Applications of Optimal Mass Transportation

David Gu[1]

[1]Computer Science Department
Stony Brook University, USA

Center of Mathematical Sciences and Applications
Harvard University

Geometric Computation and Applications
Trinity College, Dublin, Ireland

Optimal Transport Map - Computational Algorithm

1. Initialize $\mathbf{h} = \mathbf{0}$
2. Compute the Power Voronoi diagram, and the dual Power Delaunay Triangulation
3. Compute the cell areas, which gives the gradient ∇E
4. Compute the edge lengths and the dual edge lengths, which gives the Hessian matrix of E, $Hess(E)$
5. Solve linear system
$$\nabla E = Hess(E) d\mathbf{h}$$
6. Update the height vector
$$(h) \leftarrow \mathbf{h} - \lambda d\mathbf{h},$$
where λ is a constant to ensure that no cell disappears
7. Repeat step 2 through 6, until $\|d\mathbf{h}\| < \varepsilon$.

Applications

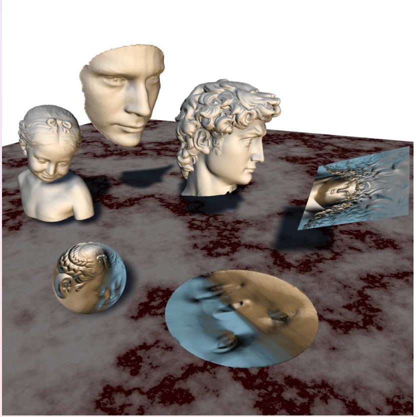
- Parameterization
- Visualization
- Registration
- Classification

Parameterizaion

Motivation

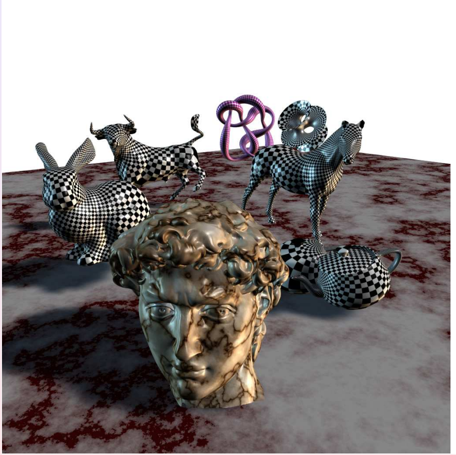
Definition (Parameterization)

Surface parameterization refers to the process of mapping a surface onto a planar (or spherical) domain.



Motivation

Parameterization can be applied for texture mapping, normal map and so on.

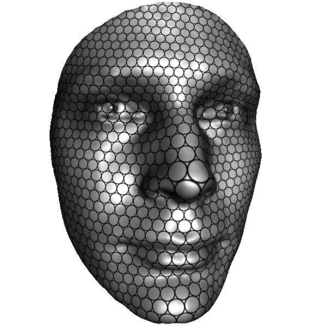


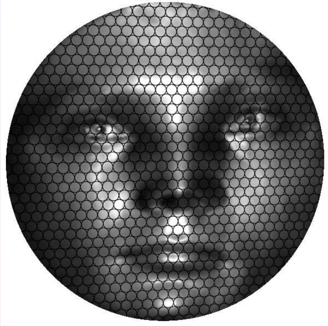
Motivation

- Unavoidably, surface parameterizations have to introduce distortions. There are two types of distortions: angle distortion and area distortion.
- If a parameterization preserves both angle and area, then the parameterization must be isometric, therefore it preserves Gaussian curvature everywhere.
- Therefore, general parameterization of curved surfaces can not preserve both angle and area.

Conformal Parameterization

Conformal parameterization: angle-preserving

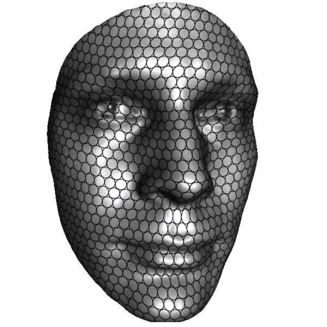


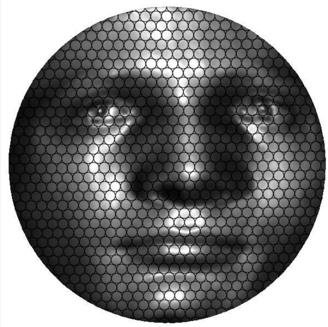


Infinitesimal circles are mapped to infinitesimal circle.

Area-preserving Parameterization

Area-preserving parameterization





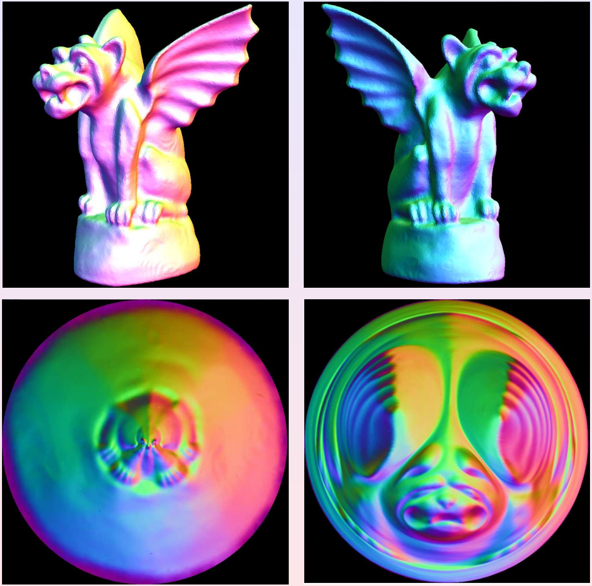
Infinitesimal ellipses are mapped to infinitesimal circles, preserving the areas.

Topological Disk Case

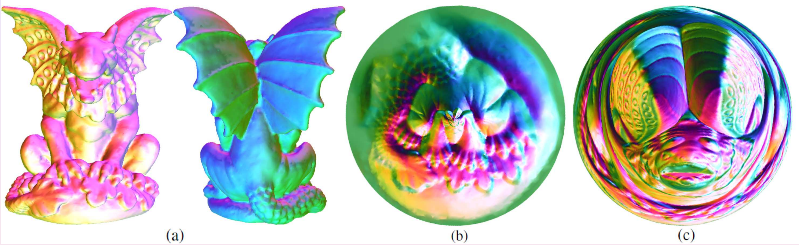
Area-Preserving Parameterization Algorithm

1. Compute conformal parameterization using Ricci flow
   $\varphi : M \to \mathbb{D}$
2. Compute the conformal factor at each vertex, treated as the measure
   $$\mu = e^{2\lambda(x,y)} dxdy$$
3. Compute the optimal mass transport map
   $$\psi : (\mathbb{D}, dxdy) \to (\mathbb{D}, e^{2\lambda} dxdy),$$
4. The composition
   $$\psi^{-1} \circ \varphi : M \to (\mathbb{D}, dxdy)$$
   is the area-preserving parameterization.

Surface Parameterization (topological disk)



Surface Parameterization (topological disk)

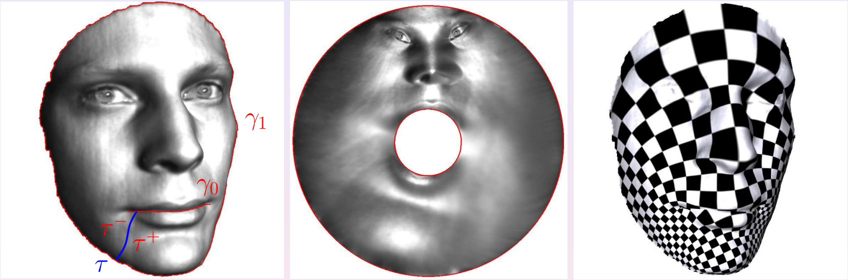


(a) Gargoyle model; (b) Angle-preserving; (c) Area-preserving.

Poly-Annulus Case

Computational Method - Koebe's iteration

Topological Annulus



Set the target curvatures to be zeros everywhere. Cut the surface along τ, isometrically embed the surface onto a planar rectangle, use complex exponential map to map it onto a planar annulus.

Koebe's Iteration - I

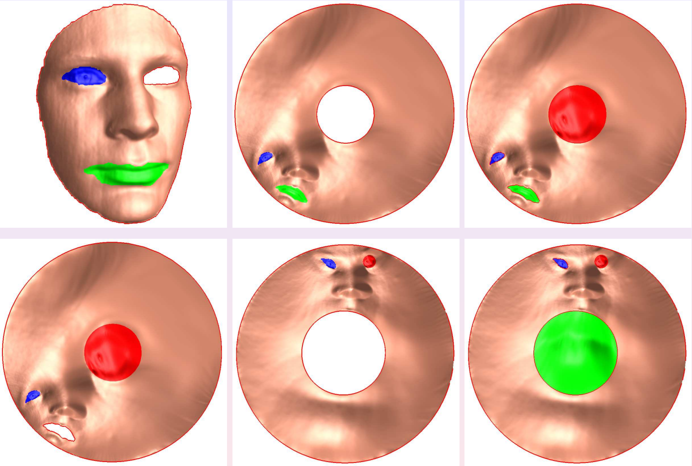


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - II

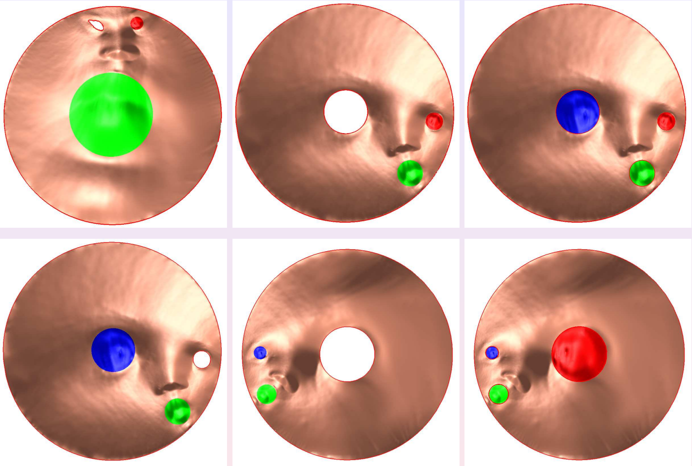


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

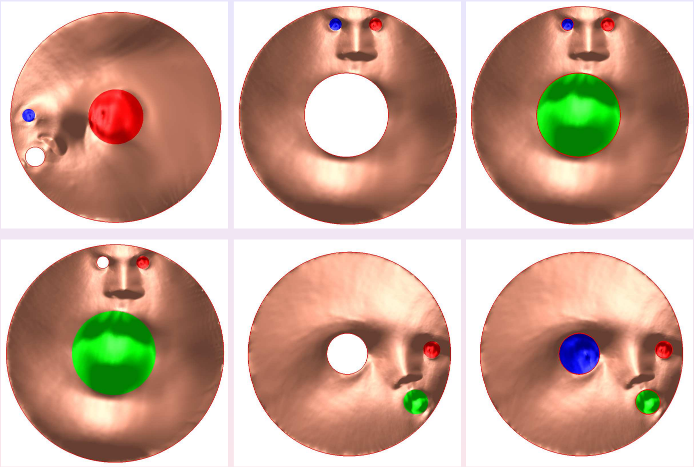


Figure: Koebe's method for computing conformal maps for multiply connected domains.

Theorem (Gu and Luo 2009)

Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

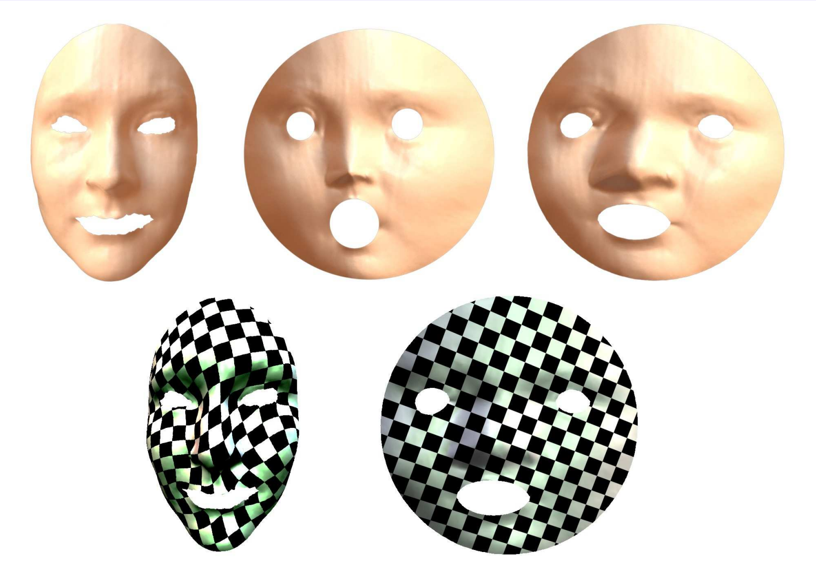
$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

Poly-Annulus Area-preserving Algorithm

1. Compute conformal parameterization, maps the mesh onto a planar circle domain
$$\varphi : M \to \mathbb{A}$$
2. Compute the conformal factor $e^{2\lambda}$ induced by the conformal mapping
3. Assign measure μ on $\mathbb{A}$,
$$\mu(p) = \begin{cases} e^{2\lambda(p)}dxdy & p \in \varphi(M) \\ \varepsilon & p \notin \varphi(M) \end{cases}$$
4. Compute the optimal mass transportation map
$$\psi : (\mathbb{D}, dxdy) \to (\mathbb{D}, \mu)$$
5. The composition $\psi^{-1} \circ \varphi : M \to \mathbb{D}$ is the area-preserving parameterization.

Area-Preserving Parameterization



Angle-preserving VS. Area-preserving

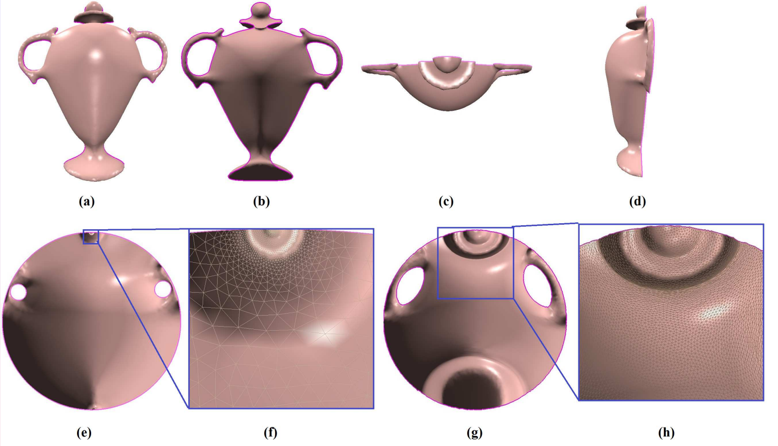
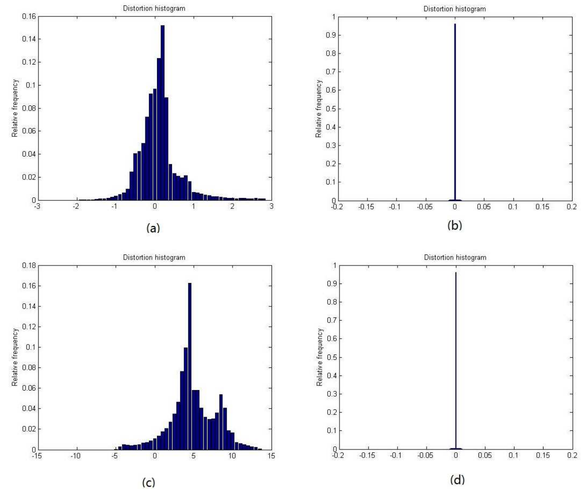


Figure: Angle-preserving and area-preserving parameterization - world cup model.

Distortions

Histograms of angle and area distortions of the face parameterizations.



Angle-preserving VS. Area-preserving

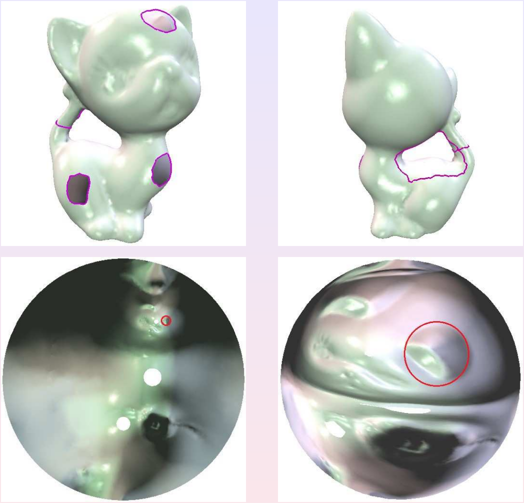


Figure: Angle-preserving and area-preserving parameterization - kitten model.

Angle-preserving VS. Area-preserving

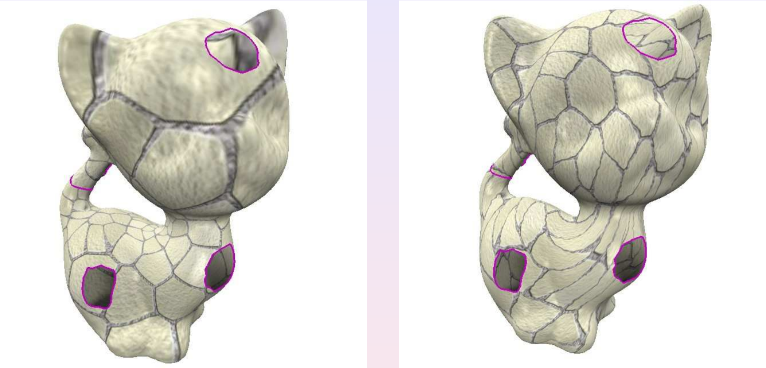


Figure: Angle-preserving and area-preserving parameterization - kitten model.

Angle-preserving VS. Area-preserving

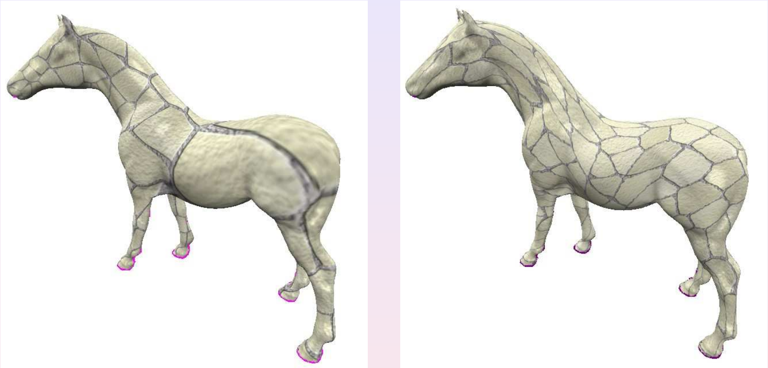


Figure: Angle-preserving and area-preserving texture mapping - horse model.

Area-Preserving texture mapping

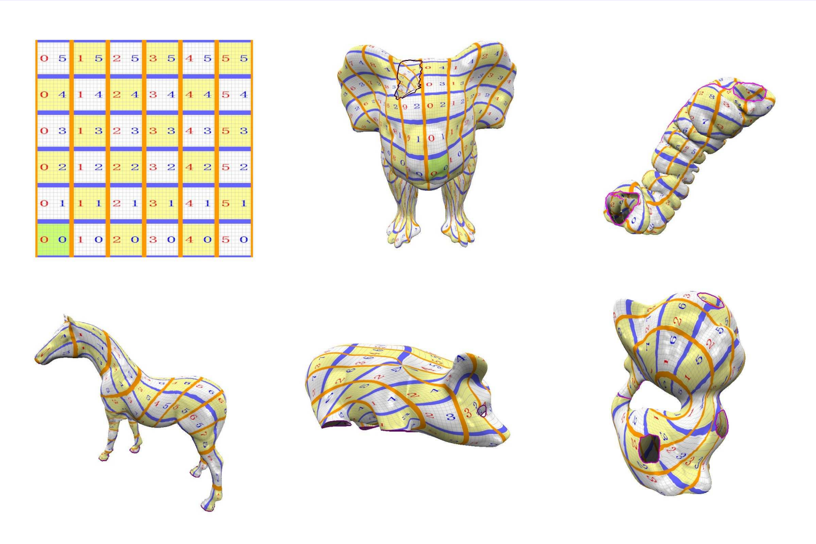


Figure: Texture mapping result based on area-preserving

Spherical Surface Case

Algorithm Pipeline

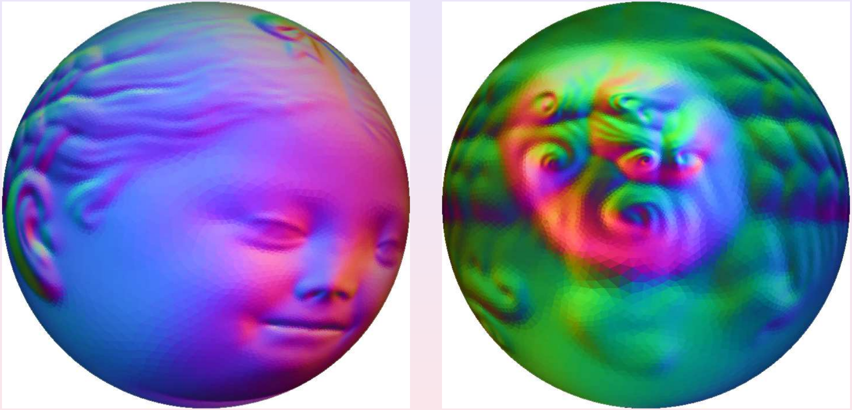
1. Choose three points $\{p_1, p_2, p_3\} \in S$, compute the unique comformal map $\phi : S \to \hat{\mathbb{C}}$, which maps $\{p_1, p_2, p_3\}$ to $\{0, 1, \infty\}$ respectively. where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
2. Compute the conformal factor $e^{2\lambda}$ induced by the conformal mapping.
3. Consider the unit sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$, Compute the stereographic projection $\psi : \mathbb{S}^2 \to \hat{\mathbb{C}}$. The induced metric is $\mathbf{h} = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}$.
4. Compute the the optimal mass transport map $\tau : ((\hat{\mathbb{C}}, \frac{4dzd\bar{z}}{(1+z\bar{z})^2}) \to (\hat{\mathbb{C}}, e^{2\lambda} dzd\bar{z}))$;
5. The composition $\psi^{-1} \circ \tau^{-1} \circ \phi : (S, \mathbf{g}) \to (\mathbb{S}^2, \mathbf{h})$ is an area-preserving mapping.

Spherical Area-Preserving Parameterization

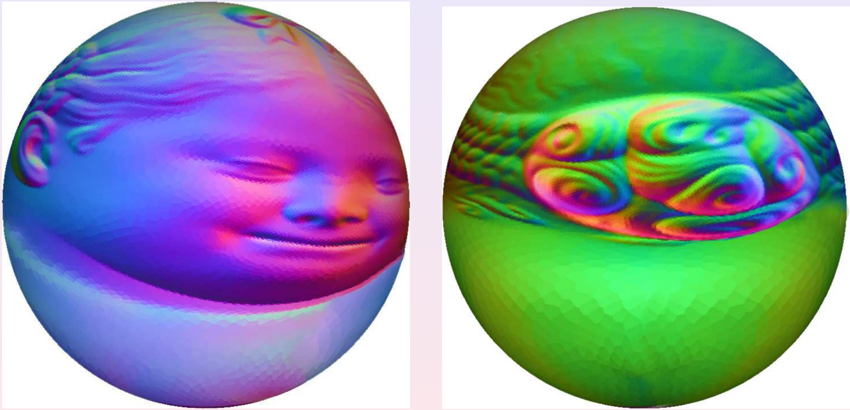
$$\begin{array}{ccc}
(S,\mathbf{g}) & \xrightarrow{\psi^{-1}\circ\tau^{-1}\circ\phi} & (\mathbb{S}^2,\mathbf{h}) \\
\Big\downarrow \phi & & \Big\downarrow \psi \\
(\hat{\mathbb{C}}, e^{2\lambda}dzd\bar{z}) & \xrightarrow{\tau^{-1}} & (\hat{\mathbb{C}}, \dfrac{4dzd\bar{z}}{(1+z\bar{z})^2})
\end{array}$$

Spherical Area-Preserving Parameterization


Spherical Area-Preserving Parameterization



Spherical Area-Preserving Parameterization



Spherical Area-Preserving Parameterization

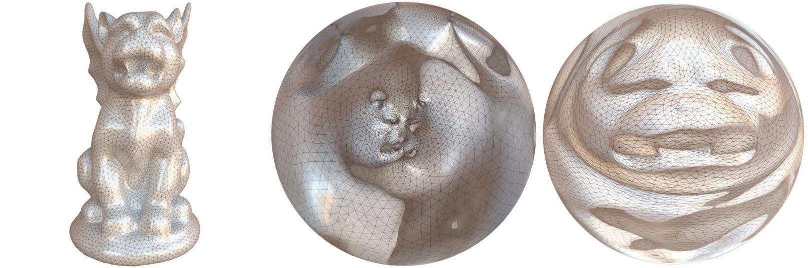


Figure: area-preserving parameterization of gargoyle model. left: gargoyle model; middle: initial conformal map; right: area-preserving map

Spherical Area-Preserving Parameterization

Figure: area-preserving parameterization of maxplanck model. left: maxplanck model; middle: initial conformal map; right: area-preserving map

Volume Preserving Parameterization

Topological Ball Case

Algorithm Pipeline

1. Use spherical harmonic map f to map the boundary of the topological ball ∂M to the unit sphere $\mathbb{S}^2$;
2. Compute a volumetric harmonic map $F : M \to \mathbb{D}^3$ with the Dirichlet boundary condition f;
3. Compute the discrete measures and construct the target point set P.
4. Compute the discrete optimal mass transportation map $\phi : \mathbb{D}^3 \to P$.
5. $\phi^{-1} \circ F : M \to \mathbb{D}^3$ gives the volume preserving parameterization.

Volumetric Parameterization

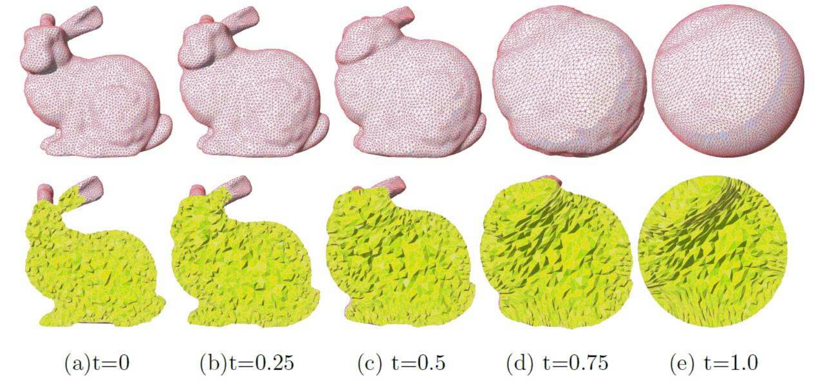


Figure: Volumetric morphing using our method.

Volumetric Parameterization

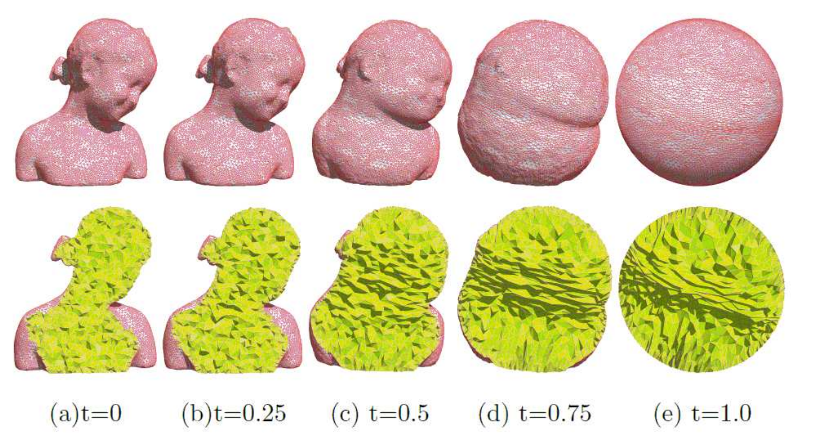


Figure: Volumetric morphing using our method.

Volumetric Parameterization

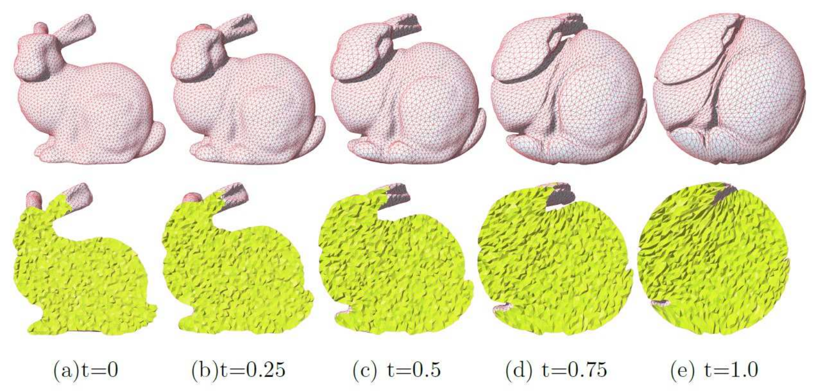
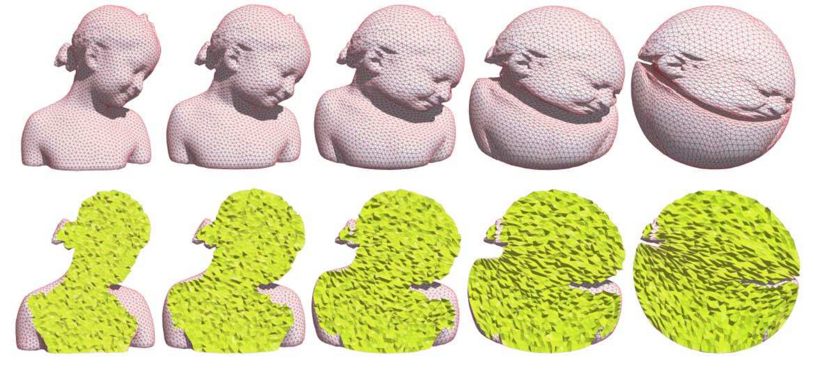


Figure: Volumetric morphing using Lévy's method.

Bruno Lévy, "A numerical algorithm for L^2 semi-discrete optimal transport in 3D", ESAIJ: Mathematical Modelling & Numerical Analysis 49(6) (2015) 1693-1715.

Volumetric Parameterization

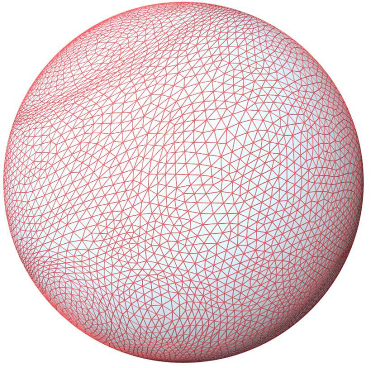


(a)t=0 (b)t=0.25 (c) t=0.5 (d) t=0.75 (e) t=1.0

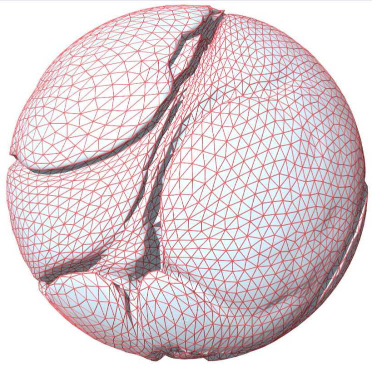
Figure: Volumetric morphing using Lévy's method.

Bruno Lévy, "A numerical algorithm for L^2 semi-discrete optimal transport in 3D", ESAIJ: Mathematical Modelling & Numerical Analysis 49(6) (2015) 1693-1715.

Volumetric Parameterization



our method



Levy's method

Figure: Volumetric morphing using different methods.

Complicated Topology Case

Volume-Preserving Parameterization

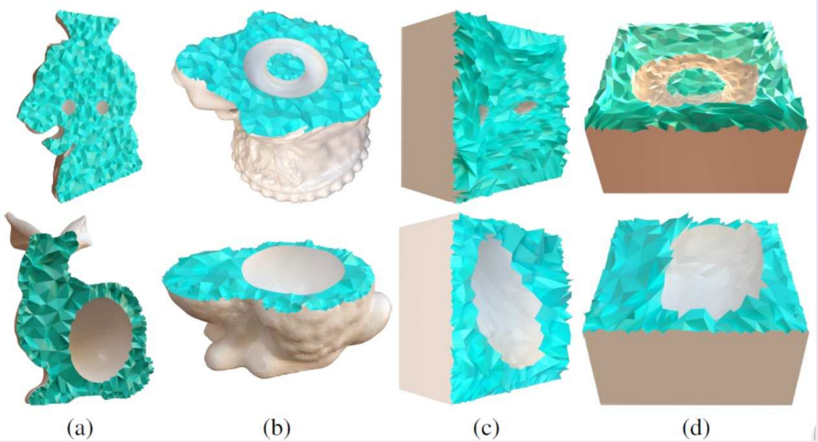


Figure: Volume-preserving parameterization of solids with complicated topologies.

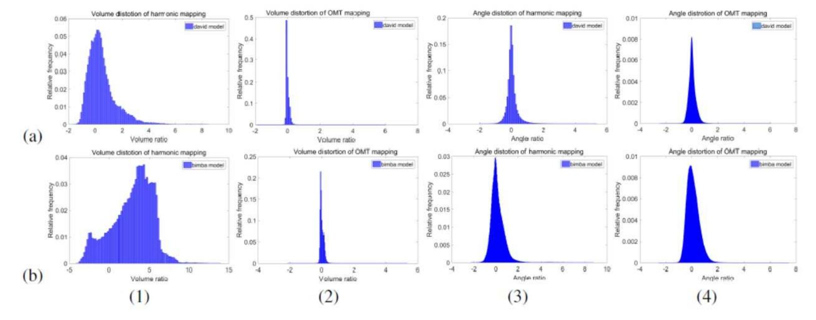


Figure: The comparison of volume distortions and dihedral angle distortions of Harmonic map and OMT map.

Visualization

Visualization

(a) Front view (b) Angle-preserving (c) Area-preserving (d) Back view

Angle-perserving parameterization vs. area-preserving parameterization

(a) 2x (b) 3x (c) 4x (d) 6x

Importance driven parameterization. The Buddha's head region is magnified by different factors

Visualization

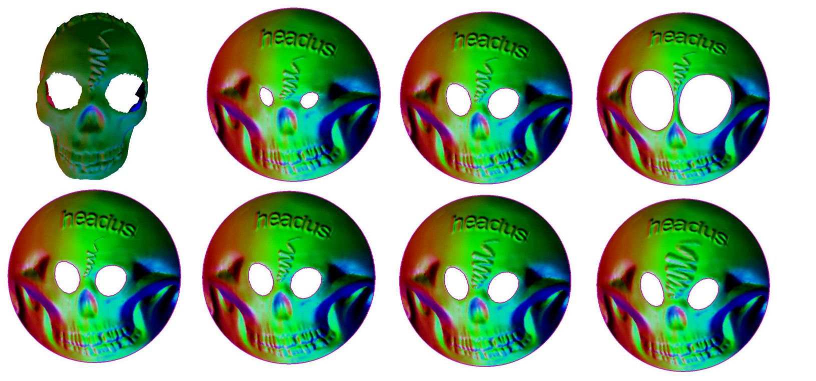


Figure: Adjust the target areas of ROIs.

Visualization

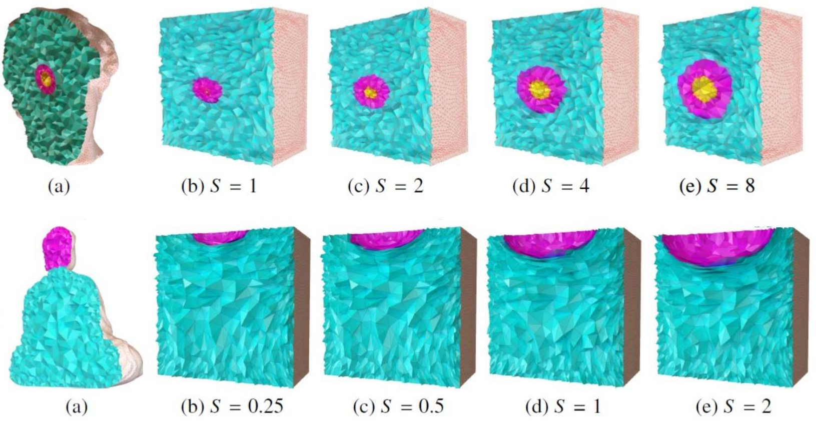


Figure: Importance driven results of ROIs with different scale factors.

Visualization

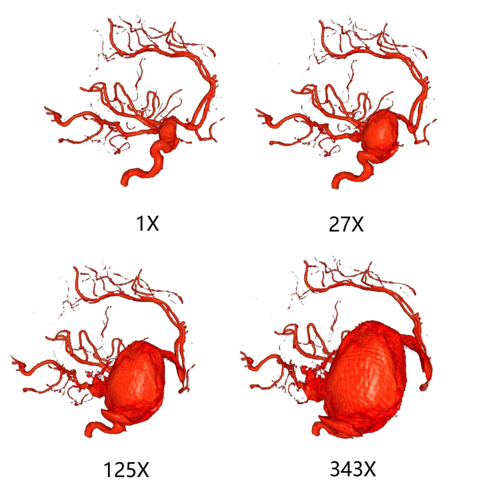
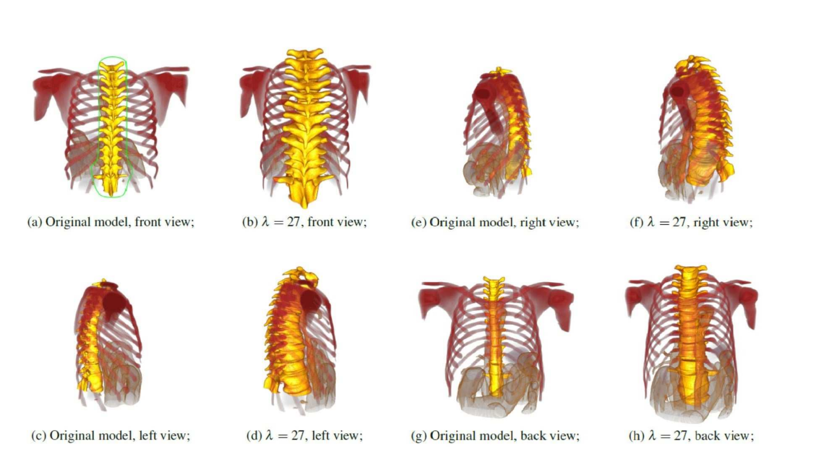


Figure: The volumetric Aneurism is magnified by large scales using the OMT technique.

Visualization



(a) Original model, front view; (b) $\lambda = 27$, front view; (e) Original model, right view; (f) $\lambda = 27$, right view;

(c) Original model, left view; (d) $\lambda = 27$, left view; (g) Original model, back view; (h) $\lambda = 27$, back view;

Figure: Adjust the target volumes of irregular regions of interest.

Visualization

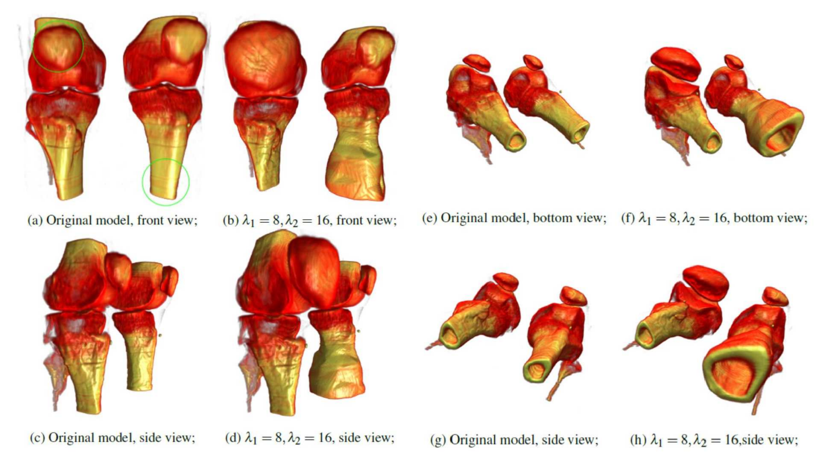


Figure: Adjust the target volumes of multiple focus regions.

Registration

Motivation

3D Surface registration plays a fundamental role in computer vision. It lays down the foundation of facial recognition, expression analysis, dynamic surface tracking, shape analysis, geometric data retrieval and many other important applications.

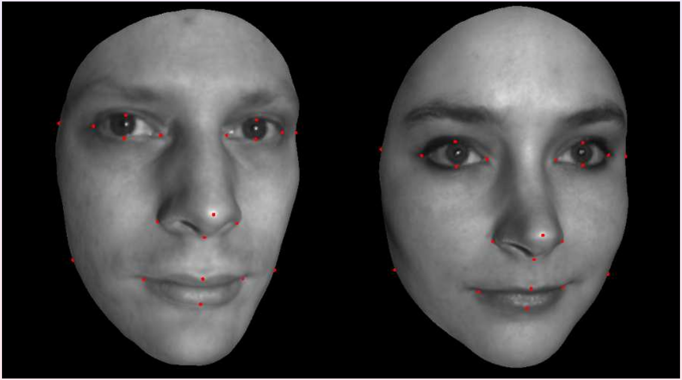


Figure: The marked metric 3D surfaces.

Surface Registration

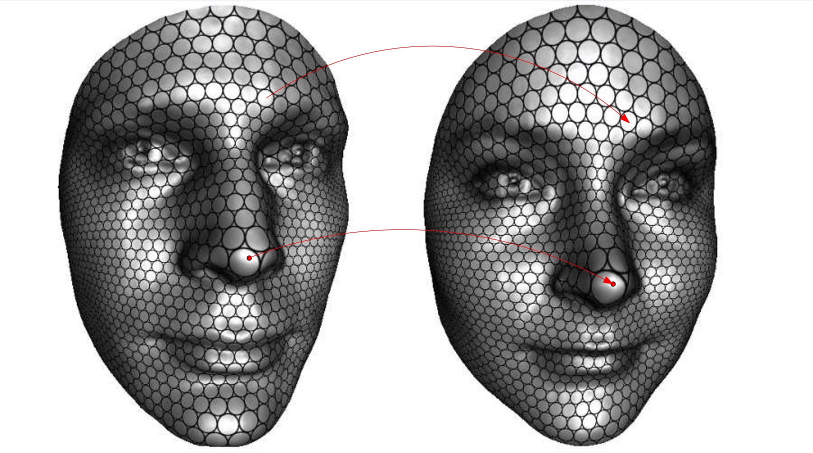
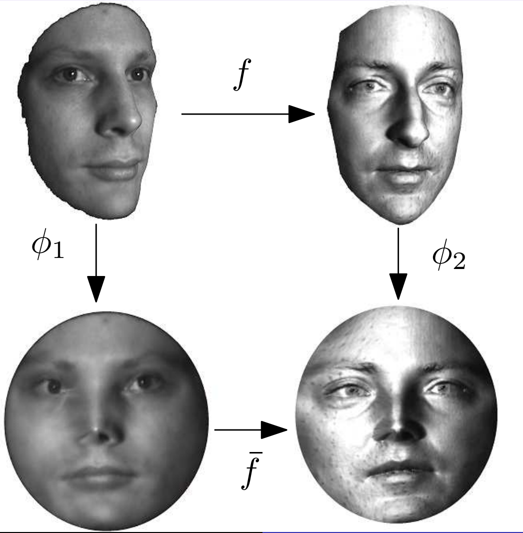


Figure: Corresponding features are mapped to each other, each small disk is mapped onto the corresponding ellipse.

Conformal Parameterization for Surface Matching

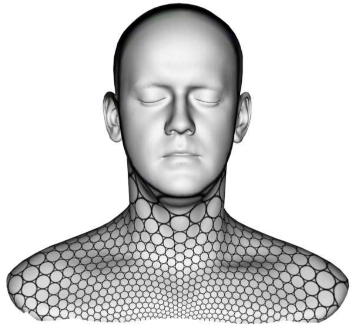
Existing method, 3D surface matching is converted to image matching by using conformal mappings.

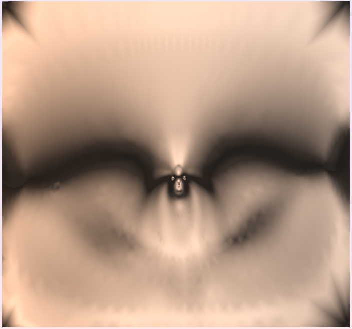


$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ \downarrow \phi_1 & & \downarrow \phi_2 \\ D_1 & \xrightarrow{\bar{f}} & D_2 \end{array}$$

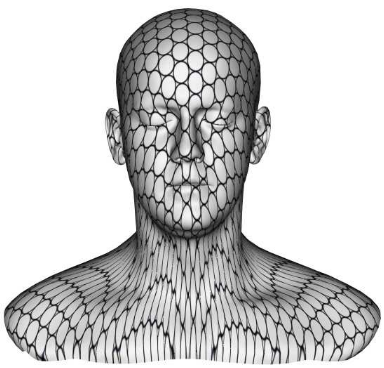
Conformal Parameterization for Surface Matching

Disadvantages: conformal parameterization may induce exponential area shrinkage, which produces numerical instability and matching mistakes.

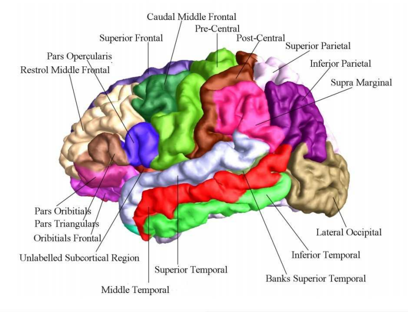


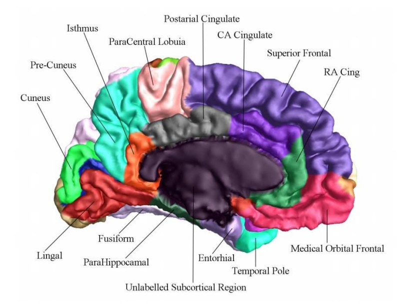


Advantage: the parameterization is area-preserving, improves the robustness.

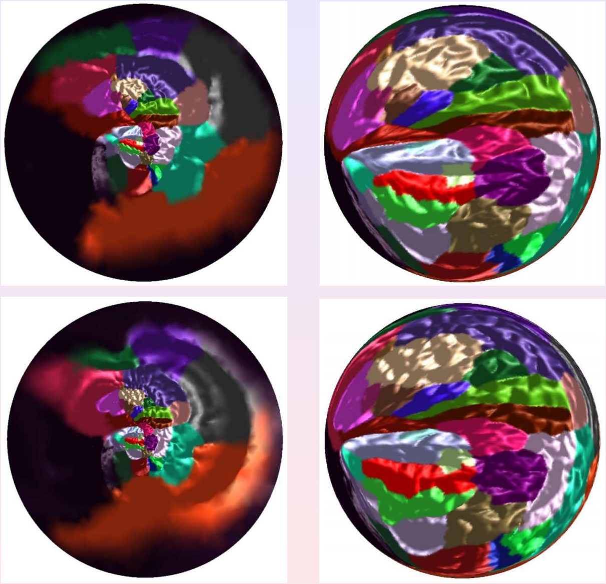


Optimal Transport Map Examples





Optimal Transport Map Examples



Registration with Large Deformations

Disadvantages: Registration based on optimal mass transportation mapping is good for isometric surface registration; for surfaces with large deformations, land mark constraints need to be added. Conventional optimal mass transportation map can not incorporate the marker constraints directly.

Central Task

Main Problem

Find robust algorithm to register surfaces with large deformations and landmark constraints.

Solutions

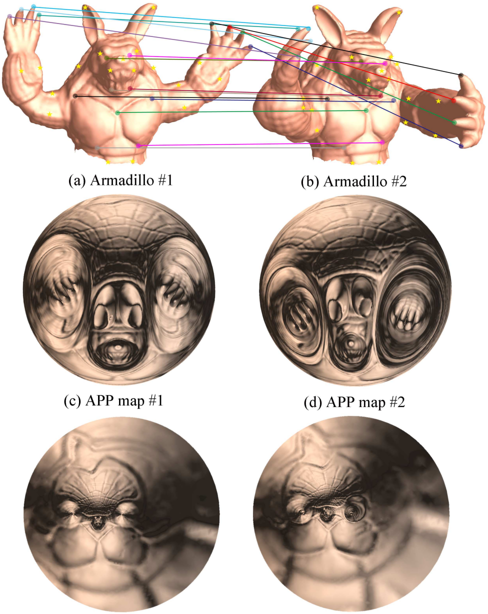
Optimal mass transportation map plus Teichmüller map.

Registration based on OMT and Teichmüller Map

Assume the surfaces are S_1 and S_2, with landmarks $\{p_i\} \subset S_1$ and $\{q_j\} \subset S_2$,

1. Compute conformal mappings $\varphi_k : S_k \to \mathbb{D}$, $k = 1, 2$, the conformal factors are $\lambda_k : S_k \to \mathbb{R}$;
2. Compute the optimal transportation maps $\psi_k : (\mathbb{D}_k, dxdy) \to (\mathbb{D}, e^{2\lambda_k} dxdy)$;
3. Compute the Teichmüller map with landmark constraints
$$\tau : \mathbb{D} \to \mathbb{D}, \tau(\psi_1 \circ \varphi_1(p_i)) = \psi_2 \circ \varphi_2(q_i).$$
4. The registration is given by
$$\varphi_2^{-1} \circ \psi_2 \circ \tau \circ \psi_1^{-1} \circ \varphi_1 : (S_1, \{p_i\}) \to (S_2, \{q_i\}).$$

Surface Registration

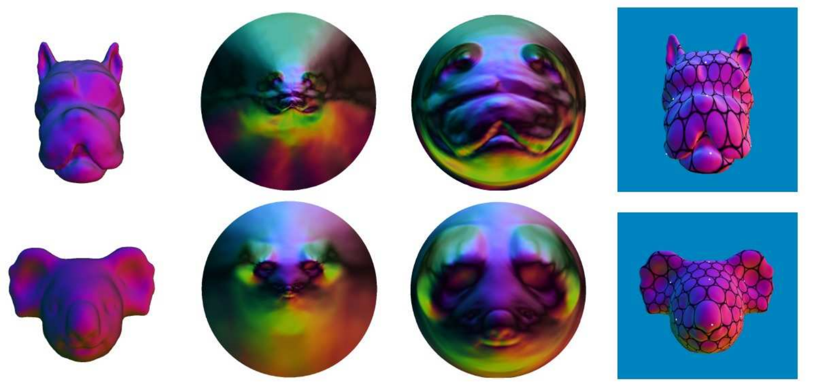


(a) Armadillo #1 (b) Armadillo #2

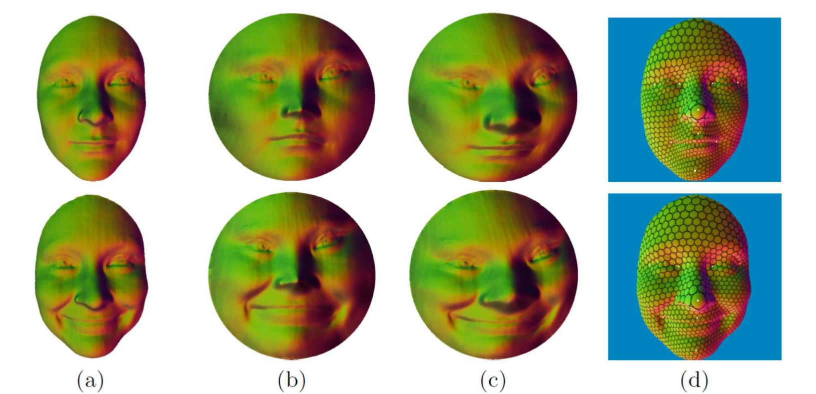
(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

Registration based on OMT and Teichmüller Map



Registration based on OMT and Teichmüller Map



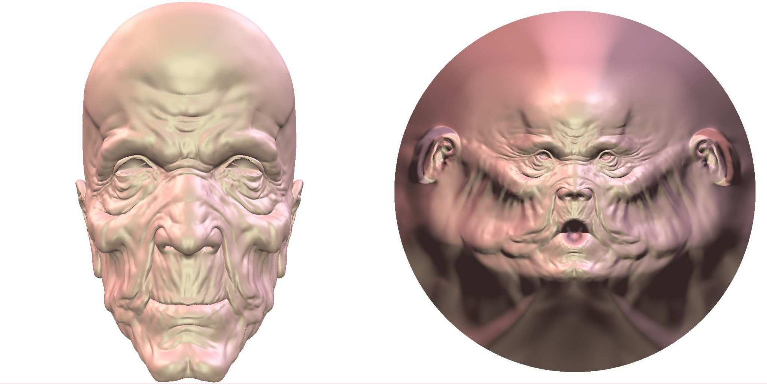
Curvature Sensitive Remeshing

Curvature Sensitive Remeshing

Algorithm Pipeline

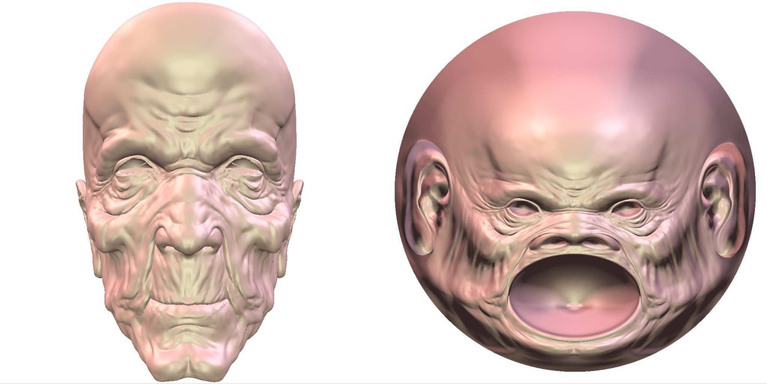
1. Compute the conformal parameterization of the input surface $\varphi : (S, \mathbf{g}) \to (\mathbb{D}, dzd\bar{z})$,
2. Compute the optimal mass transportation map $\psi : (\mathbb{D}, dxdy) \to (\mathbb{D}, \mu)$, where μ is the combination of the surface area element $e^{2\lambda}dxdy$ and the absolute value of the Gaussian curvature measure $|K(x,y)|dxdy$,
3. Uniformly sample on the preimage of the OMT map $\psi^{-1}(\mathbb{D})$,
4. Pull back the samples to the conformal parameter domain $\varphi(S)$, compute the Delaunay triangulation T,
5. Pull back the triangulation T to the original surface S, which induces the remeshing of S.

Curvature Sensitive Parameterization(CSP)



Original face mesh Conformal parameteraiztion

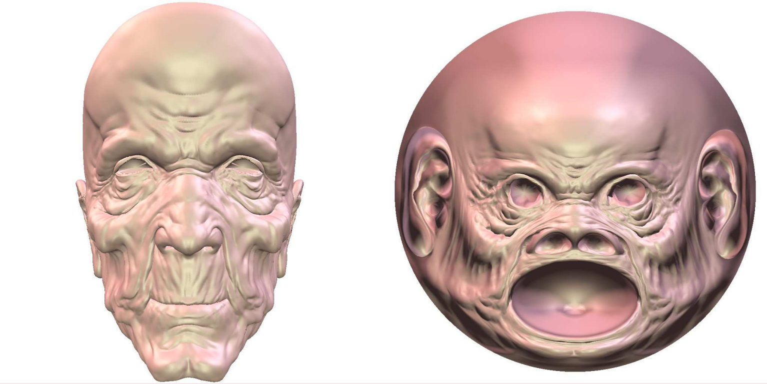
Curvature Sensitive Parameterization(CSP)



Original face mesh

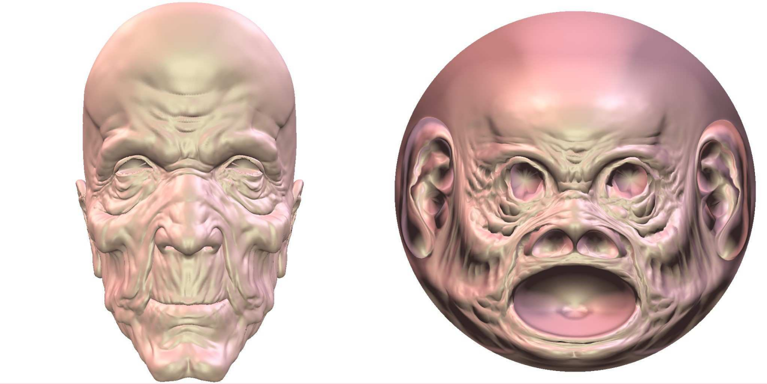
Area preserving parameteraiztion

Curvature Sensitive Parameterization(CSP)



Original face mesh   CSP: area + curvature$\times$0.1

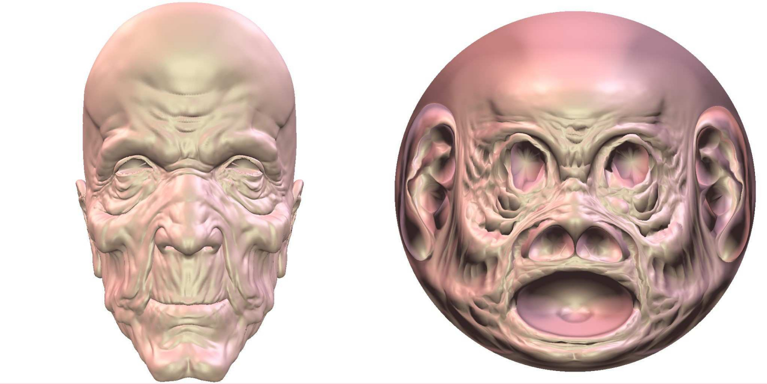
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature $\times$ 0.2

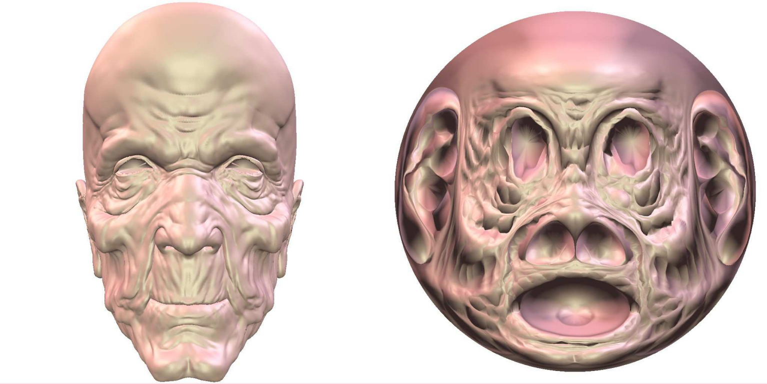
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.4

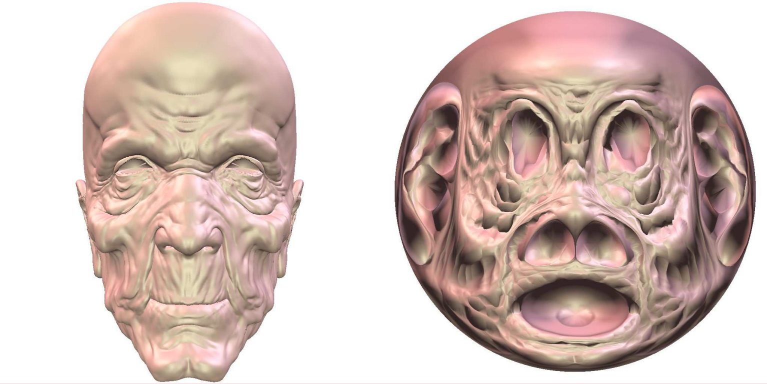
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$0.8

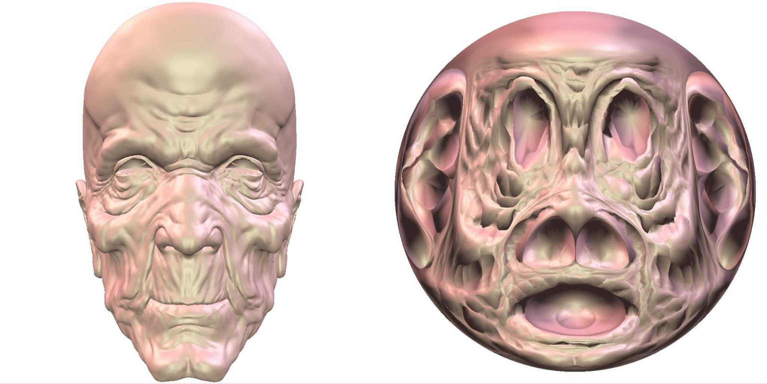
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature $\times 1.0$

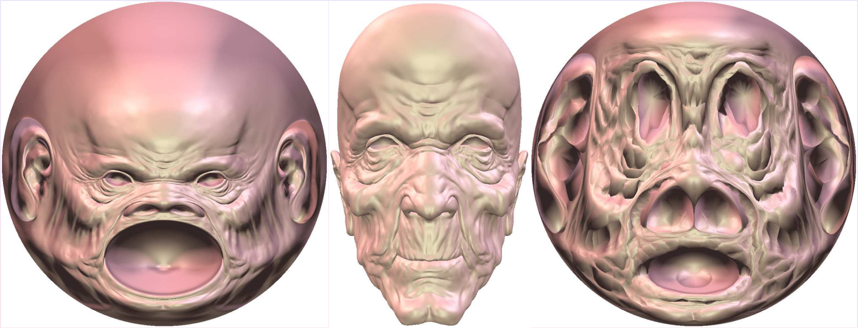
Curvature Sensitive Parameterization(CSP)



Original face mesh

CSP: area + curvature$\times$2.0

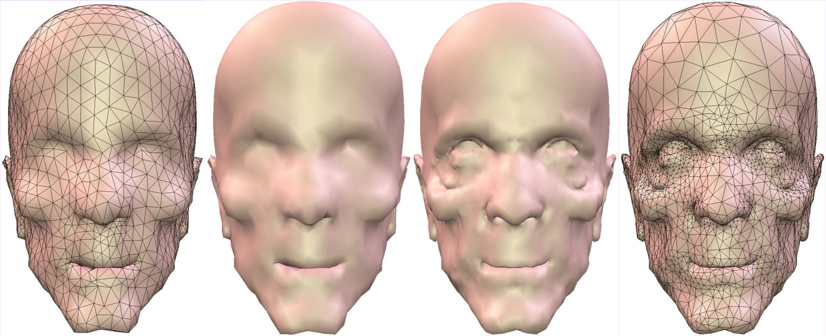
Compare Multi-scale remeshing between APP and CSP



(a) APP (b) original mesh(140K) (c) CSP

Compare Multi-scale remeshing between APP and CSP

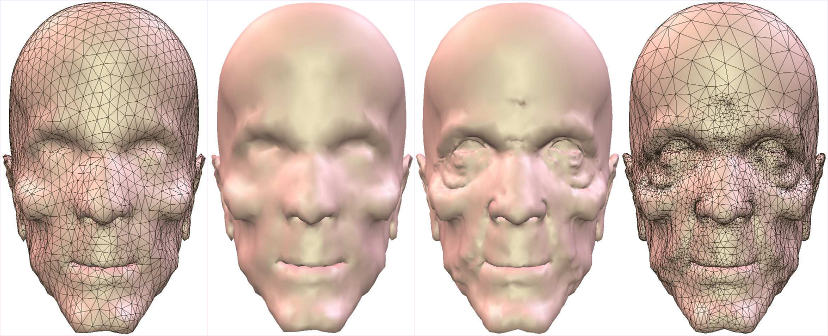
Remeshing: 1K vertices



(a) APP wireframe (b) APP smooth (c) CSP smooth (d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 2K vertices



(a) APP wireframe

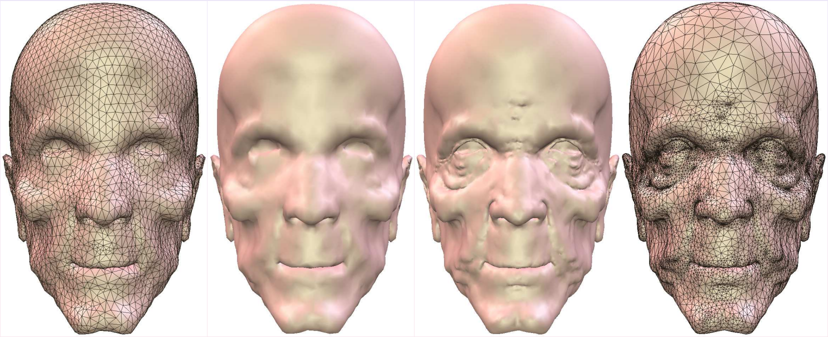
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 4K vertices



(a) APP wireframe

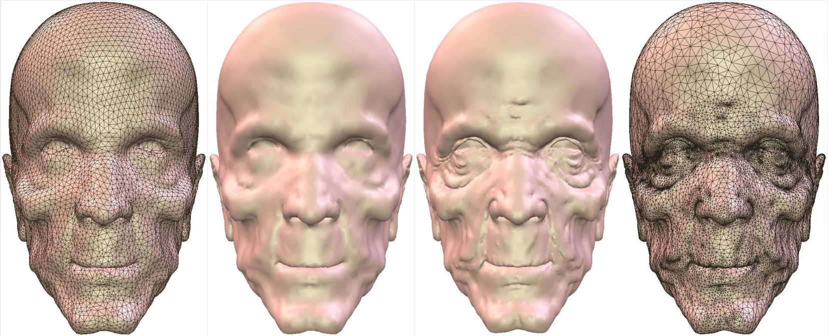
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 8K vertices



(a) APP wireframe

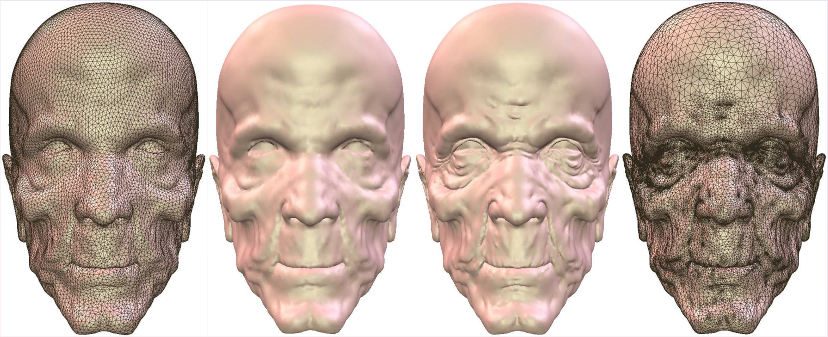
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

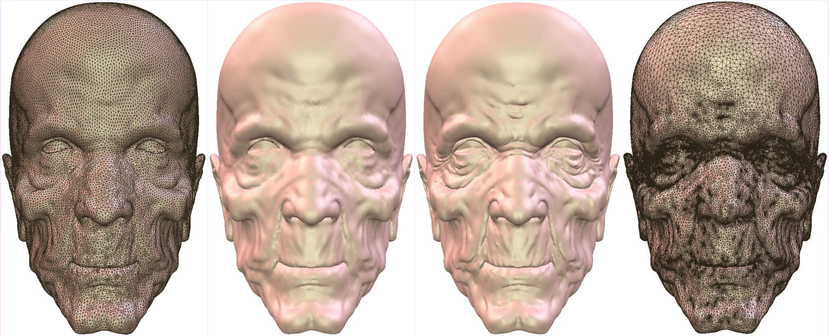
Remeshing: 16K vertices



(a) APP wireframe | (b) APP smooth | (c) CSP smooth | (d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 32K vertices



(a) APP wireframe

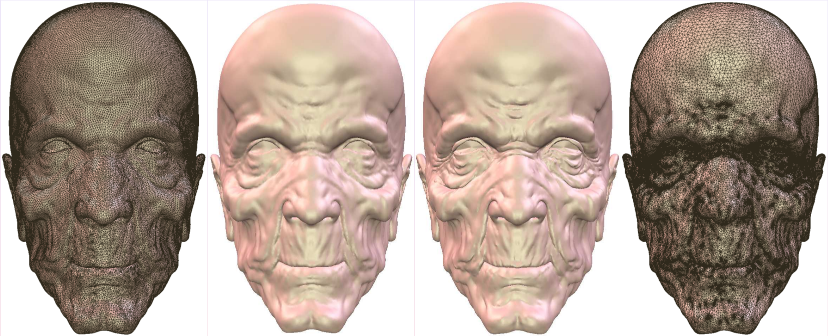
(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Compare Multi-scale remeshing between APP and CSP

Remeshing: 64K vertices



(a) APP wireframe

(b) APP smooth

(c) CSP smooth

(d) CSP wireframe

Normal Map

Normal Map

In game industry, the speed of rendering is determined by the number of the faces of a mesh. In 3D computer graphics, normal mapping is a technique used for faking the lighting of bumps and dents. It is used to add details without using more polygons. A common use of this technique is to greatly enhance the appearance and details of a low polygon model by generating a normal map from a high polygon model.

Normal Map

Algorithm Pipeline

1. Simplify the original high resolution mesh M to a low resolution mesh $\tilde{M}$;
2. Parameterize the low resolution model to a planar domain using OMT map, $\varphi : \tilde{M} \to \mathbb{D}$;
3. Compute the normal at each vertex of the high resolution model;
4. Compute the closest point map from the high-res model to the low-res model $\psi : M \to \tilde{M}$; This composes the map from the high-res model to the parameter domain of the low-res model $\varphi \circ \psi : M \to \mathbb{D}$;
5. Copy the normal to the high-res model to the parameter domain to generate the normal map.

Normal Map

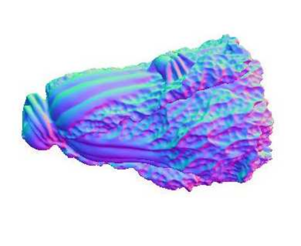
original mesh with 490k faces

simplified mesh with 1k faces

original mesh with 5k faces

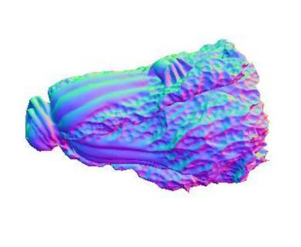
simplified mesh with 10k faces

Normal Map

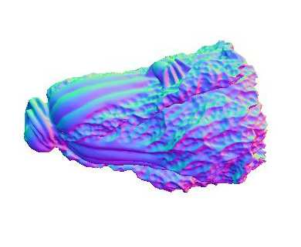


original mesh with 323k faces

simplified mesh with 1k faces



original mesh with 5k faces



simplified mesh with 10k faces

Normal Map

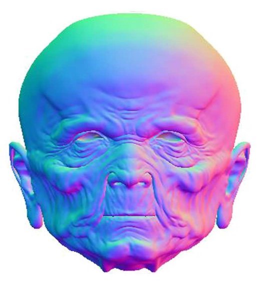
original mesh with 95k faces

simplified mesh with 1k faces

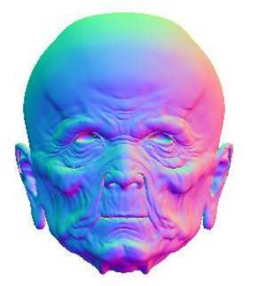
original mesh with 5k faces

simplified mesh with 10k faces

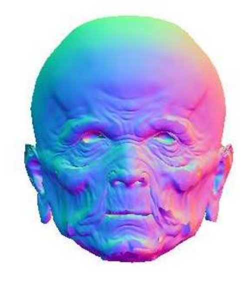
Normal Map



original mesh
with 270k faces



simplified mesh
with 7k faces



simplified mesh
with 1600 faces

Shape Classification

Introduction

- Shape classification plays a fundamental role in computer vision and medical imaging.
- We use Wasserstein distance based on optimal mass transport theory for shape classification.
- We apply our method to expression classification, brain morphometry, and demonstrate the potential of our method on the discriminative analysis of hippocampal shape in epilepsy.

Wasserstein Distance

Given a metric surface $(S, \mathbf{g})$, a Riemann mapping $\varphi : (S, \mathbf{g}) \to \mathbb{D}^2$, the conformal factor $e^{2\lambda}$ gives a probability measure on the disk. The shape distance is given by the Wasserstein distance.

The transportation cost of the optimal mass transport map defines the Wasserstein distance between two surfaces.

It intrinsically measures the dissimilarities between two surfaces.

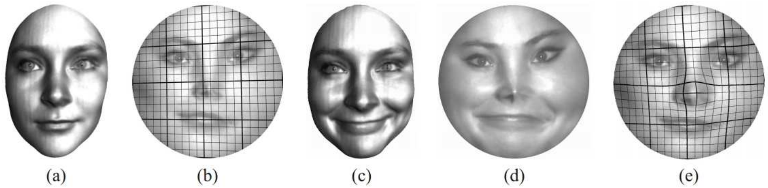


Fig. 9: **The computation of Wasserstein distance**

Wasserstein Distance Algorithm

Input: Two topological disk surfaces $(M_1, g_1), (M_2, g_2)$.
Output: The Wasserstein distance between M_1 and M_2.

1. Scale and normalize M_1 and M_2 such that the total area of each surface is π.
2. Compute the conformal maps ϕ_1: $M_1 \to \mathbb{D}_1$ and ϕ_2: $M_2 \to \mathbb{D}_2$, where $\mathbb{D}_1$and $\mathbb{D}_2$ are the unit planar disks.
3. Construct a convex planar domain (Ω, μ) from $\mathbb{D}_1$, where μ is computed by

$$\mu_{(M,\mathbf{g})} := e^{2\lambda(x,y)} dx \wedge dy$$

4. Discretize $\mathbb{D}_2$ into a planar point set with measure (P, ν), where ν is computed by

$$\nu_i = \frac{1}{3} \sum_{[v_i, v_j, v_k] \in M} area([v_i, v_j, v_k])$$

5. With $(\Omega,\ \mu)$ and (P, ν) as inputs of the Optimal Mass Transport Map algorithm, compute the optimal mass transport map map $f\!:\Omega \ \to P,\ W_i \to p_i$, where $p_i \in P, i = 1,2, \ldots, k$.
6. The Wasserstein distance between M_1 and M_2 can be computed by

$$Wasserstein(\mu, \nu) = \sum_{i=1}^{n} \int_{W_i} (x - p_i)^2 \mu(x) dx$$

Expression Classification

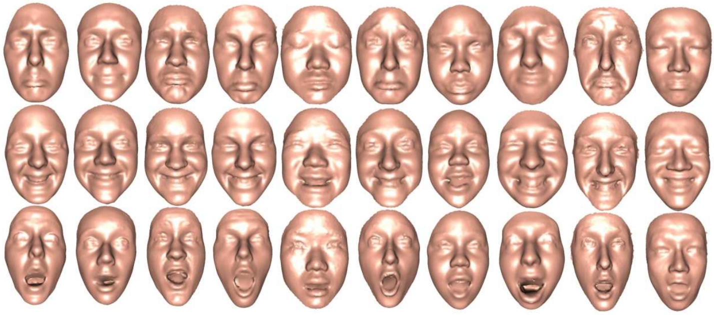
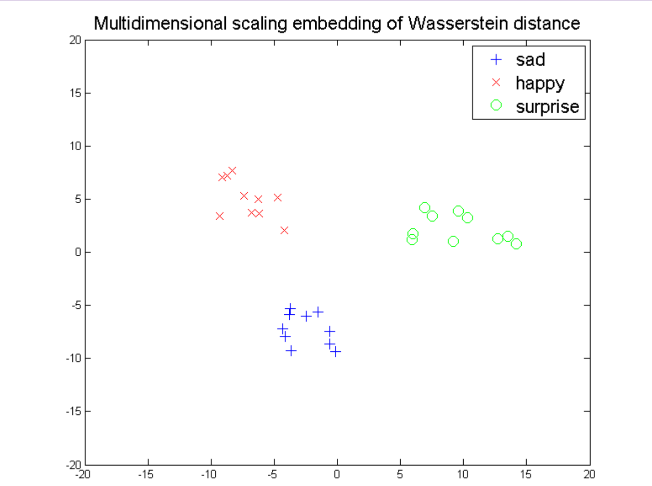


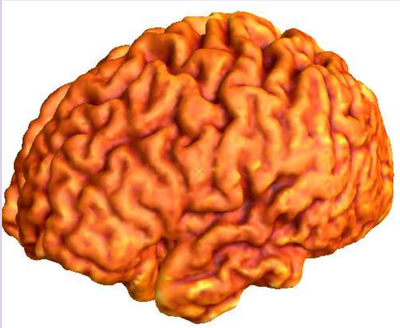
Fig. 10: **Face surfaces for expression clustering. The first row is "sad", the second row is "happy" and the third row is "surprise".**

Expression Classification

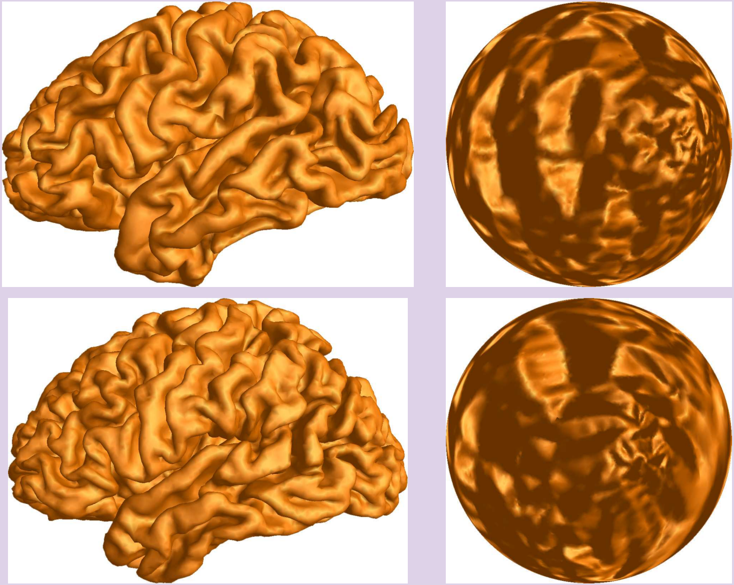
Compute the Wasserstein distances among all the facial surfaces, isometrically embed on the plane using MDS method, perform clustering.



Can we tell the IQ from the shape of the cortical surface?



From Shape to IQ



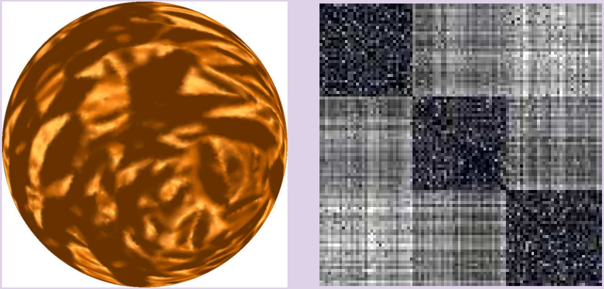


Figure: Wasserstein distance matrix.

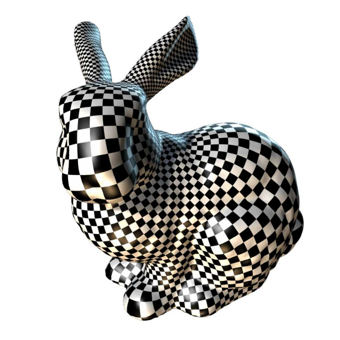
The Wasserstein distance and the IQ distance are highly correlated.

Summary

1. OMT can be applied in measure controllable parameterization, texture mapping, visualization, registration, image warping and so on
2. General framework for shape comparison/classification based on Wasserstein distance

Thanks

For more information, please email to gu@cmsa.fas.harvard.edu.



Thank you!