# The space of oriented geodesics in 3-dimensional real space forms

Dr. Nikos Georgiou

Waterford Institute of Technology

June 20, 2018

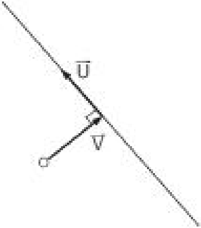
# The space of oriented geodesics

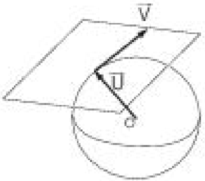
Consider the following 3-manifolds $\mathbb{M} = \mathbb{R}^3, \mathbb{S}^3$, or $\mathbb{H}^3$ and, define the set of all oriented geodesics in $\mathbb{M}$:

$$\mathbb{L}(\mathbb{M}) = \{\text{oriented geodesics in } \mathbb{M}\}.$$

Then $\mathbb{L}(\mathbb{M})$ has a structure of a 4-dimensional manifold. In particular, we have

- $\mathbb{L}(\mathbb{R}^3) = \{(\overrightarrow{U}, \overrightarrow{V}) \in \mathbb{R}^3 \times \mathbb{R}^3 | \ \overrightarrow{U} \cdot \overrightarrow{V} = 0, \ |\overrightarrow{U}| = 1\} = T\mathbb{S}^2.$



# The space of oriented geodesics

- $\mathbb{L}(\mathbb{H}^3) = \mathbb{S}^2 \times \mathbb{S}^2 - \Delta$, where $\Delta = \{(\mu_1, \mu_2) \in \mathbb{S}^2 \mid \mu_2 = -\mu_1\}$.

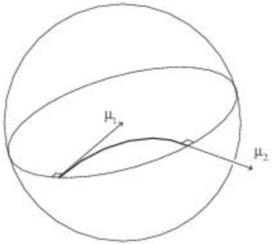



FIGURE 1
Hyperbolic geodesics in the ball model

- $\mathbb{L}(\mathbb{S}^3) = \mathbb{S}^2 \times \mathbb{S}^2$.

# The space of oriented geodesics

The dimension of the space of oriented geodesics of $\mathbb{M}^3$ is 4.

## Jacobi Fields

A **Jacobi field along the geodesic** $\gamma \subset M$ is a vector field on $M$ that describes the difference between the geodesic and an "infinitesimally close" geodesic.

For $\gamma \in \mathbb{L}(\mathbb{M}^3)$, the tangent space $T_\gamma \mathbb{L}(\mathbb{M}^3)$ is

$$T_\gamma \mathbb{L}(\mathbb{M}^3) = \{X \subset TM \mid X \text{ is an orthogonal Jacobi Field along } \gamma\}.$$

Hitchin, in 1982, has proved that rotations of orthogonal Jacobi fields along a geodesic $\gamma$ remains an orthogobal Jacobi field along $\gamma$.

# Complex structure

If $\gamma$ is an oriented geodesic, we define the rotation $\mathbb{J}_\gamma : T_\gamma M \to T_\gamma M$ about $+\pi/2$.

Note that for $X \in T_\gamma M$ we have $\mathbb{J}_\gamma \circ \mathbb{J}_\gamma(X) = -X$.

Define the endomorphism

$$\mathbb{J} : T\mathbb{L}(\mathbb{M}^3) \longrightarrow T\mathbb{L}(\mathbb{M}^3) : X \mapsto \mathcal{R}(X),$$

where $X$ is a vector field on $T\mathbb{L}(\mathbb{M}^3)$.

**Complex structure**

Let $(\mathbb{M}^3, g)$ be a 3-dimensional real space form. The map $\mathbb{J}$ is a complex structure defined on the space of oriented geodesics $\mathbb{L}(\mathbb{M}^3)$.

In other words, $\mathbb{J}$ is a linear map such that $\mathbb{J}^2 = -\mathrm{Id}$ satisfying the integrability condition.

# Symplectic structure

Let $\nabla$ be the Levi-Civita connection of the 3-dimensional real space form $(\mathbb{M}^3, g)$.

We define the following 2-form $\Omega$ in $\mathbb{L}(\mathbb{M}^3)$:
If $\gamma$ is an oriented geodesic in $\mathbb{M}^3$ and $X, Y$ are orthogonal Jacobi along $\gamma$, we have

## Symplectic form

$$\Omega_\gamma(X, Y) := g(\nabla_{\dot\gamma} X, Y) - g(X, \nabla_{\dot\gamma} Y).$$

where $\dot\gamma$ is the velocity of $\gamma$.

- $\Omega$ is a non-degenerate.
- $\Omega$ is closed, i.e., $d\Omega = 0$.

Then $\Omega$ is a **symplectic structure** on $\mathbb{L}(\mathbb{M}^3)$

# The neutral metric

## Proposition

The complex structure $\mathbb{J}$ and the symplectic structure $\Omega$ are compatible, that is,

$$\Omega(\mathbb{J}X, \mathbb{J}Y) = \Omega(X, Y),$$

for every $X, Y \in T\mathbb{L}(\mathbb{M}^3)$.

We now define the following metric in $\mathbb{L}(\mathbb{M}^3)$:

$$\mathbb{G}(X, Y) := \Omega(\mathbb{J}X, Y)$$

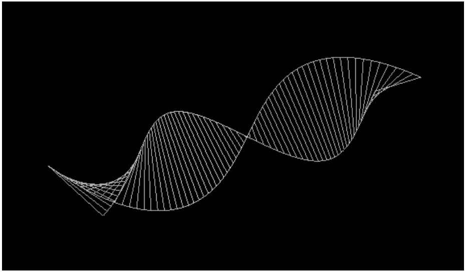
## Theorem: Properties of the metric $\mathbb{G}$

The pseudo-Riemannian metric $\mathbb{G}$ satisfies the following properties:

1. $\mathbb{G}$ is neutral, that is, it has signature $(++--)$.
2. $(\mathbb{L}(\mathbb{M}^3), \mathbb{G})$ is locally conformally flat and scalat flat.
3. $\mathbb{G}$ is invariant under the natural action of the isometry group of $(M, g)$.

# Curves in the space of oriented geodesics

A curve in $\mathbb{L}(\mathbb{M}^3)$ is a 1-parameter family of oriented geodesics. They correspond to ruled surfaces in $M$.



### Geodesics in $\mathbb{L}(\mathbb{M}^3)$

Every geodesic in $(\mathbb{L}(\mathbb{M}^3), \mathbb{G})$ is a minimal ruled surface in $M$. In particular, a geodesic in $(\mathbb{L}(\mathbb{M}^3), \mathbb{G})$ is null if and only if the corresponding ruled surface in $M$ is totally geodesic.

# The surface theory

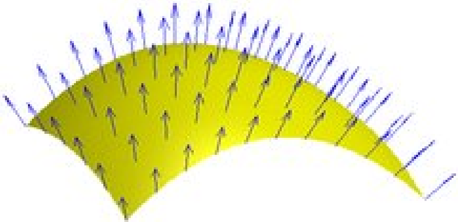
A surface $\Sigma$ in $\mathbb{L}(\mathbb{M}^3)$ is a 2-parameter family of oriented geodesics.

Using the symplectic form $\Omega$ we define the following surfaces:

**Lagrangian surfaces**

Let $f : \Sigma \to \mathbb{L}(\mathbb{M}^3)$ be an immesion of a 2-manifold in $\mathbb{L}(\mathbb{M}^3)$. A point $\gamma \in \Sigma$ is said to be a *Lagrangian point* if $(f^*\Omega)(\gamma) = 0$. If all points of $\Sigma$ are Lagrangian, then $\Sigma$ is said to be a *Lagrangian surface*.

We now consider a surface $S$ in $\mathbb{M}^3$ and take the oriented geodesics normal to $S$.



# The surface theory

The set of oriented geodesics normal to $S$ is a surface in $\mathbb{L}(\mathbb{M}^3)$ which will be denoted as $\Sigma$. The relation between $S$ and $\Sigma$ is given by the following:

### B. Guilfoyle & W. Klingenberg (2005), N. Georgiou & B. Guilfoyle (2010)

Let $S$ be an oriented surface in $M$ and let $\Sigma$ be the set of all oriented geodesics that are normal to $S$. Then $\Sigma$ is a Lagrangian surface. Furthermore, the metric $\mathbb{G}_\Sigma$ induced on $\Sigma$ is Lorentzian.

Using the complex structure $\mathbb{J}$ we define the following:

### Complex points/ Complex curve

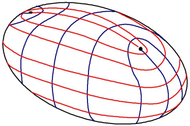
Let $\Sigma$ be a surface in $\mathbb{L}(\mathbb{M}^3)$ by $f$. A point $\gamma \in \Sigma$ is said to be a *complex point* if the complex structure $\mathbb{J}$ preserves the tangent plane $T_\gamma\Sigma$. If all points of $\Sigma$ are complex, then $\Sigma$ is said to be a **complex curve**.

# Umbilic points

Let $S$ be a surface in $\mathbb{M}^3$. A point $p \in S$ is said to be **umbilic** if the principal curvatures are equal. They are points that are looks spherical.

Example of umbilic points:

- All points of a sphere are umbilic.
- The following ellipsoide has four umbilic points.



- The rugby ball has two umbilic points.

# Umbilic points

There exists an important relation between complex points and umbilic points.

## Umbilic points

Let $S$ be an oriented surface in $M$ and let $\Sigma$ be the Lagrangian surface formed by the normal oriented geodesics to $S$. Then $p \in S$ is an umbilic point if and only if the oriented geodesic $\gamma$ orthogonal to $S$ at $p$ is a complex point.

The previous result, gives new tools to study the 90 year old Conjecture due to Carathéodory:

## Carathéodory Conjecture

Any $C^3$-smooth closed convex surface in $\mathbb{R}^3$ admits at least two umbilic points.

.

# Weingarten surfaces

A surface in $\mathbb{M}^3$ is said to be *Weingarten* if the principal curvatures are functionally related.

Surfaces in $\mathbb{R}^3$ such as

- the standard torus,
- the round spheres of radius $r > 0$,
- Constant Mean Curvature (CMC) surfaces,
- rotationally symmetric surfaces

are all Weingarten.

**Weingarten surfaces – B. Guilfoyle & W. Klingenberg (2006), N. Georgiou & B. Guilfoyle (2010)**

Let $S$ be an oriented surface in $M$ and let $\Sigma$ be the set of all oriented geodesics that are normal to $S$. Then $S$ is Weingarten if and only if the Gauss curvature of $\Sigma$ is zero.

# Minimal surfaces

Generally, a submanifold is said to be **minimal** if its volume is critical with respect to any variation.
A submanifold is minimal if and only if the mean curvature is zero.

**Minimal surfaces – R. Harvey &.B. Lawson (1982)**

Any complex curve in $\mathbb{L}(\mathbb{M})$ is a minimal surface.

For minimal Lagrangian surfaces in the space of oriented geodesics we have the following result:

**Minimal Lagrangian surfaces – H. Anciaux & B. Guilfoyle (2009), N. Georgiou (2012)**

Let $S$ be an oriented surface in the 3-dimensional real space form and $\Sigma$ be the set of all oriented geodesics normal to $S$. Then $\Sigma$ is minimal if and only if $S$ is the equidistant tube along a geodesic.

# Hamiltonian minimal surfaces

A variation $\Phi_t$ of a surface $\Sigma$ in $\mathbb{LM}$ is said to be Hamiltonian if the initial velocity $\partial_t \Phi_t|_{t=0}$ is a Hamiltonian vector field, that is, the one form $\Omega(X,.)$ is exact.

**Hamiltonian variations – N. Georgiou & G. A. Lobos (2016)**

Let $\phi_t$ be a smooth one-parameter deformation of a surface $\Sigma$ in $\mathbb{M}$. Then, the corresponding Gauss maps $\Phi_t$ form a Hamiltonian variation in $\mathbb{L}(\mathbb{M})$.

A Lagrangian submanifold is said to be **Hamiltonian minimal** if its volume is critical under Hamiltonian variations.

**Minimal Lagrangian surfaces – N. Georgiou & G. A. Lobos (2016)**

$\Sigma$ is Hamiltonian minimal if and only if $S$ is a critical point of the functional

$$\mathcal{W}(S) = \iint_S \sqrt{H^2 - K + c}\, dA,$$

where $H, K$ are respectively the mean and Gaussian curvature of $S$ and $c$ is the constant curvature of the space form $\mathbb{M}$.