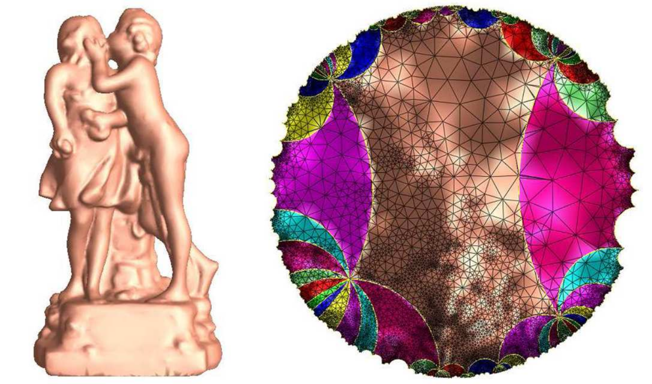
- Hessian of E denoted as Δ,

$$d\mathbf{K} = \Delta d\mathbf{u}.$$

Genus Two Example



Genus Three Example

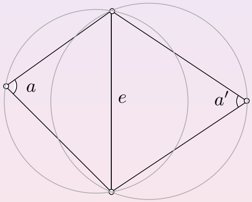


Existence Theorem

Definition (Delaunay Triangulation)

Each PL metric d on (S,V) has a Delaunay triangulation T, such that for each edge e of T,

$$a + a' \leq \pi,$$



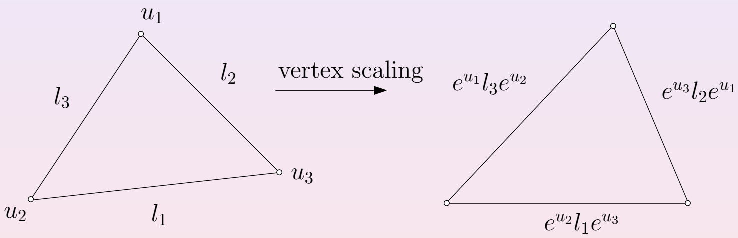
It is the dual of Voronoi decomposition of (S,V,d)

$$R(v_i) = \{x | d(x,v_i) \leq d(x,v_j) \ for \ all \ v_j\}$$

Discrete Conformality

Definition (Conformal change)

Conformal factor $u : V \to \mathbb{R}$. Discrete conformal change is vertex scaling.



proposed by physicists Rocek and Williams in 1984 in the Lorenzian setting. Luo discovered a variational principle associated to it in 2004.

Definition (Discrete Yamabe Flow)

The discrete conformal factor deforms proportional to the difference between the target curvature and the current curvature

$$\frac{du(v_i)}{dt} = \bar{K}(v_i) - K(v_i),$$

the triangulation is updated to be Delaunay during the flow.

Definition (Discrete Conformal Equivalence)

PL metrics d, d' on (S,V) are discrete conformal,

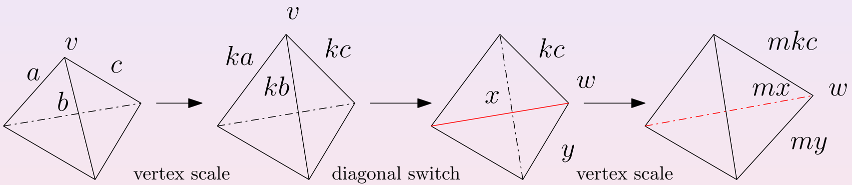
$$d \sim d'$$

if there is a sequence $d = d_1, d_2, \cdots, d_k = d'$ and $T_1, T_2, \cdots, T_k$ on (S,V), such that

1. T_i is Delaunay in d_i
2. if $T_i \neq T_{i+1}$, then $(S,d_i) \cong (S,d_{i+1})$ by an isometry homotopic to *id*
3. if $T_i = T_{i+1}$, $\exists u : V \to \mathbb{R}$, such that $\forall$ edge $e = [v_i, v_j]$,

$$l_{d_{i+1}}(e) = e^{u(v_i)} l_{d_i} e^{u(v_j)}$$

Discrete conformal metrics



Main Theorem

Theorem (Gu-Luo-Sun-Wu (2013))

$\forall$ PL metrics d on closed (S,V) and $\forall\bar{K}:V\to(-\infty,2\pi)$, such that $\sum\bar{K}(v)=2\pi\chi(S)$, $\exists$ a PL metric $\bar{d}$, unique up to scaling on (S,V), such that

1. *$\bar{d}$ is discrete conformal to d*
2. *The discrete curvature of $\bar{d}$ is $\bar{K}$.*

Furthermore, $\bar{d}$ can be found from d from a discrete curvature flow.

Remark

$\bar{K}=\frac{2\pi\chi(S)}{|V|}$, discrete uniformization.

Main Theorem

1. The uniqueness of the solution is obtained by the convexity of discrete surface Ricci energy and the convexity of the admissible conformal factor space (u-space).
2. The existence is given by the equivalence between PL metrics on (S, V) and the decorated hyperbolic metrics on (S, V) and the Ptolemy identity.

X. Gu, F. Luo, J. Sun, T. Wu, "A discrete uniformization theorem for polyhedral surfaces", Journal of Differential Geometry, Volume 109, Number 2 (2018), 223-256. (arXiv:1309.4175).

PL Metric Teichmüller Space

Definition (Marked Surface)

Suppose Σ is a closed topological surface, $V = \{v_1, v_2, \dots, v_n\} \subset \Sigma$ is a set of disjoint points on Σ, satisfying $\chi(\Sigma - V) < 0$.

Definition (Metric Equivalence)

Two polyhedral metrics d and d' are equivalent, if there is an isometric transformation $h : (\Sigma, V, d) \to (\Sigma, V, d')$, h is homotopic to the identity of the marked surface (Σ, V).

Definition (PL Teichmüller Space)

All the equivalence classes of the PL metrics on the marked surface (Σ, V) consist the Teichmüller space

$$T_{PL}(\Sigma, V) := \{d | polyhedralmetricon(\Sigma, V)\} / \{isometry \sim id(\Sigma, V)\}.$$

Definition (Local Chart for PL Teichmüller Space)

Assume $\mathscr{T}$ is a triangulation of (Σ, V), the edge length function determines a unique PL metric,

$$\Phi_{\mathscr{T}} : \mathbb{R}_{\triangle}^{E(\mathscr{T})} \to T_{PL}(\Sigma, V),$$

this gives a local coordinates of the PL Teichmüller space, where the domain

$$\mathbb{R}_{\triangle}^{E(\mathscr{T})} = \left\{ x \in \mathbb{R}_{>0}^{E(\mathscr{T})} | \forall \Delta = \{e_i, e_j, e_k\}, x(e_i) + x(e_j) > x(e_k) \right\}$$

is a convex set. We use $\mathscr{P}_{\mathscr{T}}$ to represent the image of $\Phi_{\mathscr{T}}$, then $(\mathscr{P}_{\mathscr{T}}, \Phi_{\mathscr{T}}^{-1})$ form a local chart of $T_{PL}(\Sigma, V)$.

PL Teichmüller Space

Definition (Atlas of PL metric Teichmüller Space)

Given a closed marked surface (Σ, V),the atlas of $T_{PL}(\Sigma, V)$ consists of all local charts $(\mathscr{P}_{\mathscr{T}}, \Phi_{\mathscr{T}}^{-1})$, where $\mathscr{T}$ exhaust all possible triangulations,

$$\mathscr{A}(T_{pl}(S,V)) = \bigcup_{\mathscr{T}} (\mathscr{P}_{\mathscr{T}}, \Phi_{\mathscr{T}}^{-1}).$$

From $|V|+|F|-|E| = 2-2g$ and $3|F| = 2|E|$, we obtain $|E| = 6g-6+3|V|$.

Theorem (Troyanov)

Given a closed marked surface (Σ, V), the PL metric Teichmüller space $T_{PL}(\Sigma, V)$ and the Euclidean space $\mathbb{R}^{6g-6+3|V|}$ is diffeomorphic.

Complete Hyperbolic Metric Teichmüller Space

Poincare Disk Model

The unit disk is with hyperbolic Riemannian metric

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2},$$

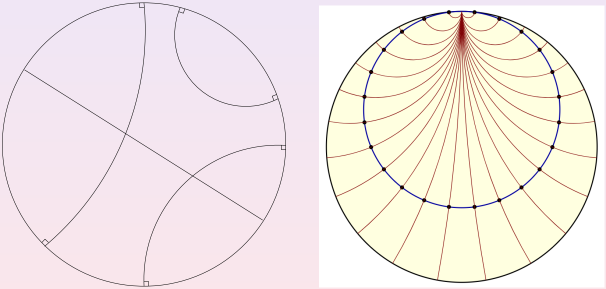


Figure: Hyperbolic geodesics in the Poincare model.

Upper Half Plane Model

The upper half plane is with hyperbolic Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

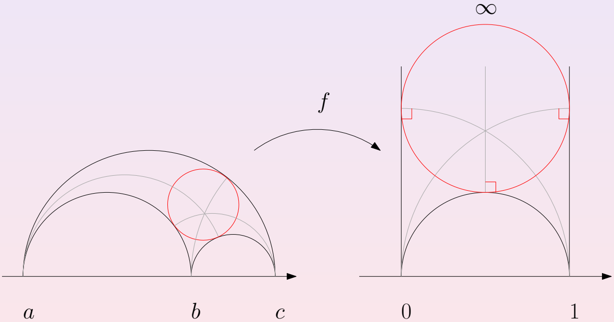


Figure: All hyperbolic ideal triangles are isometric¡$

Definition (Thurston's Shear Coordinates)

Given an ideal quadrilateral, Thurston's shear coordinates equal to the oriented distance from L to R along the diagonal.

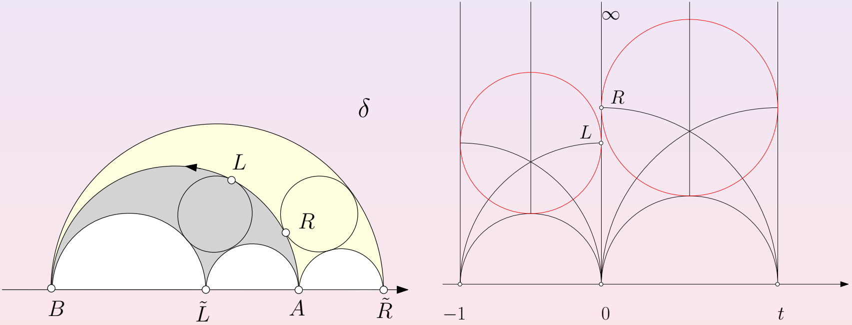


Figure: Hyperbolic Ideal Quadrilateral.

Definition (Thurston's Shear Coordinates)

Given an ideal quadrilateral, Thurston's shear coordinates equal to the oriented distance from L to R along the diagonal.

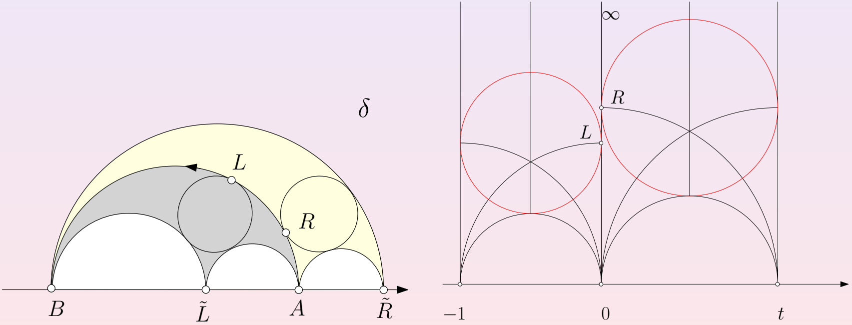


Figure: Hyperbolic Ideal Quadrilateral.

Construction of Hyperbolic Metric

Assume a genus g surface with n vertices removed, $\Sigma = \Sigma_g - \{v_1, v_2, \ldots, v_n\}, n \geq 1, \chi(\Sigma) < 0, (\Sigma, \mathscr{T})$ is a triangulation. Given a function defined on edges, $x : E(\mathscr{T}) \to \mathbb{R}$, construct a hyperbolic structure $\pi(X)$

1. for every triangle $\Delta \in \mathscr{T}$, construct a hyperbolic ideal triangle, $\Delta \to \Delta^*$;
2. for every edge $e \in E(\mathscr{T})$, adjacent to two faces $\Delta_1 \cap \Delta_2 = e$, glue two ideal triangles Δ_1^*ºÍΔ_2^* along e isometrically, the shear coordinates on e equals to $x(e)$.

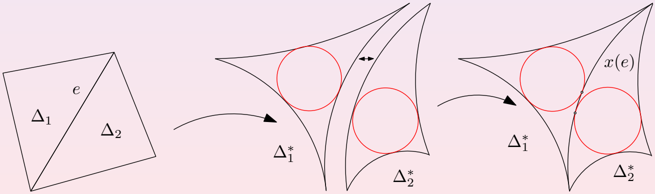


Figure: Construction of a complete metric.

Lemma

If $\pi(x)$ is a complete metric with finite area, namely each vertex becomes a cusp, then for each $v \in \{v_1, v_2, \dots, v_n\}$,

$$\sum_{e \sim v} x(e) = 0.$$

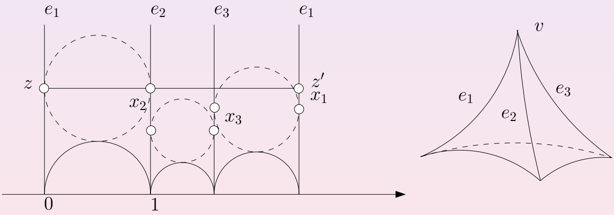


Figure: Condition for complete hyperbolic metric.

Define linear space:

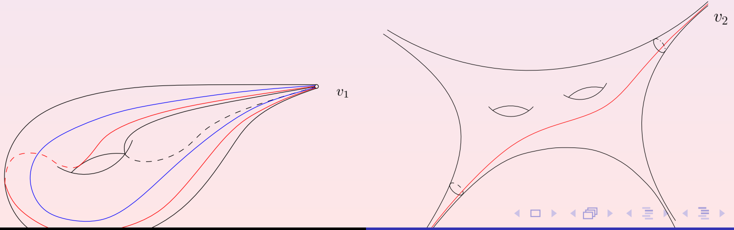
$$\mathbb{R}^E_P = \left\{ x \in \mathbb{R}^E | \forall v \in V, \sum_{v \sim e} x(e) = 0 \right\}$$

Theorem (Thurston)

The mapping

$$\Phi_{\mathscr{T}} : \mathbb{R}^E_P \to T(\Sigma), x \mapsto [\pi(x)]$$

is injective and surjective, $\Phi_{\mathscr{T}}(x)$ *under* $\mathscr{T}$ *has shear coordinates* $x(e)$*.*



Definition (Complete Hyperbolic Metric Teichmüller Space)

Given a closed marked surface (Σ, V) with genus g, $\chi(\Sigma - V) < 0$, all the complete hyperbolic metrics defined on $\Sigma - V$ with finite area, and each $v \in V$ being a cusp, form the hyperbolic metric Teichmüller space of $\Sigma - V$, denoted as $T_H(\Sigma, V)$.

From $|V| + |F| - |E| = 2 - 2g$ and $3|F| = 2|E|$, we obtain $|E| = 6g - 6 + 3|V|$. The cusp condition removes $|V|$ freedoms.

Corollary

The hyperbolic metric Teichmüller Space $T(\Sigma, V)$ is a real analytic manifold, diffeomorphic to $\mathbb{R}^{6g-6+2|V|}$, where g is the genus of the closed surface Σ.

Complete Hyperbolic Teichmüller Space

Definition (Complete Hyperbolic Metric Equivalence)

Two complete hyperbolic metrics $\mathbf{h}$ and $\mathbf{h}'$ on a closed marked surface (Σ, V) with finite total area are equivalent, if there is an isometric transformation

$$h : (\Sigma - V, \mathbf{h}) \rightarrow (\Sigma - V, \mathbf{d}'),$$

furthermore h is homotopic to the identity map of $\Sigma - V$.

Definition (Complete Hyperbolic Teichmüller Space)

Given a closed marked surface (Σ, V), $\chi(\Sigma - V) < 0$, all the equivalence classes of the complete hyperbolic metrics with finite area on (Σ, V) form the Teichmüller space:

$$T_H(\Sigma - V) = \{\mathbf{h} | \mathbf{h} compelete, finitearea\} / \{\text{isometry} \sim idof(\Sigma - V)\} \quad (4)$$

Complete Hyperbolic Metric Teichmüller Space

Definition (Local Chart of $T_H(\Sigma - V)$)

Assume $\mathscr{T}$ is a triangulation of (Σ, V), its shear coordinates determines a unique complete hyperbolic metric with finite area,

$$\Theta_{\mathscr{T}} : \Omega_{\mathscr{T}} \to T_H(\Sigma - V) \tag{5}$$

this gives a local chart of the Teichmüller space, where the domain $\Omega_{\mathscr{T}}$ is a sublinear space in $\mathbb{R}^{E(\mathscr{T})}$, satisfying the cusp conditions. Then $(\Omega_{\mathscr{T}}, \Theta_{\mathscr{T}}^{-1})$ form a local chart of $T_H(\Sigma - V)$.

Definition (Atlas of $T_H(\Sigma - V)$)

Each triangulation $\mathscr{T}$ of (Σ, V) corresponds to a local chart $(\Omega_{\mathscr{T}}, \Theta_{\mathscr{T}}^{-1})$. By exhausting all possible triangulations, the union of all local charts gives the atlas of $T_H(\Sigma - V)$:

$$\mathscr{A}(T_H(\Sigma - V)) = \bigcup_{\mathscr{T}} \left(\Omega_{\mathscr{T}}, \Theta_{\mathscr{T}}^{-1}\right).$$

Decorated Hyperbolic Metric Teichmüller Space

Decorated Ideal Hyperbolic Triangle

τ is a decorated ideal hyperbolic triangle, three infinite vertices are $v_1, v_2, v_3 \in \partial\mathbb{H}^2$. Each v_i is associated with a horoball H_i, the length of $\partial H_i \cap \tau$ is α_i; the oriented length of e_i is l_i: if $H_j \cap H_k = \emptyset$ then $l_i > 0$, otherwise $l_i < 0$. Penner's λ-length L_i is defined as

$$L_i := e^{\frac{1}{2}l_i}.$$

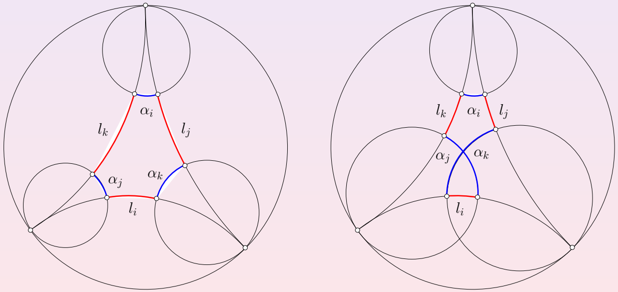


Figure: Decorated ideal hyperbolic triangle, left frame $l_i > 0$, right frame $l_i < 0$.

Definition (Decorated Hyperbolic Metric)

A decorated hyperbolic metric on a marked closed surface (Σ, V) is represented as (d, w):

1. d is a complete, with finite area hyperbolic metric;
2. each cusp v_i is associated with a haroball H_i. The center of H_i is v_i, the length of ∂H_i is w_i. $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n_{>0}$

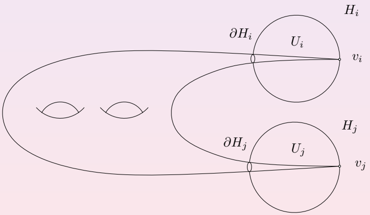


Figure: Decorated hyperbolic metric.

Definition (Decorated Hyperbolic Metric Equivalence)

Two decorated hyperbolic metric (d_1, w_1) and (d_2, w_2) on (Σ, V) are equivalent, if there is an isometric transformation h between them, h preserves all the horoballs and is isotopic to the identity map of $\Sigma - V$.

Definition (Decorated Hyperbolic Metric Teichmüller Space)

Given a closed marked surface (Σ, V), $\chi(S - V) < 0$, the decorated hyperbolic metric Teichmüller space of (Σ, V) is defined as

$$T_D(\Sigma, V) := \frac{\{(d,w) | \text{decorated hyperbolic metric}\}}{\{\text{isometry homotopic to id, preserves horoballs}\}}$$

Mappings Among Teichmüller Spaces

Theorem

Given a closed marked surface (Σ, V), $\chi(\Sigma - V) < 0$, the decorated hyperbolic metric Teichmüller space and the complete hyperbolic metric Techihmüller space has the relation:

$$T_D(\Sigma, V) = T_H(\Sigma, V) \times \mathbb{R}^{|}V|_{>0}.$$

where $\mathbb{R}^{|}V|_{>0}$ represents the length of the decoration ∂H_i.

Fix a triangulation $\mathscr{T}$ of (Σ, V), construct a mapping between the local charts determined by $\mathscr{T}$,

$$\Phi_{\mathscr{T}} : T_{PL}(\Sigma, V) \to T_D(\Sigma, V), x(e) \mapsto 2\ln x(e).$$

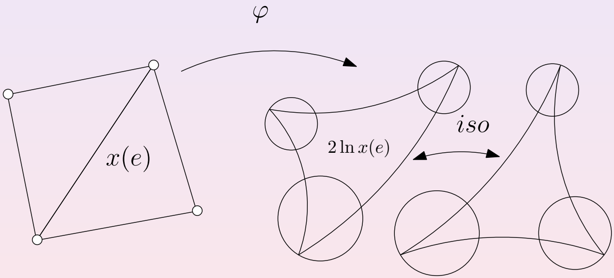


Figure: Euclidean metric to decorated hyperbolic metric.

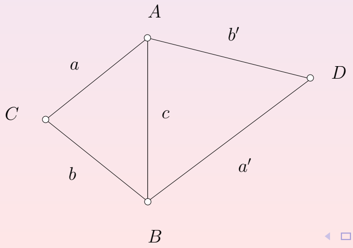
Euclidean Metric to Complete Hyperbolic Metric

Definition (Cross Ratio)

Given a marked surface with a PL metric and a triangulation $(\Sigma, \mathbf{d}, \mathscr{T})$, for a pair of adjacent faces$\{A,C,B\}$ ºÍ$\{A,B,D\}$ sharing the edge $\{A,B\}$, the cross ratio on the common edge is defined as:

$$\mathrm{Cr}(\{A,B\}) := \frac{aa'}{bb'},$$

where a, a', b, b' are the lengths of the edges $\{A,C\}, \{B,D\}, \{B,C\}, \{A,D\}$ under the PL metric $\mathbf{d}$.



Length cross ratio of $(\Sigma, V, d, \mathcal{T})$ satisfies the cusp condition, hence we can construct a mapping $\Psi_{\mathcal{T}} : T_{PL}(\Sigma, V) \to T_H(\Sigma, V)$, such that the shear coordinates of the complete hyperbolic metric equals to the length cross ratio of the PL metric.

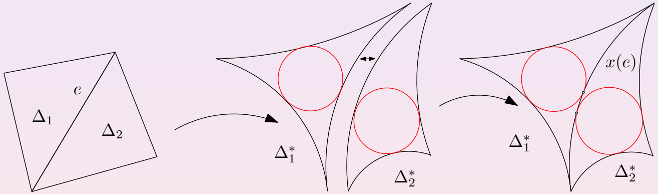


Figure: Euclidean metric to complete hyperbolic metric.

Consistency among the transformations

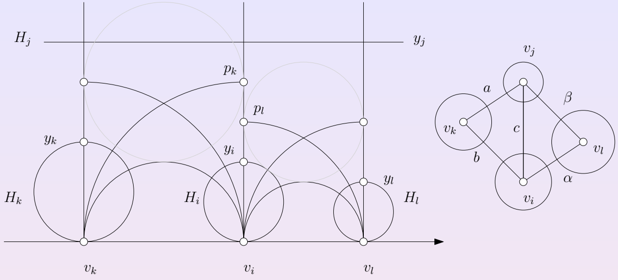


Figure: Cross ratio, Penner's λ length,shear coordinates.

Fix a triangulation $\mathscr{T}$,

$$\begin{array}{ccc} T_{PL}(\Sigma, V) & \xrightarrow{Cr} & T_{PL}(\Sigma, V) \\ \downarrow \Phi_{\mathscr{T}} & & \downarrow \Psi_{\mathscr{T}} \\ T_D(\Sigma, V) & \xrightarrow{Sh} & T_H(\Sigma, V) \end{array}$$

The above diagram commutes.

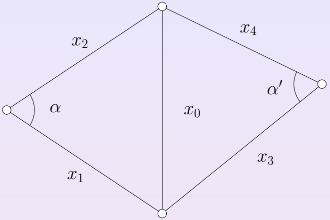


Figure: Euclidean Delaunay triangulation.

Definition (Euclidean Delaunay Triangulation)

Given a marked surface with a PL metric $(\Sigma, V, \mathbf{d})$, Delaunay triangulation $\mathscr{T}$ satisfies condition, for all edges $\alpha + \alpha' \leq \pi$. Equivalently $\cos\alpha + \cos\alpha' \geq 0$,

$$\frac{x_1^2 + x_2^2 - x_0^2}{2x_1x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3x_4} \geq 0. \quad (6)$$

Decorated Hyperbolic Delaunay Triangulation

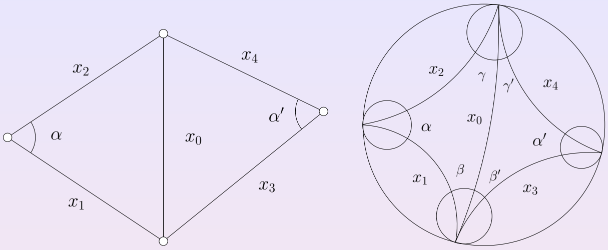


Figure: Delaunay triangulations.

Lemma

The transformation $\Phi_{\mathscr{T}} : T_{PL}(\Sigma, V) \to T_D(\Sigma, V)$ preserves Delaunay triangulations.

Since both situations:

$$\frac{x_1^2 + x_2^2 - x_0^2}{2x_1x_2} + \frac{x_3^2 + x_4^2 - x_0^2}{2x_3x_4} \geq 0. \quad (7)$$

Ptolemy Conditions

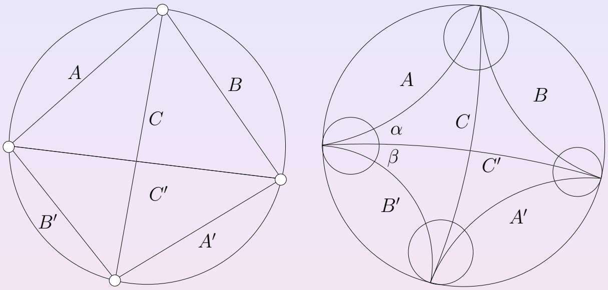


Figure: Ptolemy conditions.

Let A, A', B, B', C, C' are edge lengths of the Euclidean quadrilateral and the Penner's λ-length of the decorated hyperbolic ideal quarilateral, then both of them satisfy the Ptolemy conditions:

$$CC' = AA' + BB'.$$

Global defined mapping

The mapping $\Phi_{\mathscr{T}} : T_{PL}(\Sigma, V) \to T_D(\Sigma, V)$ is defined on each local chart, by Tolemy condition. By Ptolemy condition, all the locally defined mappings $\Phi_{\mathscr{T}}$ can be glued together to form a global map

$$\Phi : T_{PL}(\Sigma, V) \to T_D(\Sigma, V),$$

Ptolemy condition shows that the global mapping is continuous. Further computation shows that Φ is globally C^1.

Global Mapping

Define the cell decomposition of the Teichmüller spaces

$$T_{PL}(\Sigma, V) = \bigcup_{\mathscr{T}} C_{PL}(\mathscr{T})$$

where

$$C_{PL}(\mathscr{T}) := \{[d] \in T_{PL} | \mathscr{T} \text{is Delaunay under} d\}.$$

Similarly

$$T_D(\Sigma, V) = \bigcup_{\mathscr{T}} C_D(\mathscr{T})$$

where

$$C_D(\mathscr{T}) := \{[d] \in T_D | \mathscr{T} \text{is Delaunay under} d\}.$$

Inside the cells, the mapping $\Phi_{\mathscr{T}} : C_{PL}(\mathscr{T}) \to C_D(\mathscr{T})$ is a diffeomorphism.

Global Mapping

On the boundary of the cells, restricted on $C_{PL}(\mathscr{T}) \cap C_{PL}(\mathscr{T}')$, where four points are cocircle,

$$\begin{array}{ccc} C_{PL}(\mathscr{T}) & \xrightarrow{Euclidean\ Ptolemy} & C_{PL}(\mathscr{T}') \\ \downarrow \Phi_{\mathscr{T}} & & \downarrow \Phi_{\mathscr{T}'} \\ C_D(\mathscr{T}) & \xrightarrow{Hyperbolic\ Ptolemy} & C_D(\mathscr{T}') \end{array}$$

Furthermore,

$$\begin{array}{ccc} C_{PL}(\mathscr{T}) & \xrightarrow{Euclidean\ Ptolemy} & C_{PL}(\mathscr{T}') \\ \downarrow \nabla\Phi_{\mathscr{T}} & & \downarrow \nabla\Phi_{\mathscr{T}'} \\ C_D(\mathscr{T}) & \xrightarrow{Hyperbolic\ Ptolemy} & C_D(\mathscr{T}') \end{array}$$

the diagram commutes. So the piecewise diffeomorphism $\Phi_{\mathscr{T}}$ can be glued together to form a global C^1 map:

$$\Phi : T_{PL}(\Sigma, V) \to T_D(\Sigma, V).$$

Existence of Solution to Discrete Surface Ricci Flow

Existence Proof

Domain Ω_u is the space of discrete conformal factor,

$$\Omega_u = \mathbb{R}^n \cap \left\{ \mathbf{u} | \sum_{i=1}^{n} u_i = 0 \right\}.$$

The range Ω_K is the space of discrete curvatures,

$$\Omega_K = \left\{ \mathbf{K} \in (-\infty, 2\pi)^n | \sum_{i=1}^{n} K_i = 2\pi\chi(S) \right\}$$

both of them are open sets in $\mathbb{R}^{|V|-1}$. The global mapping is

$$F : \Omega_u \xrightarrow{\exp} \{p\} \times \mathbb{R}^{|V|}_{>0} \to T_D(\Sigma, V) \xrightarrow{\Phi^{-1}} T_{PL}(\Sigma, V) \xrightarrow{K} \Omega_K$$

Existence Proof

The global mapping is C^1,

$$F : \Omega_u \xrightarrow{\exp} \{p\} \times \mathbb{R}_{>0}^{|V|} \to T_D(\Sigma, V) \xrightarrow{\Phi^{-1}} T_{PL}(\Sigma, V) \xrightarrow{K} \Omega_K$$

During the flow, the triangulation is always Delaunay, the cotangent edge weight is non-negative, the discrete Laplace-Beltrami matrix is strictly positive definite. Hence the Hessian matrix of the energy

$$E(\mathbf{u}) = \int^{\mathbf{u}} \sum_{i=1}^{n} K_i du_i$$

is strictly convex. F is the gradient map of the energy,

$$F(\mathbf{u}) = \nabla E(\mathbf{u}),$$

because Ω_u is convex, the mapping is a diffeomorphism.

Convergence of Solutions to Discrete Surface Ricci Flow

Convergence Proof

Definition (δ triangulation)

Given a compact polyhedral surface (Σ, V, d), a triangulation $\mathscr{T}$ is a δ-triangulation, $\delta > 0$, if all the inner angles are in the interval $(\delta, \frac{\pi}{2} - \delta)$.

Definition ((δ, c)-triangulation)

Given a compact triangulated polyhedral surface (S, T, l^*), a geometric subdivision sequence (T_n, l_n^*) is a (δ, c) subdivision sequence, $\delta > 0$, $c > 0$, if each (T_n, l_n^*) is a δ triangulation, and the edge lengths satisfy

$$l_n^* e \in (\frac{1}{cn}, \frac{c}{n}), \forall e \in E(T_n)$$

Polyhedral surface can be replaced by a surface with a Riemannian metric, triangulation can be replaced by geodesic triangulation, then we obtain (δ, c) geodesic subdivision

Convergence Proof

Theorem (Discrete Surface Ricci Flow Convergence)

Given a simply connected Riemannian surface $(S,\mathbf{g})$ *with a single boundary, the inner angles at the three corners are* $\frac{\pi}{3}$. *Given a* (δ,c) *geodesic subdivision sequence* $(\mathcal{T}_n,L_n)$, *for any edge* $e\in E(T_n)$, $L_n(e)$ *is the geodesic length under the metric* $\mathbf{g}$. *There exists discrete conformal factor* $w_n\in\mathbb{R}^{V(\mathcal{T}_n)}$, *such that for large enough n,* $C_n=(S,\mathcal{T}_n,w_n*L_n)$ *satisfies*

a. C_n *is isometric to a planar equilateral triangle* $\triangle$, *and* C_n *is a* $\delta_S/2$*-triangulation*
b. *discrete uniformizations map* $\varphi_n:C_n\to\triangle$ *converge to the smooth uniformization map* $\varphi:(S,\mathbf{g})\to(\triangle,dzd\bar{z})$ *uniformly, such that*

$$\lim_{n\to\infty}\|\ \varphi_n|_{V(\mathcal{T}_n)}-\varphi|_{V(\mathcal{T}_n)}\ \|_\infty=0.$$

References

- X. Gu, F. Luo, J. Sun, T. Wu, “A discrete uniformization theorem for polyhedral surfaces”, Journal of Differential Geometry, Volume 109, Number 2 (2018), 223-256. (arXiv:1309.4175).
- X. Gu, F. Luo and T. Wu, “Convergence of Discrete Conformal Geometry and Computation of Uniformization Maps”, Asian Journal of Mathematics, 2017.

Computational Algorithms

Topological Quadrilateral

Topological Quadrilateral

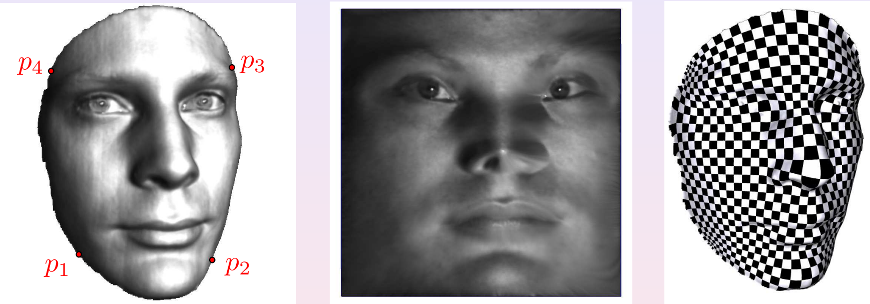


Figure: Topological quadrilateral.

Definition (Topological Quadrilateral)

Suppose S is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then S is called a topological quadrilateral, and denoted as $Q(p1, p2, p3, p4)$.

Theorem

Suppose $Q(p1, p2, p3, p4)$ is a topological quadrilateral with a Riemannian metric $\mathbf{g}$, then there exists a unique conformal map $\phi : S \to \mathbb{C}$, such that ϕ maps Q to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.

Algorithm: Topological Quadrilateral

Input: A topological quadrilateral M
Output: Conformal mapping from M to a planar rectangle
$\phi : M \to \mathbb{D}$

1. Set the target curvatures at corners $\{p_0, p_1, p_2, p_3\}$ to be $\frac{\pi}{2}$,
2. Set the target curvatures to be 0 everywhere else,
3. Run ricci flow to get the target conformal metric $\bar{\mathbf{u}}$,
4. Isometrically embed the surface using the target metric.

Topological Annulus

Topological Annulus

Figure: Topological annulus.

Definition (Topological Annulus)

Suppose S is a surface of genus zero with two boundaries, the S is called a topological annulus.

Theorem

Suppose S is a topological annulus with a Riemannian metric $\mathbf{g}$, the boundary of S are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius γ. Then $-\log\gamma$ is the conformal module of S. The mapping ϕ is unique up to a planar rotation.

Algorithm: Topological Annulus

Input: A topological annulus M, $\partial M = \gamma_0 - \gamma_1$
Output: a conformal mapping from the surface to a planar annulus $\phi : M \to \mathbb{A}$

1. Set the target curvature to be 0 every where,
2. Run Ricci flow to get the target metric,
3. Find the shortest path γ_2 connecting γ_0 and γ_1, slice M along γ_2 to obtain $\bar{M}$,
4. Isometrically embed $\bar{M}$ to the plane, further transform it to a flat annulus

$$\{z | r \leq Re(z) \leq 0\} / \{z \to z + 2k\sqrt{-1}\pi\}$$

   by planar translation and scaling,
5. Compute the exponential map $z \to \exp(z)$, maps the flat annulus to a canonical annulus.

Riemann Mapping

Conformal Module

Simply Connected Domains

Topological Disk

Definition (Topological Disk)

Suppose S is a surface of genus zero with one boundary, the S is called a topological disk.

Theorem

Suppose S is a topological disk with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical disk. The mapping ϕ is unique up to a Möbius transformation,

$$z \to e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

Algorithm: Topological Disk

Input: A topological disk M, an interior point $p \in M$
Output: Riemann mapping $\phi : M \to mathbbD$, maps M to the unit disk and p to the origin

1. Punch a small hole at p in the mesh M,
2. Use the algorithm for topological annulus to compute the conformal mapping.

Multiply connected domains

Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)

Suppose S is a surface of genus zero with multiple boundaries, then S is called a multiply connected annulus.

Theorem

Suppose S is a multiply connected annulus with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of S. Such kind of conformal mapping are unique up to Möbius transformations.

Algorithm: Multiply-Connected Annulus

Input: A multiply-connected annulus M,

$$\partial M = \gamma_0 - \gamma_1, \cdots, \gamma_n.$$

Output: a conformal mapping $\phi : M \to \mathbb{A}$, $\mathbb{A}$ is a circle domain.

1. Fill all the interior holes γ_1 to γ_n
2. Punch a hole at γ_k, $1 \leq k \leq n$
3. Conformally map the annulus to a planar canonical annulus
4. Fill the inner circular hole of the canonical annulus
5. Repeat step 2 through 4, each round choose different interior boundary γ_k. The holes become rounder and rounder, and converge to canonical circles.

Koebe's Iteration - I

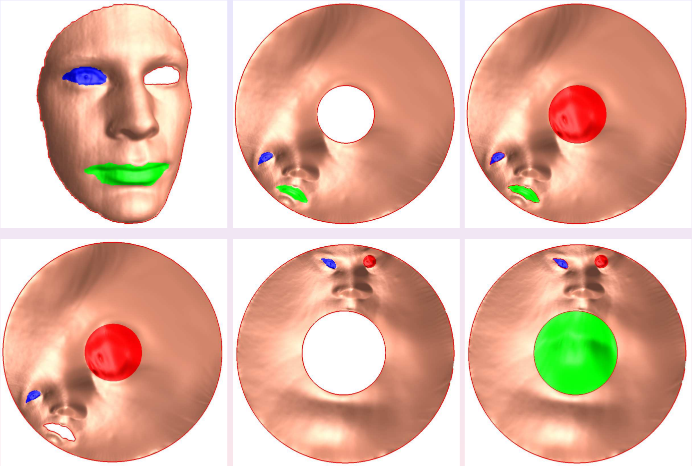


Figure: Koebe's iteration for computing conformal maps for multiply connected domains.

Koebe's Iteration - II

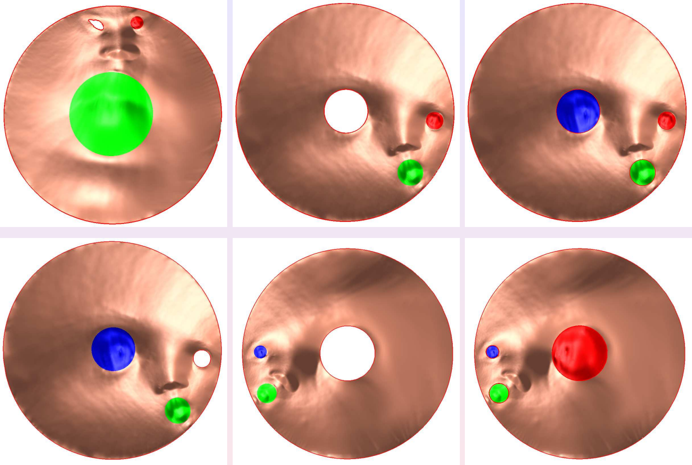


Figure: Koebe's iteration for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

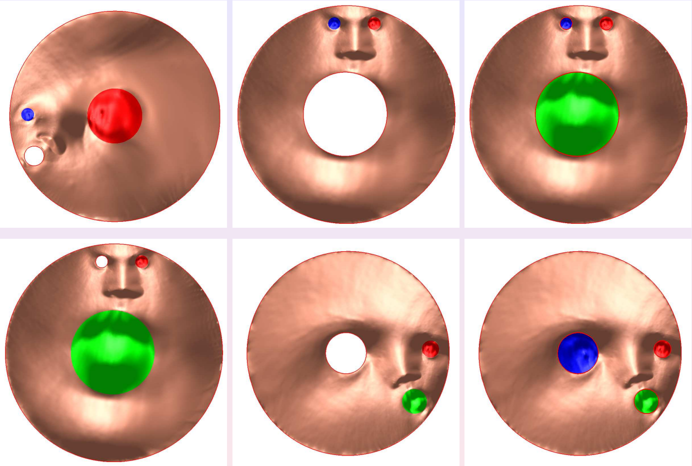


Figure: Koebe's iteration for computing conformal maps for multiply connected domains.

Convergence Analysis

Theorem (Gu and Luo 2009)

Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

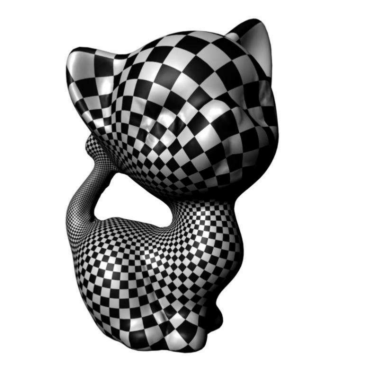
$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

W. Zeng, X. Yin, M. Zhang, F. Luo and X. Gu, "Generalized Koebe's method for conformal mapping multiply connected domains", Proceeding SPM'09 SIAM/ACM Joint Conference on Geometric and Physical Modeling, Pages 89-100.

Topological Torus

Topological torus



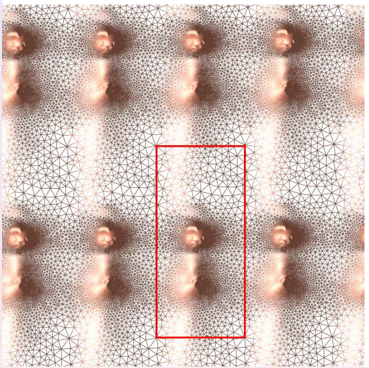


Figure: Genus one closed surface.

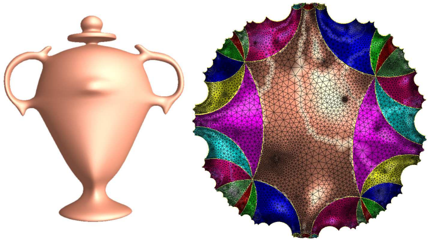
Algorithm: Topological Torus

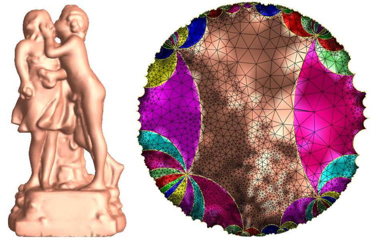
Input: A topological torus M

Output: A conformal mapping which maps M to a flat torus $\mathbb{C}/\{m+n\alpha|m,n\mathbb{Z}\}$

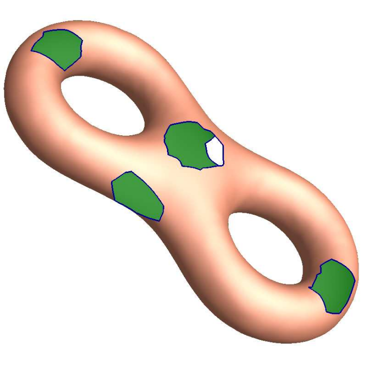
1. Compute a basis for the fundamental group $\pi_1(M)$, $\{\gamma_1,\gamma_2\}$.
2. Slice the surface along γ_1,γ_2 to get a fundamental domain $\bar{M}$,
3. Set the target curvature to be 0 everywhere
4. Run Ricci flow to get the flat metric
5. Isometrically embed $\tilde{S}$ to the plane

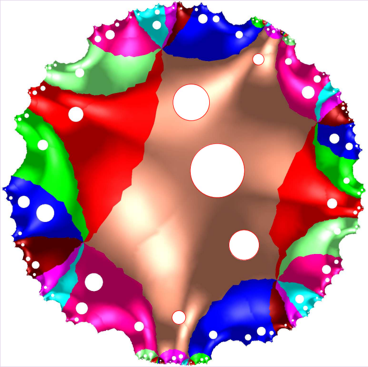
Computational results for genus 2 and genus 3 surfaces.





Hyperbolic Koebe's Iteration

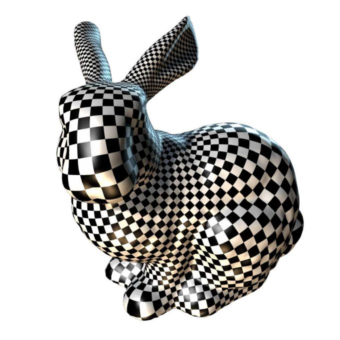




M. Zhang, Y. Li, W. Zeng and X. Gu. "Canonical conformal mapping for high genus surfaces with boundaries", Computer and Graphics, Vol 36, Issue 5, Pages 417-426, August 2012.

Thanks

For more information, please email to gu@cmsa.fas.harvard.edu.



Thank you!