Applications of Optimal Mass Transportation

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Optimal Transport Map - Computational Algorithm

1. Initialize $h = 0$
2. Compute the Power Voronoi diagram, and the dual Power Delaunay Triangulation
3. Compute the cell areas, which gives the gradient $\nabla E$
4. Compute the edge lengths and the dual edge lengths, which gives the Hessian matrix of $E$, $\text{Hess}(E)$
5. Solve linear system

$$\nabla E = \text{Hess}(E) dh$$

6. Update the height vector

$$(h) \leftarrow h - \lambda dh,$$

where $\lambda$ is a constant to ensure that no cell disappears

7. Repeat step 2 through 6, until $\|dh\| < \varepsilon$. 

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Applications

- Parameterization
- Visualization
- Registration
- Classification
Parameterization
Surface parameterization refers to the process of mapping a surface onto a planar (or spherical) domain.
Parameterization can be applied for texture mapping, normal map and so on.
Unavoidably, surface parameterizations have to introduce distortions. There are two types of distortions: angle distortion and area distortion.

If a parameterization preserves both angle and area, then the parameterization must be isometric, therefore it preserves Gaussian curvature everywhere.

Therefore, general parameterization of curved surfaces can not preserve both angle and area.
Conformal parameterization: angle-preserving

Infinitesimal circles are mapped to infinitesimal circle.
Area-preserving parameterization

Infinitesimal ellipses are mapped to infinitesimal circles, preserving the areas.
Topological Disk Case
Area-Preserving Parameterization Algorithm

1. Compute conformal parameterization using Ricci flow
   \( \varphi : M \rightarrow \mathbb{D} \)

2. Compute the conformal factor at each vertex, treated as the measure
   \[
   \mu = e^{2\lambda(x,y)} \, dx \, dy
   \]

3. Compute the optimal mass transport map
   \[
   \psi : (\mathbb{D}, dx \, dy) \rightarrow (\mathbb{D}, e^{2\lambda} \, dx \, dy),
   \]

4. The composition
   \[
   \psi^{-1} \circ \varphi : M \rightarrow (\mathbb{D}, dx \, dy)
   \]
   is the area-preserving parameterization.
Surface Parameterization (topological disk)
(a) Gargoyle model; (b) Angle-preserving; (c) Area-preserving.
Poly-Annulus Case
Computational Method - Koebe’s iteration
Set the target curvatures to be zeros everywhere. Cut the surface along $\tau$, isometrically embed the surface onto a planar rectangle, use complex exponential map to map it onto a planar annulus.
Figure: Koebe’s method for computing conformal maps for multiply connected domains.
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Figure: Koebe's method for computing conformal maps for multiply connected domains.
Convergence Analysis

Theorem (Gu and Luo 2009)

Suppose genus zero surface has \( n \) boundaries, then there exists constants \( C_1 > 0 \) and \( 0 < C_2 < 1 \), for step \( k \), for all \( z \in \mathbb{C} \),

\[
|f_k \circ f^{-1}(z) - z| < C_1 C_2^{\frac{k}{n}},
\]

where \( f \) is the desired conformal mapping.
Poly-Annulus Area-preserving Algorithm

1. Compute conformal parameterization, maps the mesh onto a planar circle domain
   \[ \varphi : M \rightarrow \mathbb{A} \]

2. Compute the conformal factor \( e^{2\lambda} \) induced by the conformal mapping

3. Assign measure \( \mu \) on \( \mathbb{A} \),
   \[ \mu(p) = \begin{cases} e^{2\lambda(p)} \, dx\,dy & p \in \varphi(M) \\ \varepsilon & p \notin \varphi(M) \end{cases} \]

4. Compute the optimal mass transportation map
   \[ \psi : (\mathbb{D}, dx\,dy) \rightarrow (\mathbb{D}, \mu) \]

5. The composition \( \psi^{-1} \circ \varphi : M \rightarrow \mathbb{D} \) is the area-preserving parameterization.

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Area-Preserving Parameterization

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Figure: Angle-preserving and area-preserving parameterization - world cup model.
Distortions

Histograms of angle and area distortions of the face parameterizations.
Figure: Angle-preserving and area-preserving parameterization - kitten model.
Figure: Angle-preserving and area-preserving parameterization - kitten model.
**Figure**: Angle-preserving and area-preserving texture mapping - horse model.
Area-Preserving texture mapping

Figure: Texture mapping result based on area-preserving
Spherical Surface Case
Choose three points \( \{p_1, p_2, p_3\} \in S \), compute the unique conformal map \( \phi : S \rightarrow \hat{\mathbb{C}} \), which maps \( \{p_1, p_2, p_3\} \) to \( \{0, 1, \infty\} \) respectively. where \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \).

Compute the conformal factor \( e^{2\lambda} \) induced by the conformal mapping.

Consider the unit sphere \( S^2 \) embedded in \( \mathbb{R}^3 \), Compute the stereographic projection \( \psi : S^2 \rightarrow \hat{\mathbb{C}} \). The induced metric is \( h = \frac{4dzd\bar{z}}{(1+z\bar{z})^2} \).

Compute the the optimal mass transport map \( \tau : ((\hat{\mathbb{C}}, \frac{4dzd\bar{z}}{(1+z\bar{z})^2}) \rightarrow (\hat{\mathbb{C}}, e^{2\lambda} dzd\bar{z})) \);

The composition \( \psi^{-1} \circ \tau^{-1} \circ \phi : (S, g) \rightarrow (S^2, h) \) is an area-preserving mapping.
Spherical Area-Preserving Parameterization

\[
(S, g) \xrightarrow{\psi^{-1} \circ \tau^{-1} \circ \phi} (S^2, h) \\
(\hat{C}, e^{2\lambda} dzd\bar{z}) \xrightarrow{\tau^{-1}} (\hat{C}, \frac{4dzd\bar{z}}{(1 + z\bar{z})^2})
\]
Spherical Area-Preserving Parameterization

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Spherical Area-Preserving Parameterization

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Figure: area-preserving parameterization of gargoyle model. Left: gargoyle model; middle: initial conformal map; right: area-preserving map
**Figure**: area-preserving parameterization of maxplanck model. left: maxplanck model; middle: initial conformal map; right: area-preserving map
Volume Preserving Parameterization
Topological Ball Case
Algorithm Pipeline

1. Use spherical harmonic map $f$ to map the boundary of the topological ball $\partial M$ to the unit sphere $S^2$;
2. Compute a volumetric harmonic map $F : M \to \mathbb{D}^3$ with the Dirichlet boundary condition $f$;
3. Compute the discrete measures and construct the target point set $P$.
4. Compute the discrete optimal mass transportation map $\phi : \mathbb{D}^3 \to P$.
5. $\phi^{-1} \circ F : M \to \mathbb{D}^3$ gives the volume preserving parameterization.

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Figure: Volumetric morphing using our method.
Figure: Volumetric morphing using our method.
Figure: Volumetric morphing using Lévy’s method.

**Figure:** Volumetric morphing using Lévy’s method.

our method

Levy's method

Figure: Volumetric morphing using different methods.
Complicated Topology Case
Figure: Volume-preserving parameterization of solids with complicated topologies.
Harmonic vs. OMT

Figure: The comparison of volume distortions and dihedral angle distortions of Harmonic map and OMT map.
Visualization
Visualization

(a) Front view
(b) Angle-preserving
(c) Area-preserving
(d) Back view

Angle-perserving parameterization vs. area-preserving parameterization

(a) 2x
(b) 3x
(c) 4x
(d) 6x

Importance driven parameterization. The Buddha's head region is magnified by different factors
Figure: Adjust the target areas of ROIs.
Figure: Importance driven results of ROIs with different scale factors.
Figure: The volumetric Aneurism is magnified by large scales using the OMT technique.
**Figure:** Adjust the target volumes of irregular regions of interest.
Figure: Adjust the target volumes of multiple focus regions.
Registration
Motivation

3D Surface registration plays a fundamental role in computer vision. It lays down the foundation of facial recognition, expression analysis, dynamic surface tracking, shape analysis, geometric data retrieval and many other important applications.

Figure: The marked metric 3D surfaces.
**Figure:** Corresponding features are mapped to each other, each small disk is mapped onto the corresponding ellipse.
Existing method, 3D surface matching is converted to image matching by using conformal mappings.
Disadvantages: conformal parameterization may induce exponential area shrinkage, which produces numerical instability and matching mistakes.
Advantage: the parameterization is area-preserving, improves the robustness.
Optimal Transport Map Examples

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Disadvantages: Registration based on optimal mass transportation mapping is good for isometric surface registration; for surfaces with large deformations, landmark constraints need to be added. Conventional optimal mass transportation map cannot incorporate the marker constraints directly.
**Main Problem**
Find robust algorithm to register surfaces with large deformations and landmark constraints.

**Solutions**
Optimal mass transportation map plus Teichmüller map.
Assume the surfaces are $S_1$ and $S_2$, with landmarks $\{p_i\} \subset S_1$ and $\{q_j\} \subset S_2$,

1. Compute conformal mappings $\varphi_k : S_k \to \mathbb{D}$, $k = 1, 2$, the conformal factors are $\lambda_k : S_k \to \mathbb{R}$;

2. Compute the optimal transportation maps $\psi_k : (\mathbb{D}_k, dx dy) \to (\mathbb{D}, e^{2\lambda_k} dx dy)$;

3. Compute the Teichmüller map with landmark constraints

$$\tau : \mathbb{D} \to \mathbb{D}, \tau(\psi_1 \circ \varphi_1(p_i)) = \psi_2 \circ \varphi_2(q_i).$$

4. The registration is given by

$$\varphi_2^{-1} \circ \psi_2 \circ \tau \circ \psi_1^{-1} \circ \varphi_1 : (S_1, \{p_i\}) \to (S_2, \{q_i\}).$$
Surface Registration

(a) Armadillo #1
(b) Armadillo #2
(c) APP map #1
(d) APP map #2
(e) Conformal map #1
(f) Conformal map #2
Registration based on OMT and Teichmüller Map

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Registration based on OMT and Teichmüller Map

(a)  (b)  (c)

(d)

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Curvature Sensitive Remeshing
Curvature Sensitive Remeshing

Algorithm Pipeline

1. Compute the conformal parameterization of the input surface $\varphi : (S, g) \rightarrow (\mathbb{D}, dzd\bar{z})$,

2. Compute the optimal mass transportation map $\psi : (\mathbb{D}, dx\,dy) \rightarrow (\mathbb{D}, \mu)$, where $\mu$ is the combination of the surface area element $e^{2\lambda} dx\,dy$ and the absolute value of the Gaussian curvature measure $|K(x, y)|\,dx\,dy$,

3. Uniformly sample on the preimage of the OMT map $\psi^{-1}(\mathbb{D})$,

4. Pull back the samples to the conformal parameter domain $\varphi(S)$, compute the Delaunay triangulation $T$,

5. Pull back the triangulation $T$ to the original surface $S$, which induces the remeshing of $S$. 

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Curvature Sensitive Parameterization (CSP)

Original face mesh

Conformal parameterization
Curvature Sensitive Parameterization (CSP)

Original face mesh

Area preserving parameterization
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 0.1$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 0.2$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 0.4$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature \times 0.8
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 1.0$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 2.0$
Compare Multi-scale remeshing between APP and CSP

(a) APP  
(b) original mesh (140K)  
(c) CSP
Compare Multi-scale remeshing between APP and CSP

Remeshing: 1K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 2K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 4K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe

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Compare Multi-scale remeshing between APP and CSP

Remeshing: 8K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 16K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 32K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe

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Compare Multi-scale remeshing between APP and CSP

Remeshing: 64K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe

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Normal Map
In game industry, the speed of rendering is determined by the number of the faces of a mesh. In 3D computer graphics, normal mapping is a technique used for faking the lighting of bumps and dents. It is used to add details without using more polygons. A common use of this technique is to greatly enhance the appearance and details of a low polygon model by generating a normal map from a high polygon model.
Algorithm Pipeline

1. Simplify the original high resolution mesh $M$ to a low resolution mesh $\tilde{M}$;

2. Parameterize the low resolution model to a planar domain using OMT map, $\varphi : \tilde{M} \rightarrow \mathbb{D}$;

3. Compute the normal at each vertex of the high resolution model;

4. Compute the closest point map from the high-res model to the low-res model $\psi : M \rightarrow \tilde{M}$; This composes the map from the high-res model to the parameter domain of the low-res model $\varphi \circ \psi : M \rightarrow \mathbb{D}$;

5. Copy the normal to the high-res model to the parameter domain to generate the normal map.
Normal Map

- Original mesh with 323k faces
- Simplified mesh with 1k faces
- Original mesh with 5k faces
- Simplified mesh with 10k faces
Normal Map

original mesh with 95k faces

simplified mesh with 1k faces

original mesh with 5k faces

simplified mesh with 10k faces
Normal Map

- original mesh with 270k faces
- simplified mesh with 7k faces
- simplified mesh with 1600 faces
Shape Classification
Shape classification plays a fundamental role in computer vision and medical imaging.

We use Wasserstein distance based on optimal mass transport theory for shape classification.

We apply our method to expression classification, brain morphometry, and demonstrate the potential of our method on the discriminative analysis of hippocampal shape in epilepsy.
Wasserstein Distance

Given a metric surface \((S, g)\), a Riemann mapping \(\varphi : (S, g) \rightarrow \mathbb{D}^2\), the conformal factor \(e^{2\lambda}\) gives a probability measure on the disk. The shape distance is given by the Wasserstein distance.

The transportation cost of the optimal mass transport map defines the Wasserstein distance between two surfaces. It intrinsically measures the dissimilarities between two surfaces.

**Fig. 9: The computation of Wasserstein distance**

(a) (b) (c) (d) (e)
Wasserstein Distance Algorithm

**Input:** Two topological disk surfaces \((M_1, g_1), (M_2, g_2)\).

**Output:** The Wasserstein distance between \(M_1\) and \(M_2\).

1. Scale and normalize \(M_1\) and \(M_2\) such that the total area of each surface is \(\pi\).
2. Compute the conformal maps \(\phi_1: M_1 \to \mathbb{D}_1\) and \(\phi_2: M_2 \to \mathbb{D}_2\), where \(\mathbb{D}_1\) and \(\mathbb{D}_2\) are the unit planar disks.
3. Construct a convex planar domain \((\Omega, \mu)\) from \(\mathbb{D}_1\), where \(\mu\) is computed by
   \[
   \mu_{\mathbb{D}_1} := e^{2\Lambda(x,y)} dx \wedge dy
   \]
4. Discretize \(\mathbb{D}_2\) into a planar point set with measure \((P, \nu)\), where \(\nu\) is computed by
   \[
   \nu_i = \frac{1}{3} \sum_{[v_i, v_j, v_k] \in M} \text{area}([v_i, v_j, v_k])
   \]
5. With \((\Omega, \mu)\) and \((P, \nu)\) as inputs of the Optimal Mass Transport Map algorithm, compute the optimal mass transport map map \(f: \Omega \to P\), \(W_i \to p_i\), where \(p_i \in P, i = 1, 2, ..., k\).
6. The Wasserstein distance between \(M_1\) and \(M_2\) can be computed by
   \[
   \text{Wasserstein}(\mu, \nu) = \sum_{i=1}^{n} \int_{W_i} (x - p_i)^2 \mu(x) dx
   \]
Fig. 10: Face surfaces for expression clustering. The first row is “sad”, the second row is “happy” and the third row is “surprise”.
Compute the Wasserstein distances among all the facial surfaces, isometrically embed on the plane using MDS method, perform clustering.
Can we tell the IQ from the shape of the cortical surface?
From Shape to IQ
The Wasserstein distance and the IQ distance are highly correlated.

Figure: Wasserstein distance matrix.
OMT can be applied in measure controllable parameterization, texture mapping, visualization, registration, image warping and so on.

General framework for shape comparison/classification based on Wasserstein distance.
For more information, please email to gu@cmsa.fas.harvard.edu.

Thank you!