Deletion in Abstract Voronoi diagrams in Expected Linear Time

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Voronoi diagram of points

$S =$ set of n point sites

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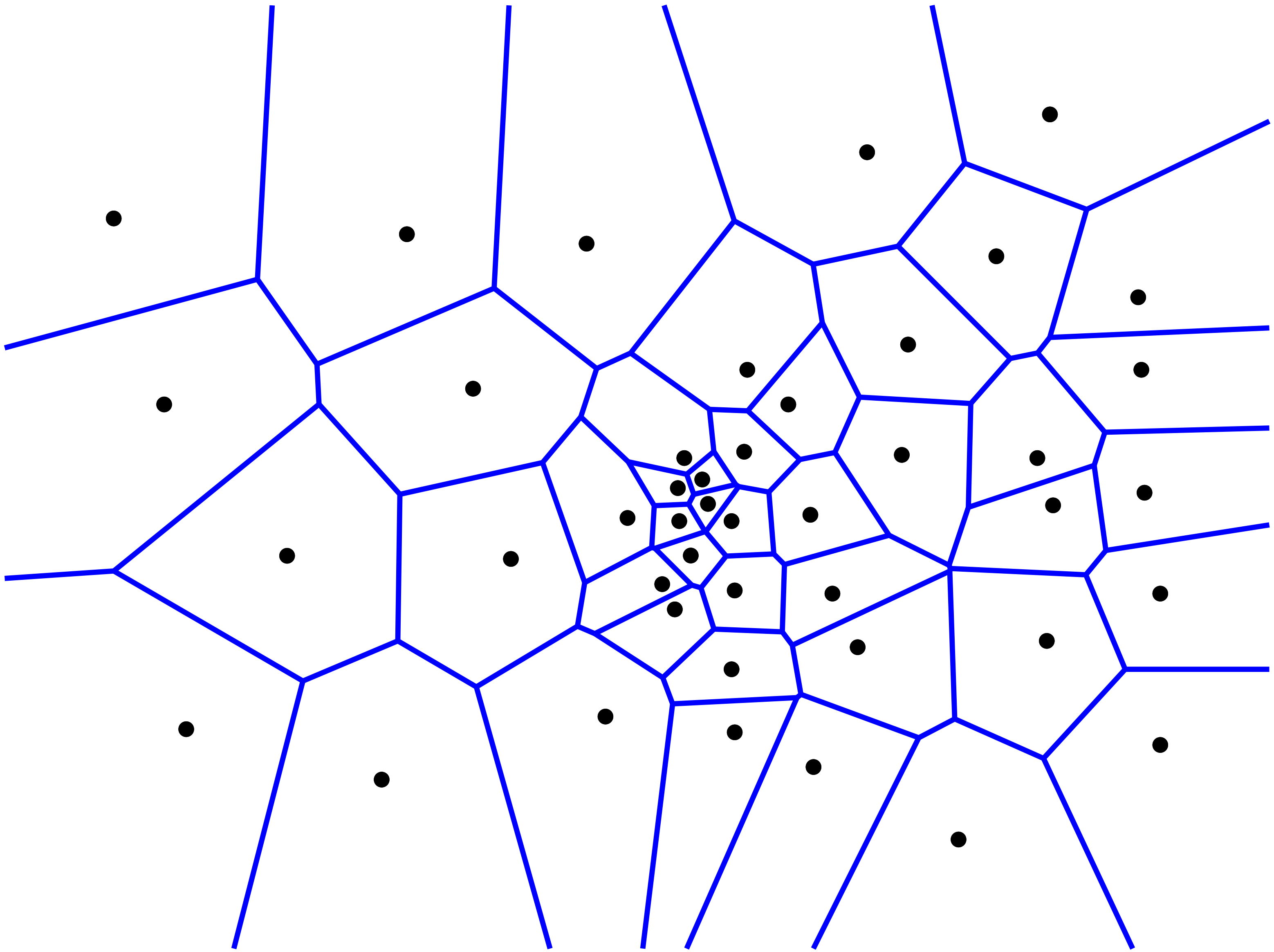

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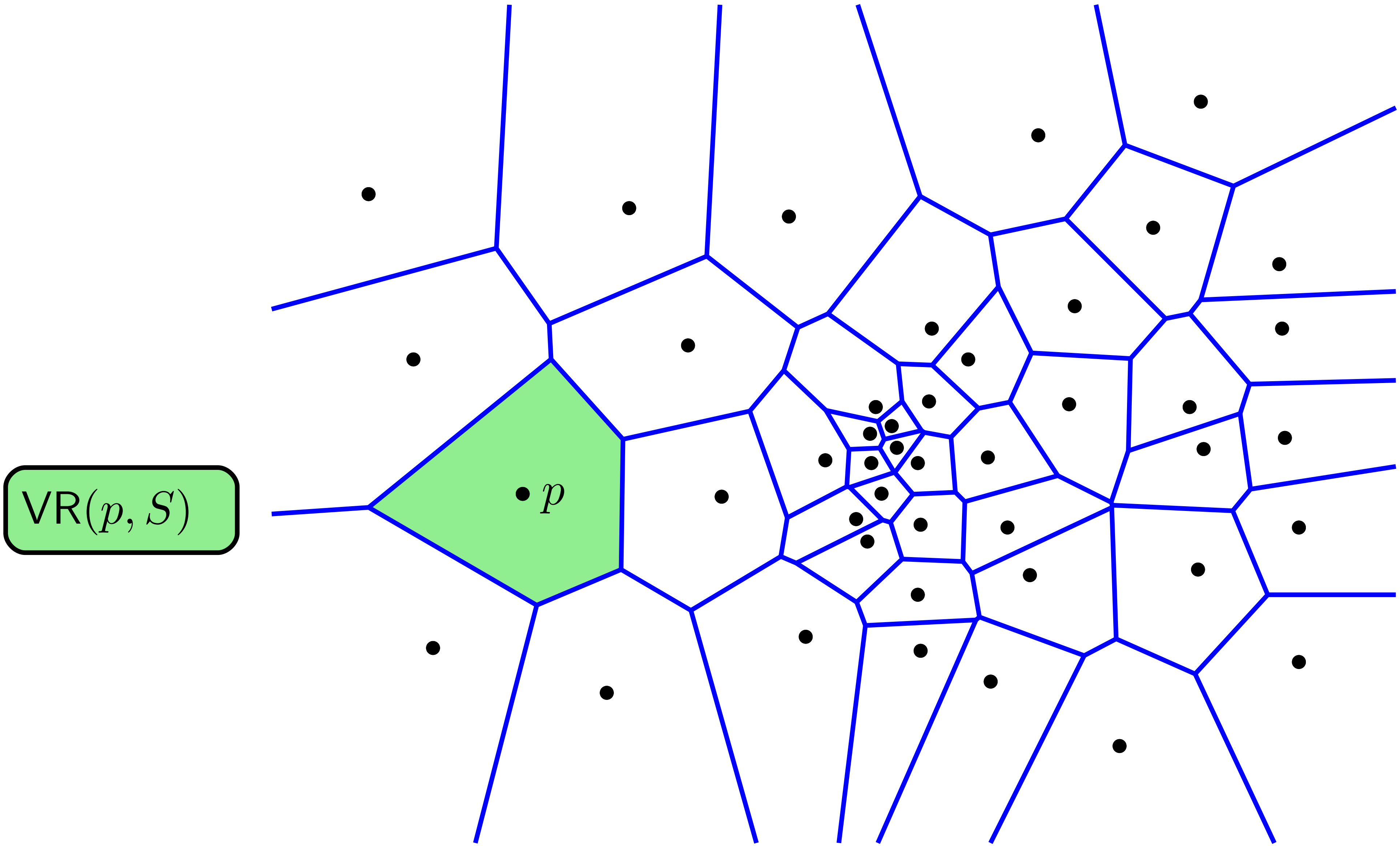

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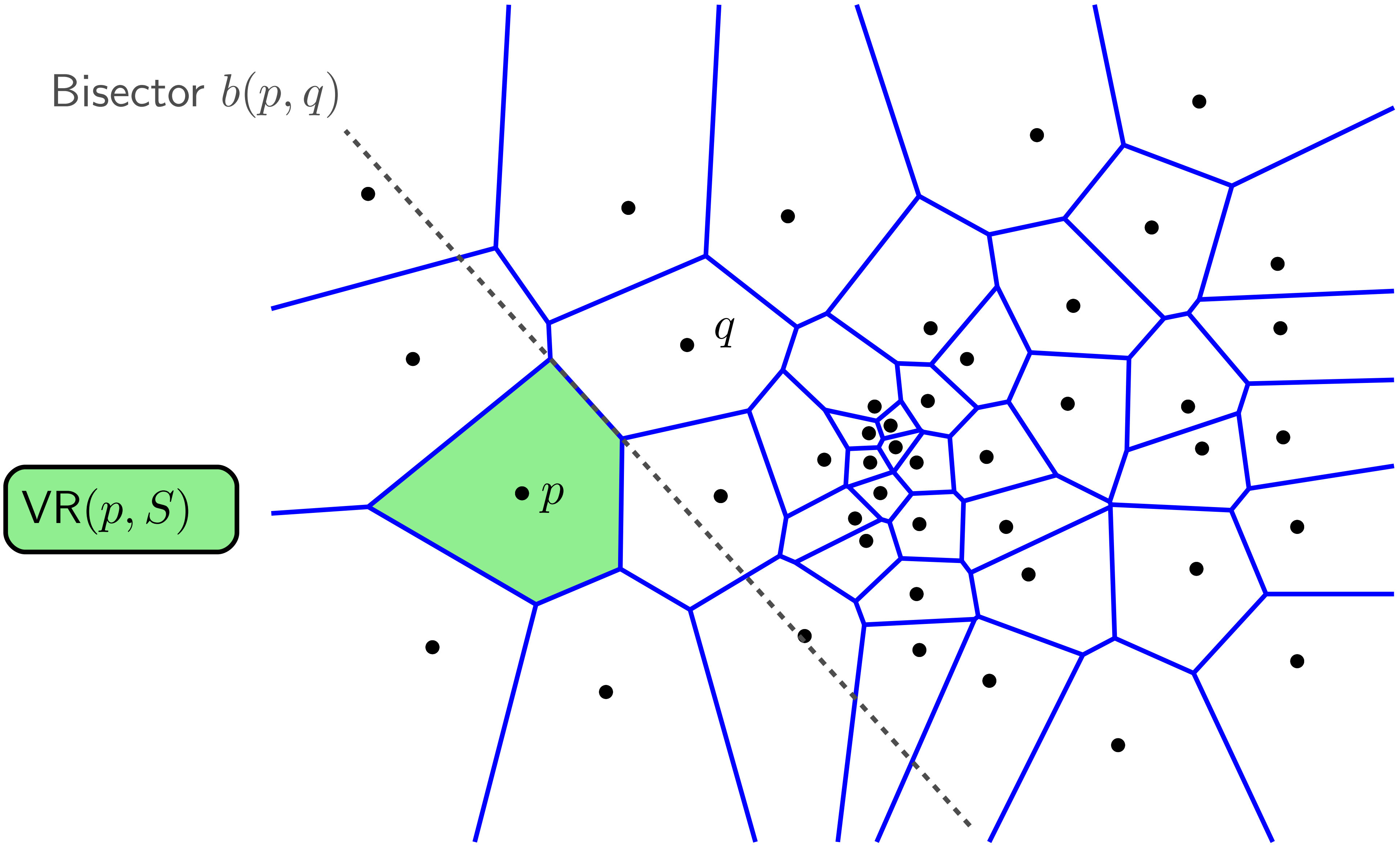

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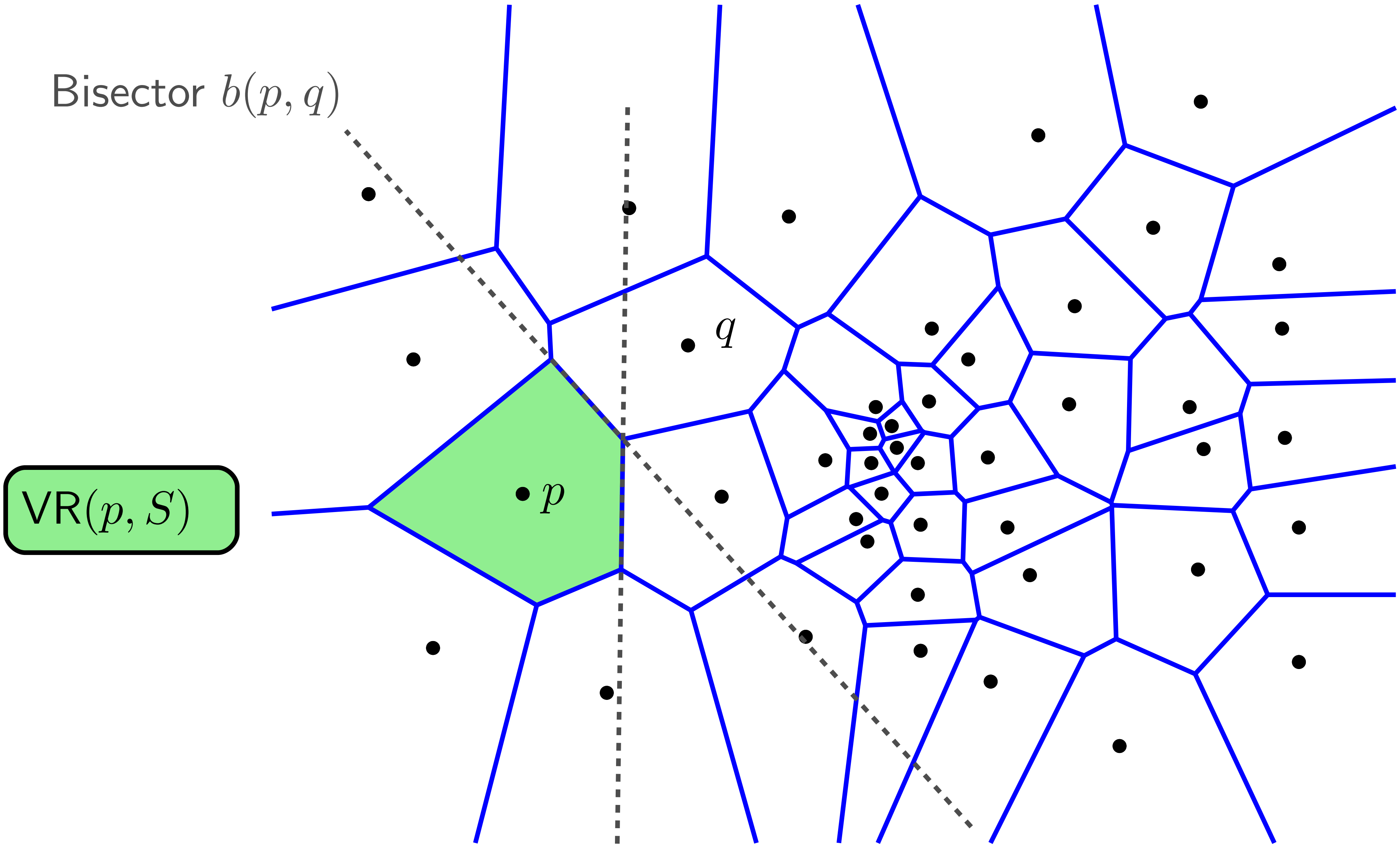

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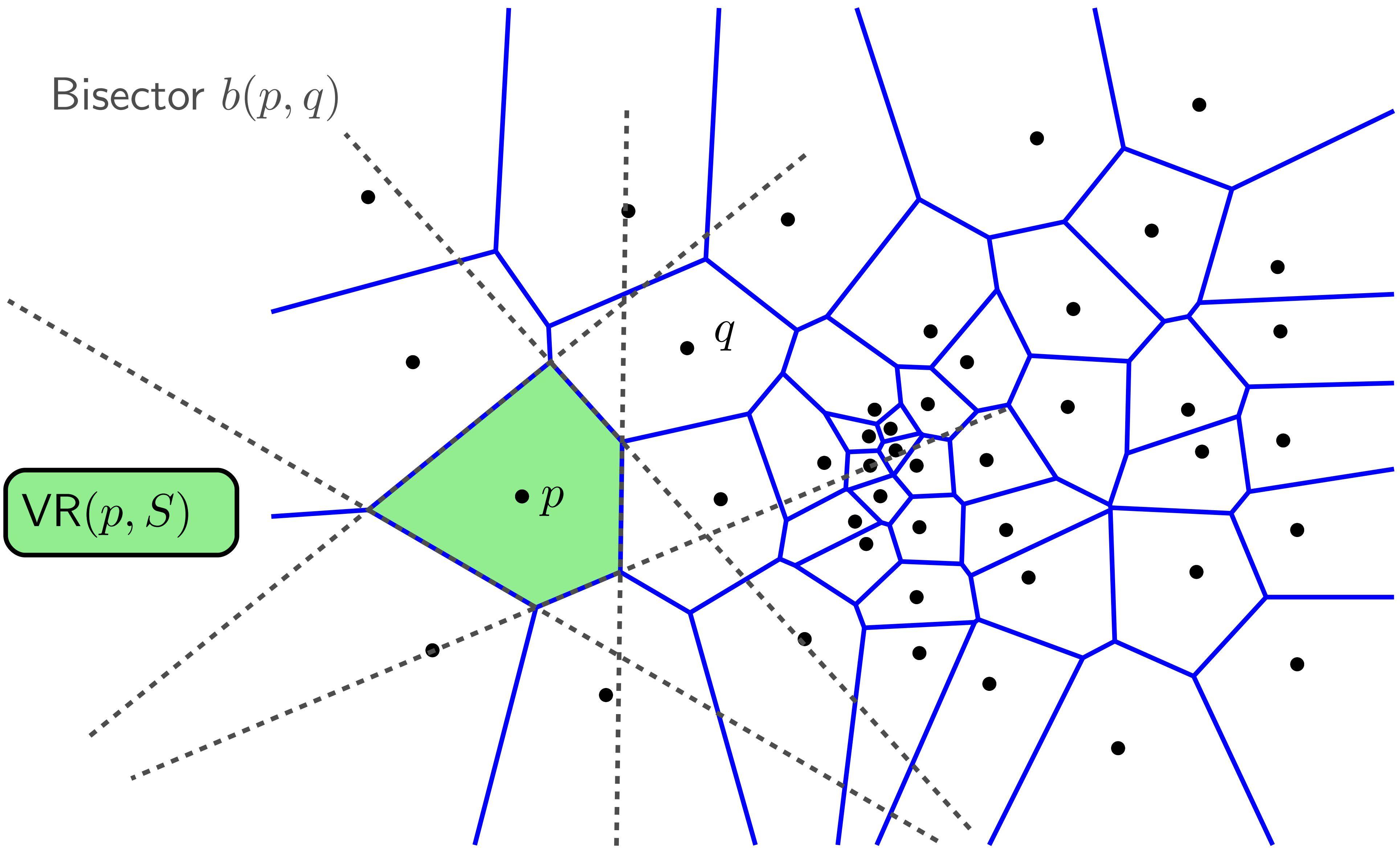

Voronoi diagram of segments

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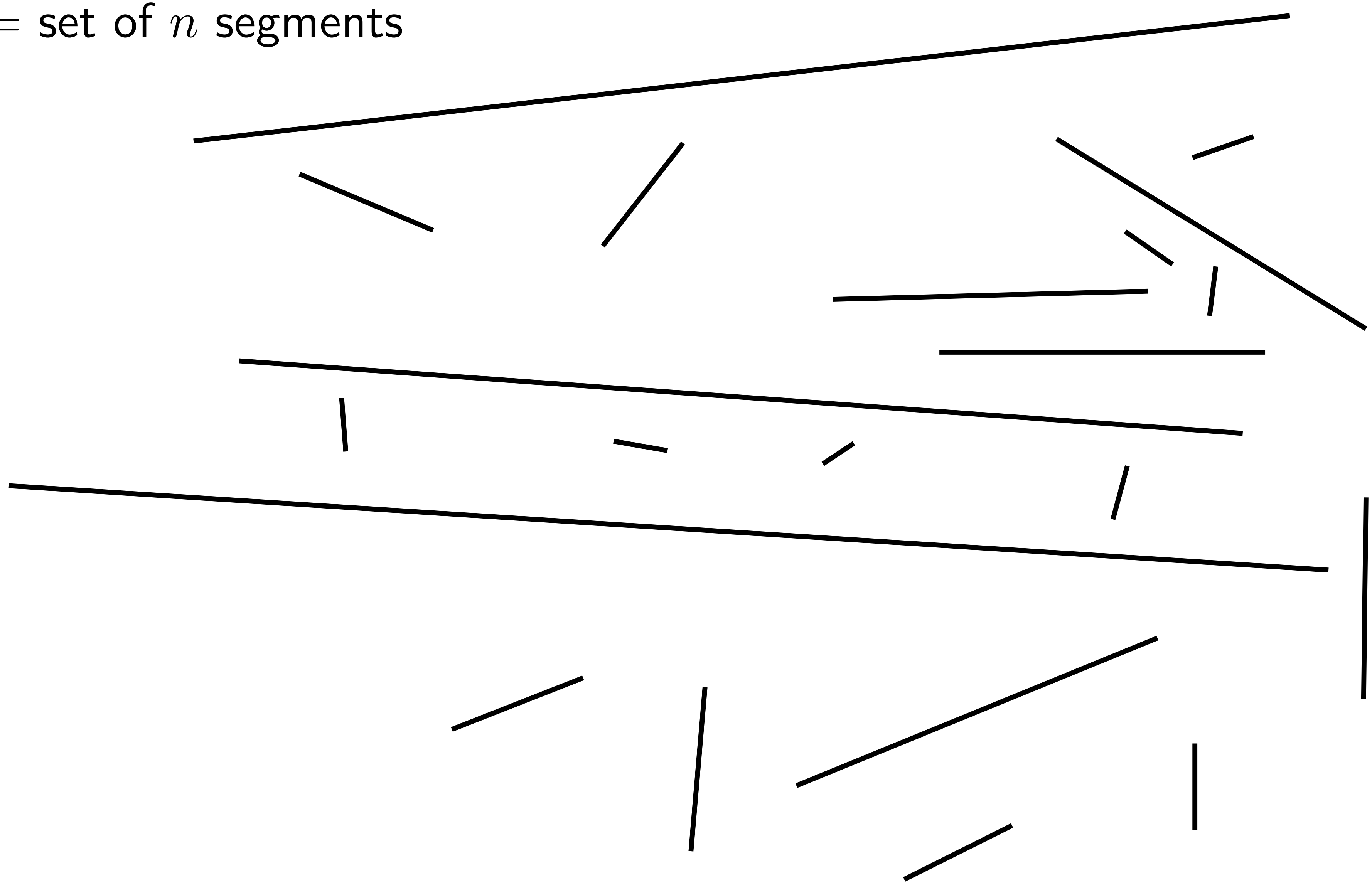

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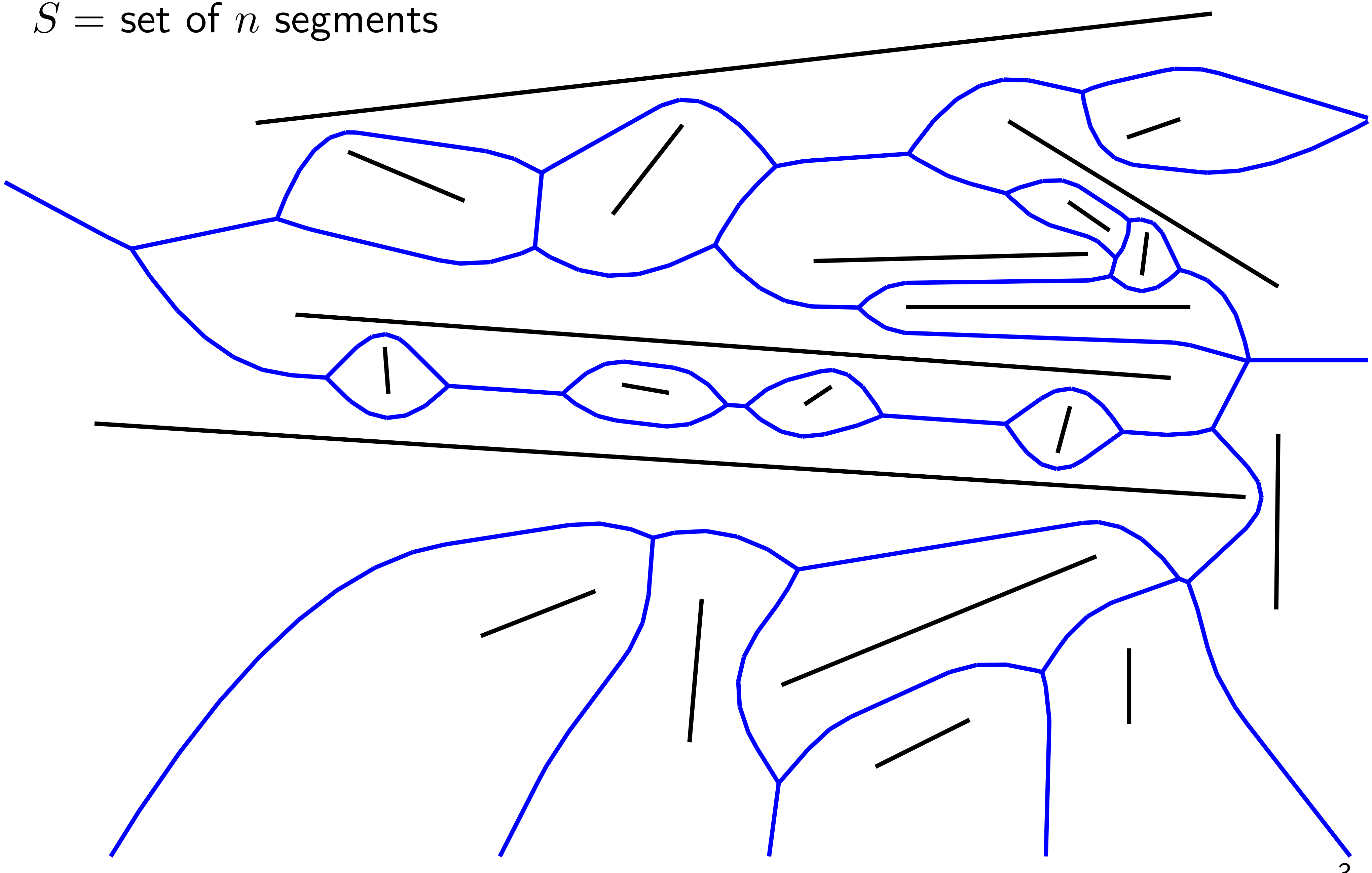
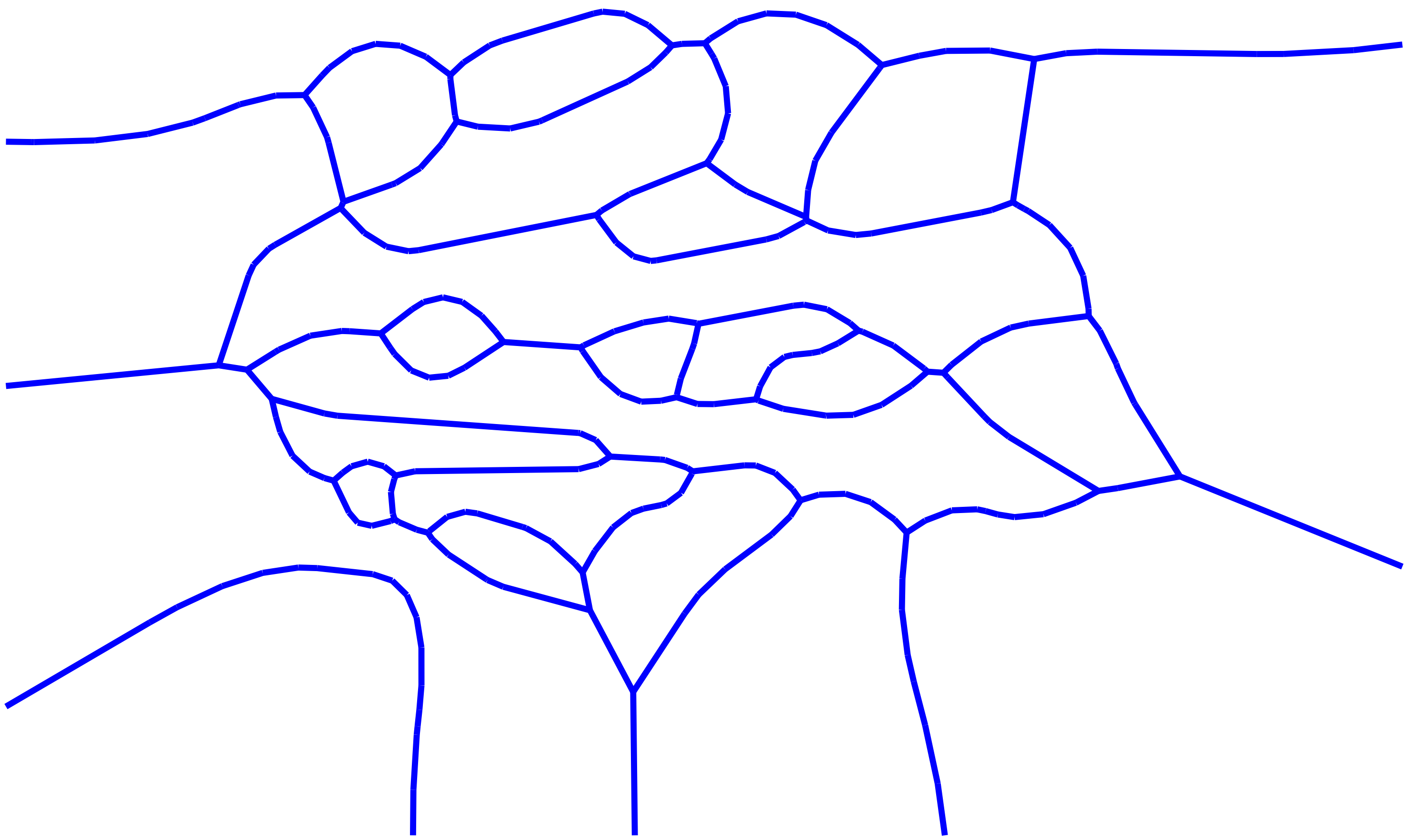


Abstract Voronoi diagram

- Offer a unifying framework to many concrete diagrams.
  Defined on bisecting curves satisfying some axioms, rather than sites.



Rolf Klein. *Concrete and Abstract Voronoi Diagrams*. 1989.

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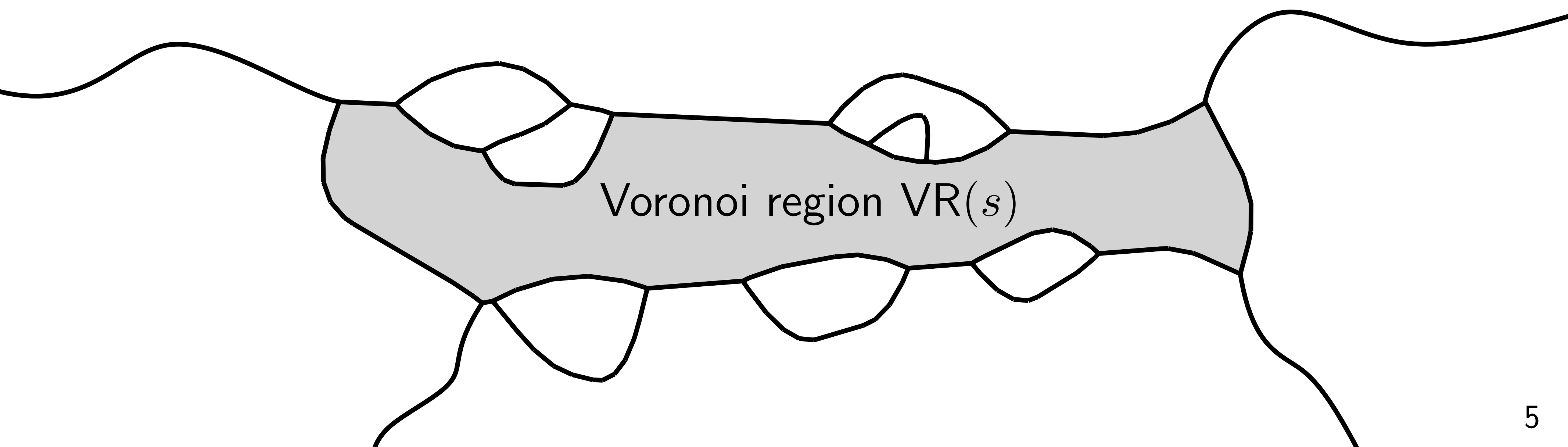

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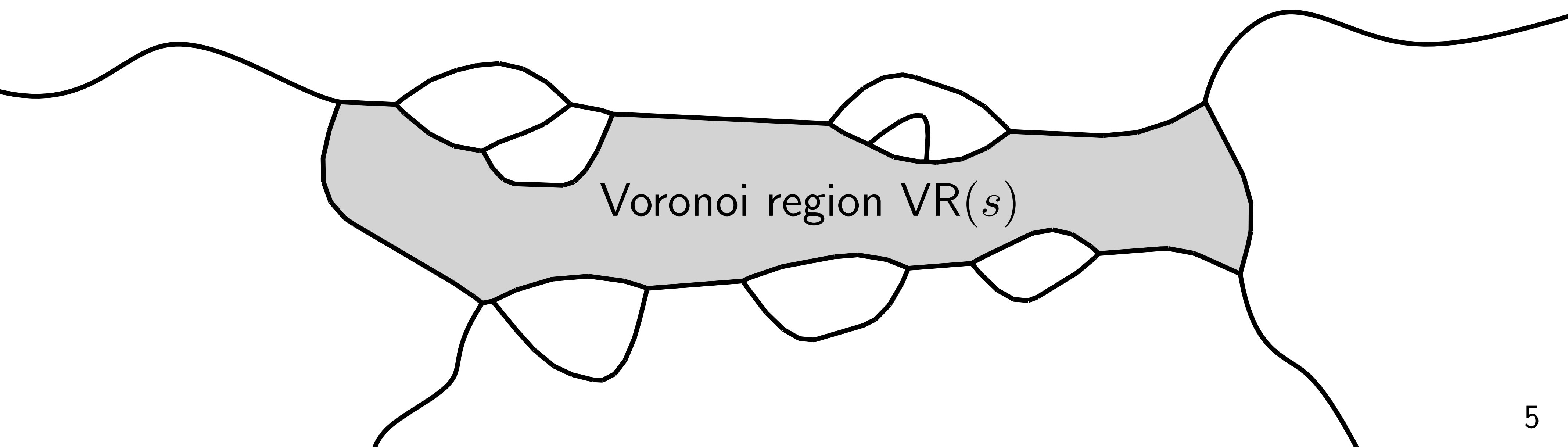

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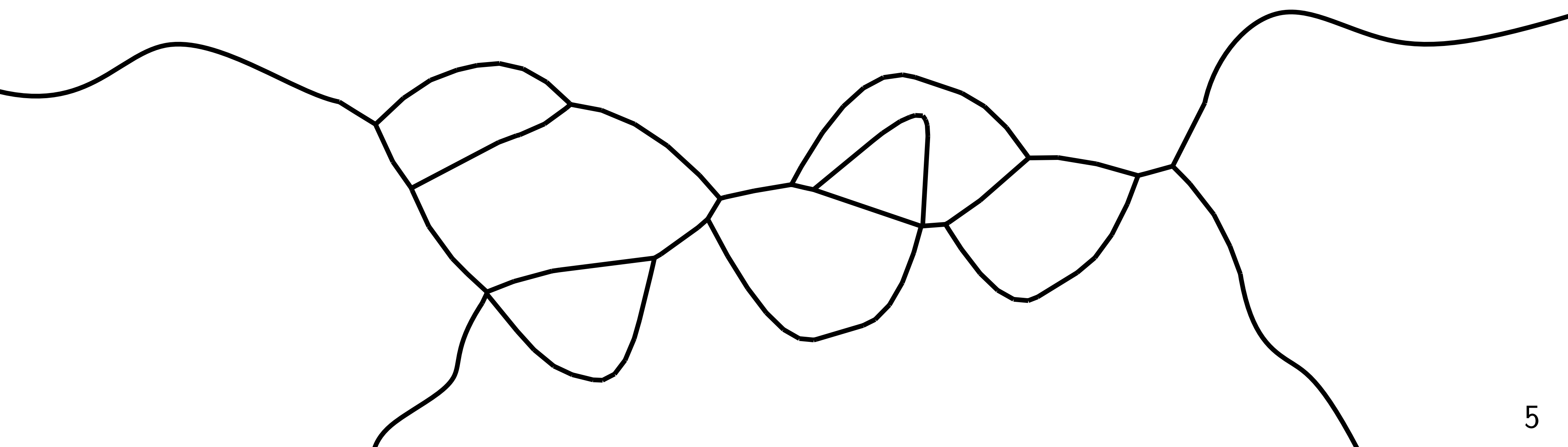

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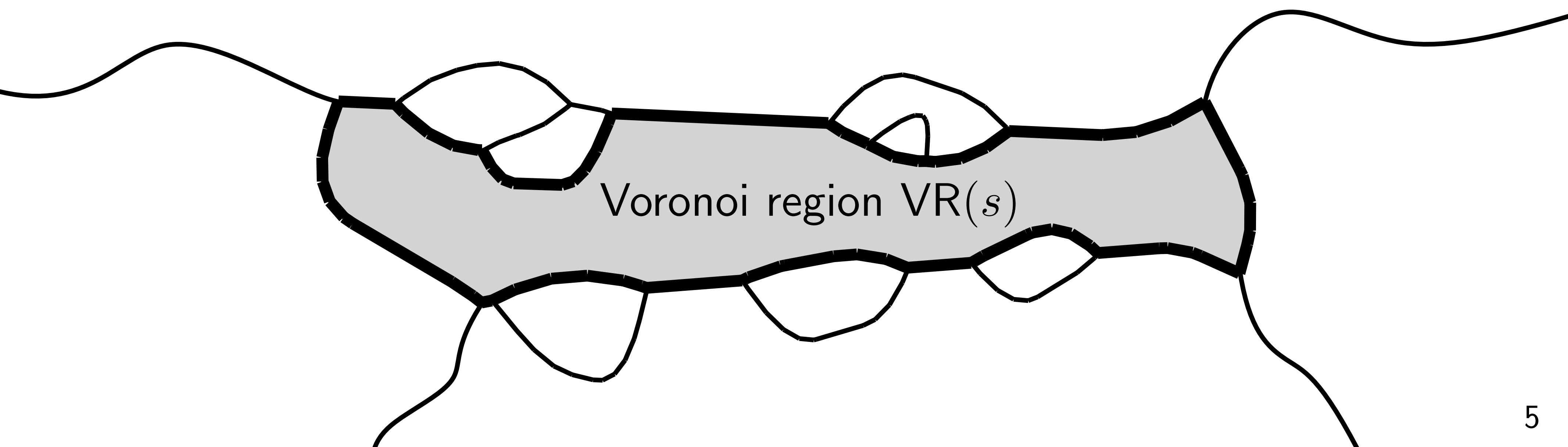
Goal: Compute $\mathcal{V}(S \setminus s)$ within VR(s), in time **linear** (randomized or deterministic) in the complexity of the region's boundary



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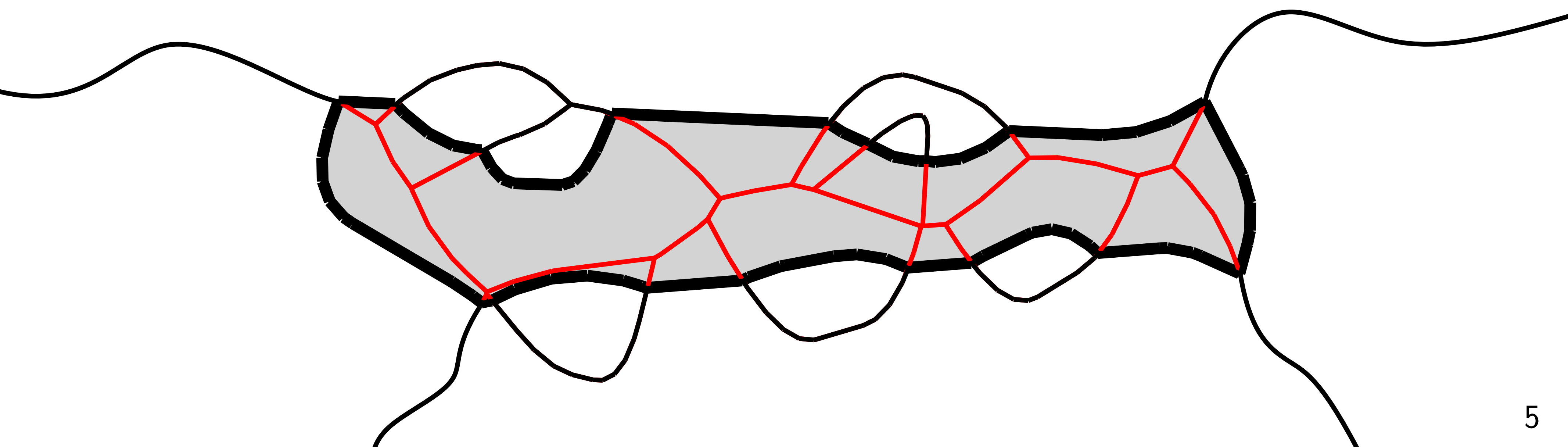

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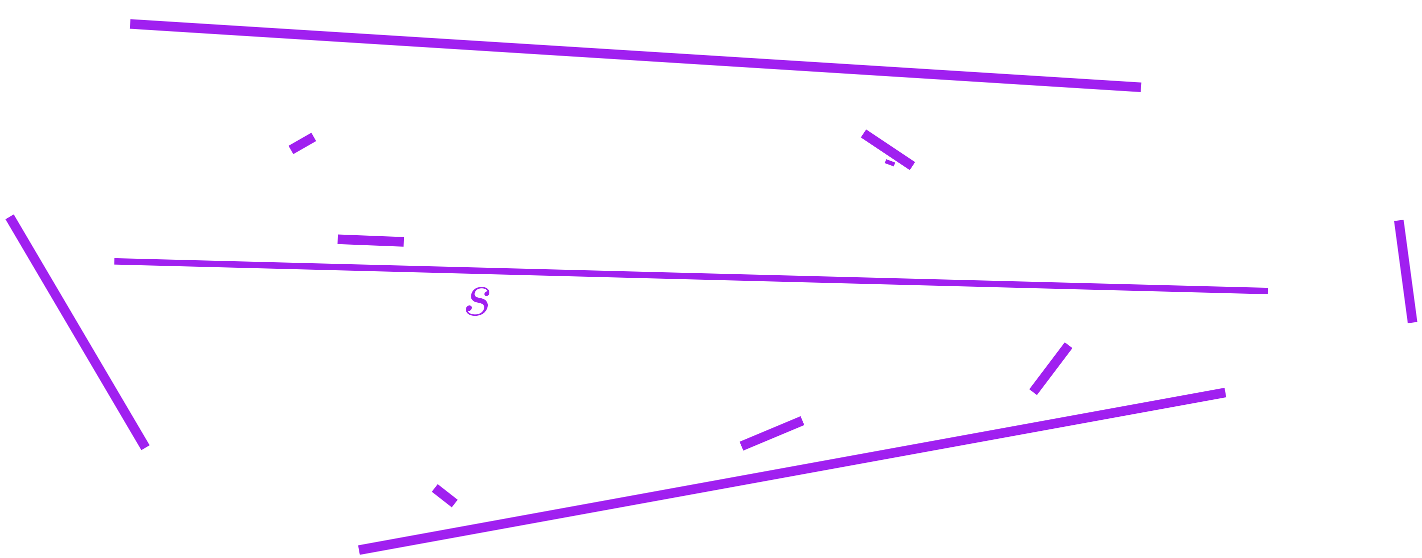

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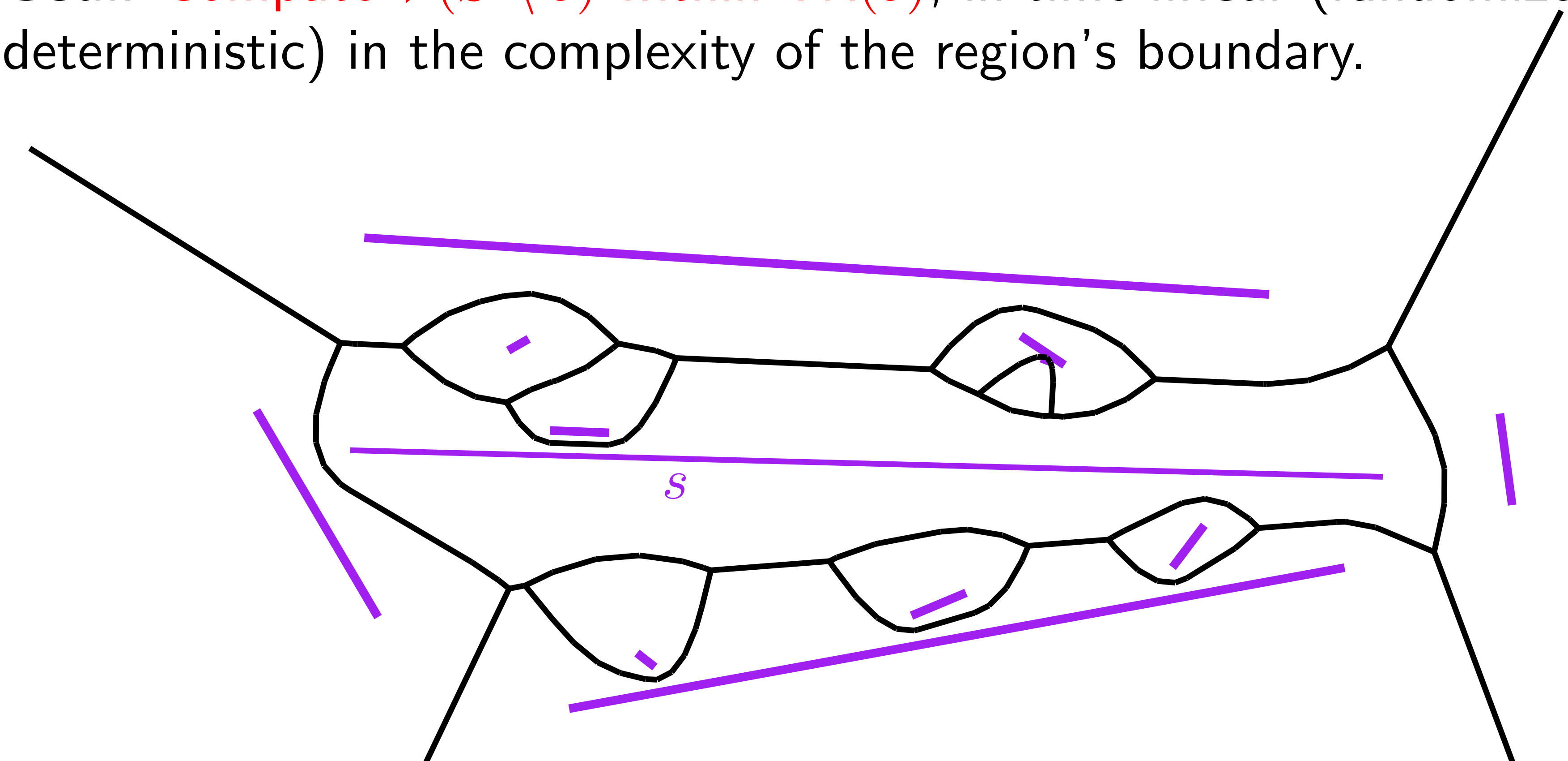

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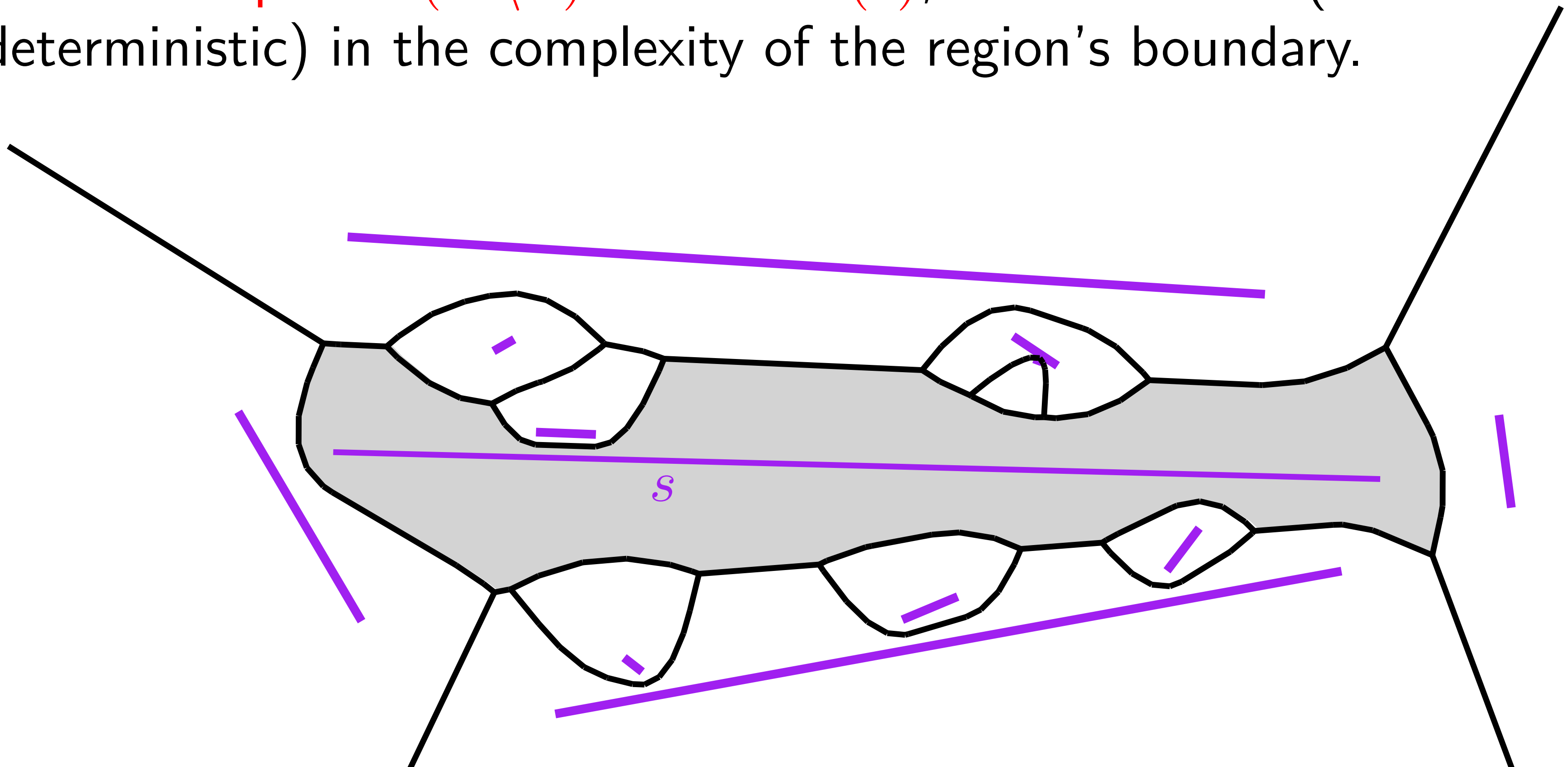

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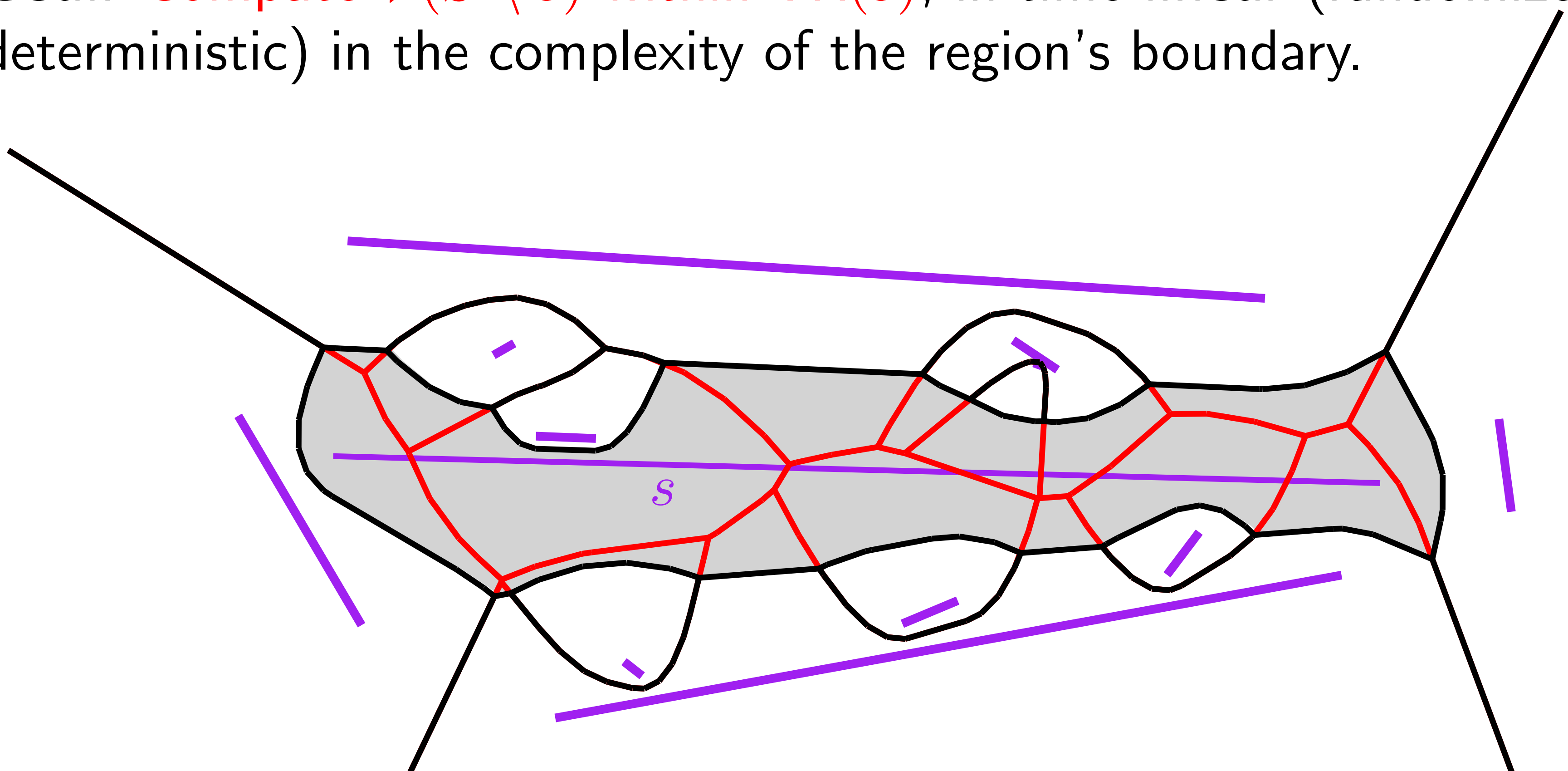

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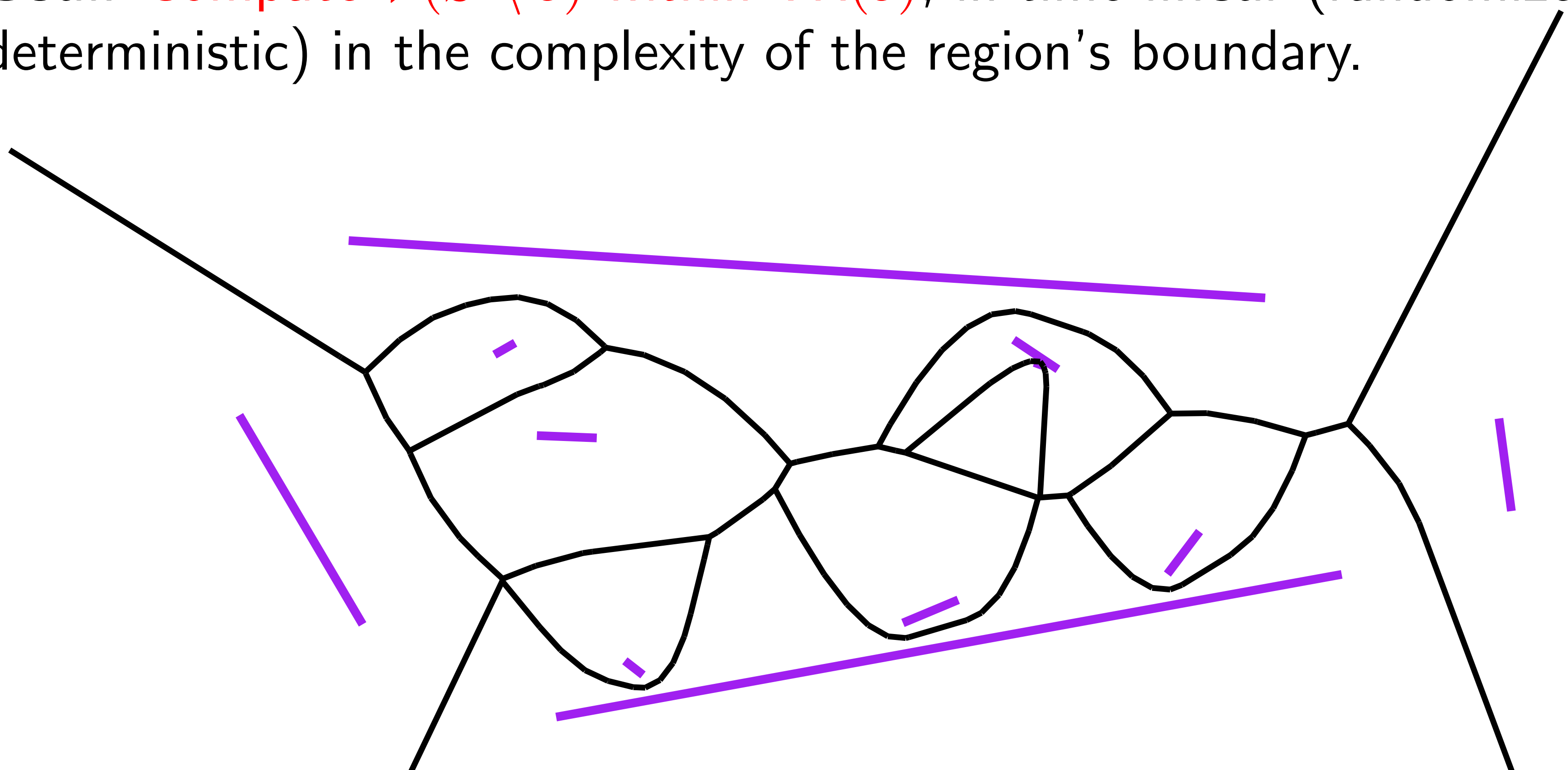

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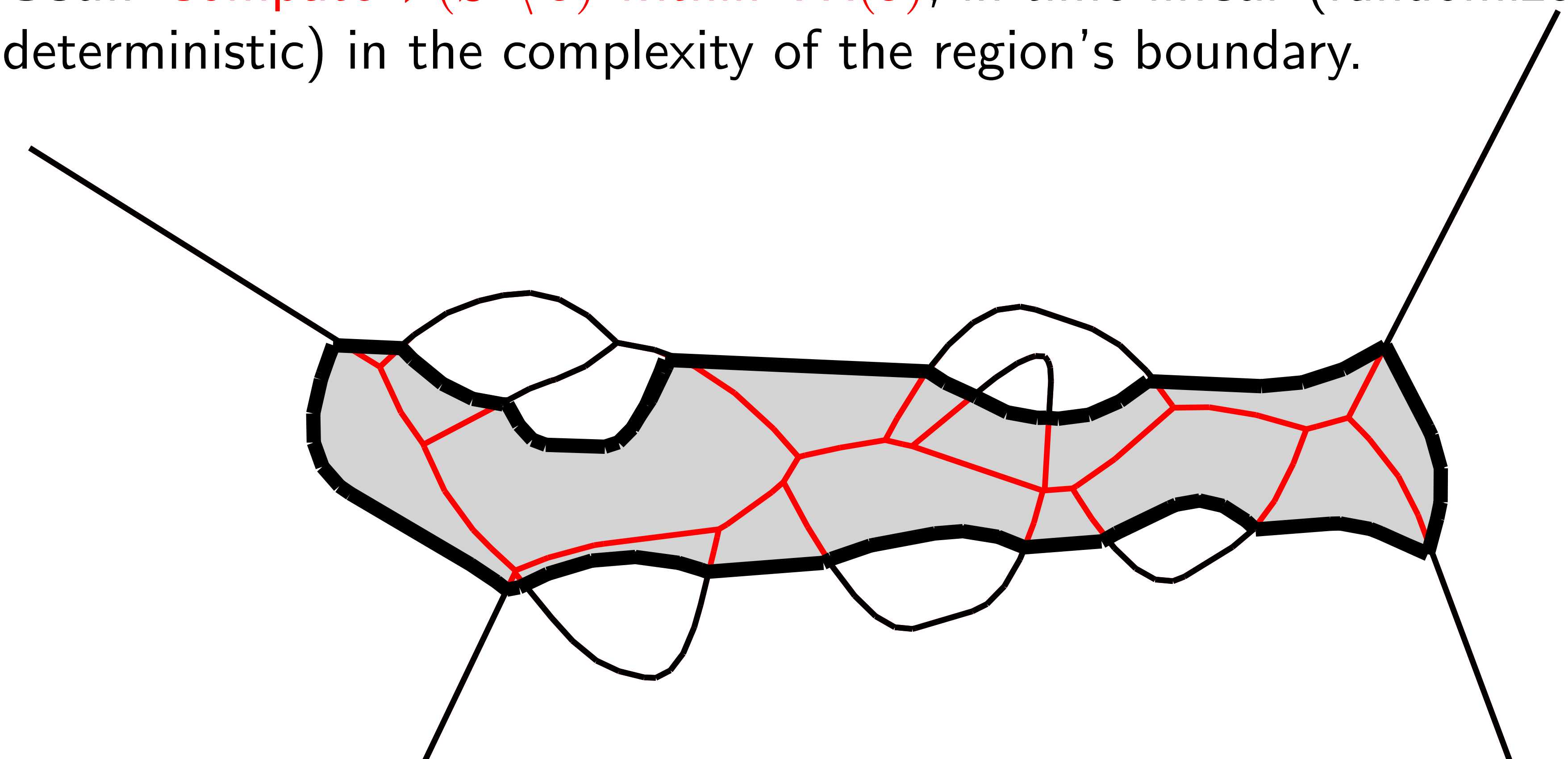

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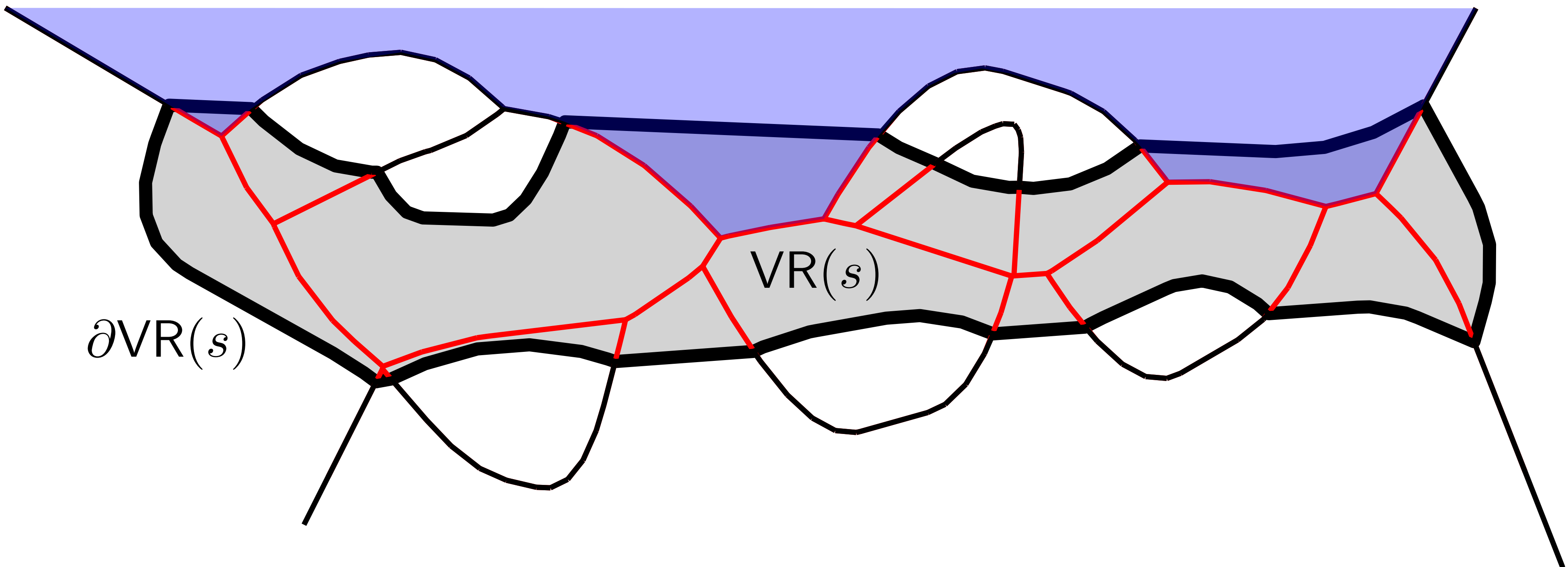
- **Open** since then for other sites (line segments, circles etc.).

- **Open** for **abstract Voronoi diagrams** (AVDs).

What is difficult?

For abstract Voronoi diagrams and non-point sites (line segments, circles):

The Voronoi region of **one site** can have **multiple faces** within $\mathrm{VR}(s)$.
– The sites along $\partial\mathrm{VR}(s)$ can **repeat**.
(AVDs: $\partial\mathrm{VR}(s)$ is a Davenport-Schinzel sequence of order 2.)



Linear-time Voronoi algorithms

- The Voronoi diagram of points in **convex position**.
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  – Update the **order-**$(k+1)$ **diagram** within an order-k region.

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  Given a Hamiltonian curve, that visits **every region exactly once** and intersects **each bisector exactly once**.

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- **Forest-Like abstract Voronoi Diagrams** [Bohler, Klein, Liu, CCCG 2014]
  Similar conditions, where no region can have multiple faces.

Linear-time Voronoi algorithms

- The **medial axis** of a simple polygon.
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Linear-time Voronoi algorithms

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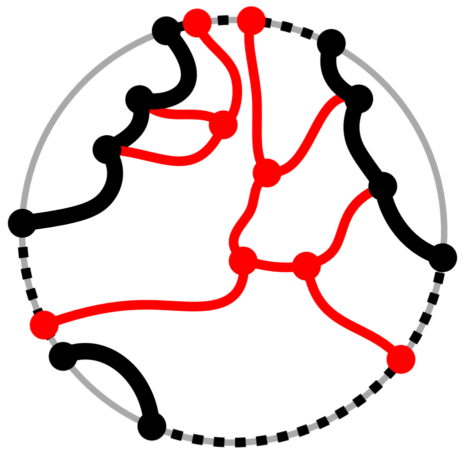
- **Expected** linear-time algorithm for the **farthest-segment** Voronoi diagram, after the sequence of faces at infinity is known.
  [Khramtcova, Papadopoulou, ISAAC 2015]

Our results



Introduce **Voronoi-like diagrams**

– **relaxed** version of a Voronoi diagram (easier to compute)

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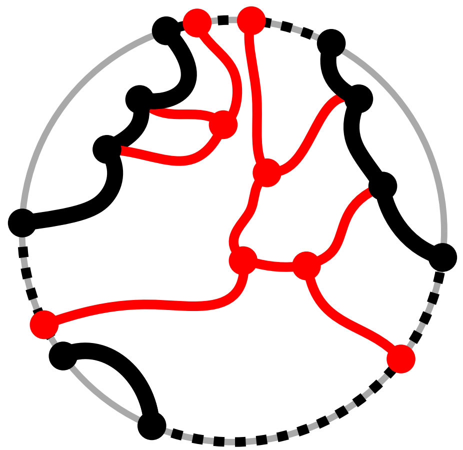

Introduce **Voronoi-like diagrams**

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A simple, **randomized incremental** algorithm for updating abstract Voronoi diagrams after **deletion** of one site **in expected linear time**.

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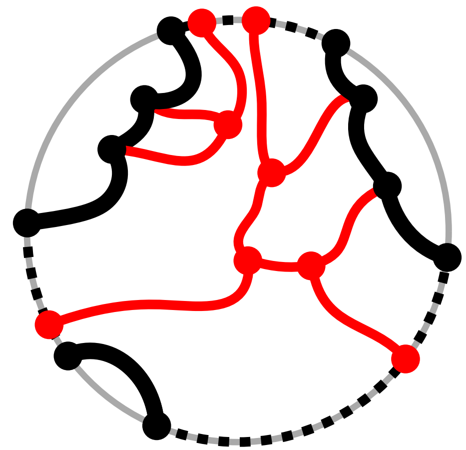


Introduce **Voronoi-like diagrams**

– **relaxed** version of a Voronoi diagram
(easier to compute)

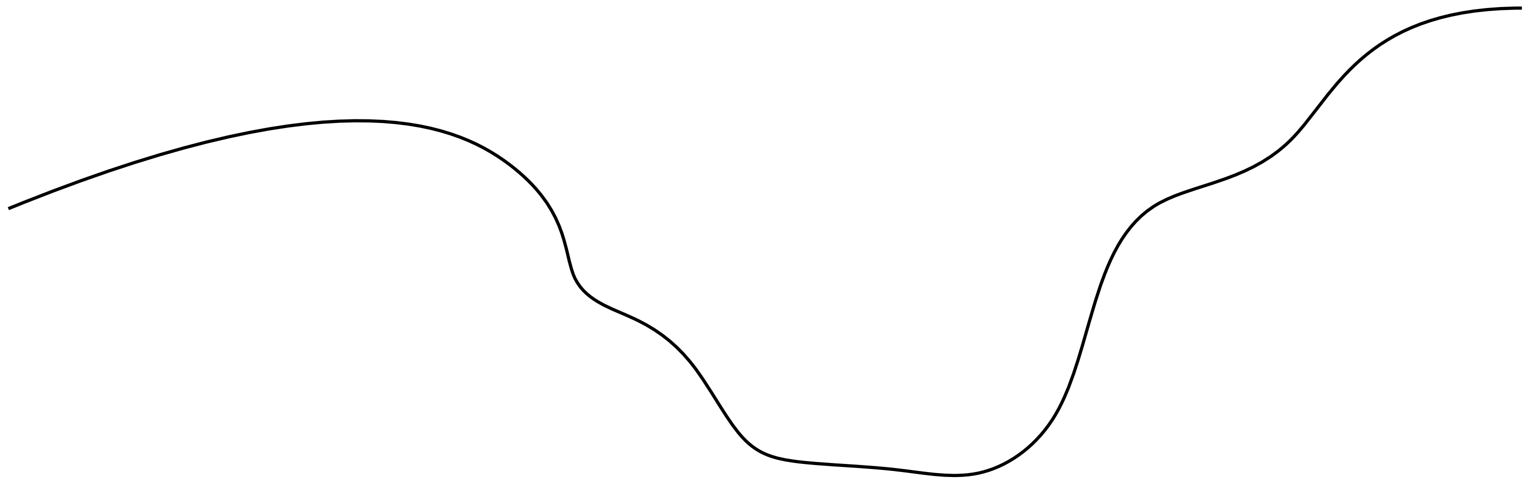
A simple, **randomized incremental** algorithm for updating abstract Voronoi diagrams after **deletion** of one site **in expected linear time**.

– adapt to the **farthest abstract Voronoi diagram**,
after the sequence of its faces at infinity is known.

Overview

- Define abstract Voronoi diagrams (AVDs).
- Define **Voronoi-like diagrams**.
- Properties of Voronoi-like diagrams.
- Define an **insertion** operation on Voronoi-like diagrams.
- Sketch a randomized incremental algorithm.

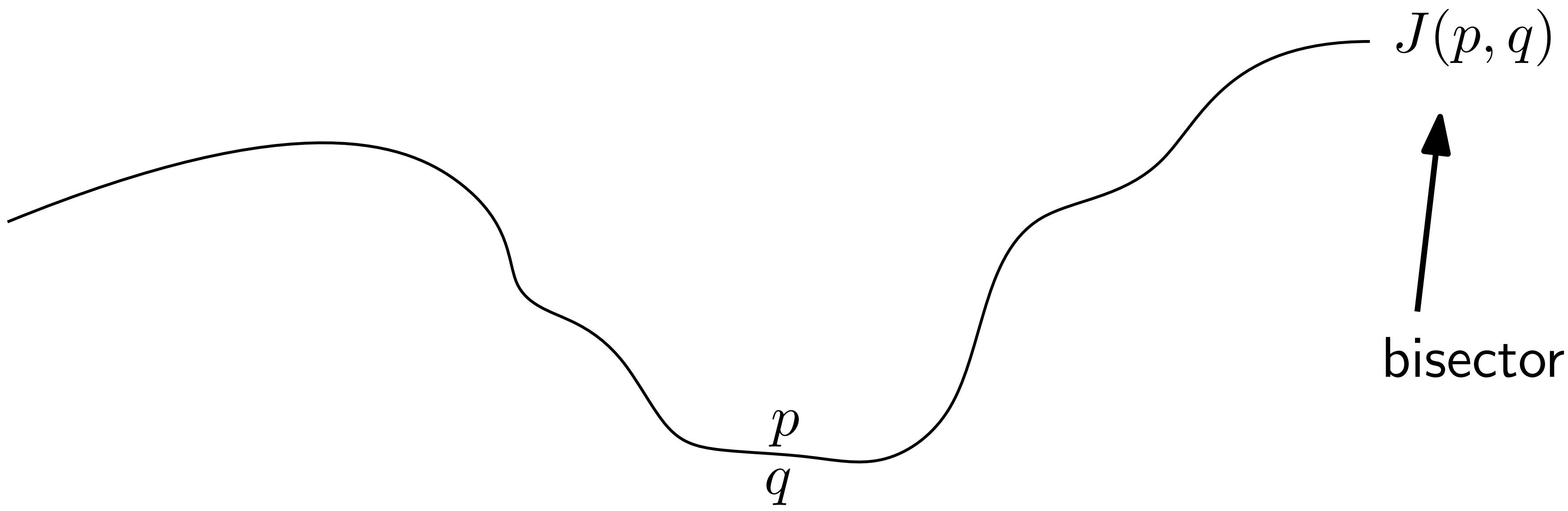
Abstract Voronoi diagrams



Rolf Klein. *Concrete and Abstract Voronoi Diagrams*. 1989.

Abstract Voronoi diagrams

S abstract sites, $n = |S|$.



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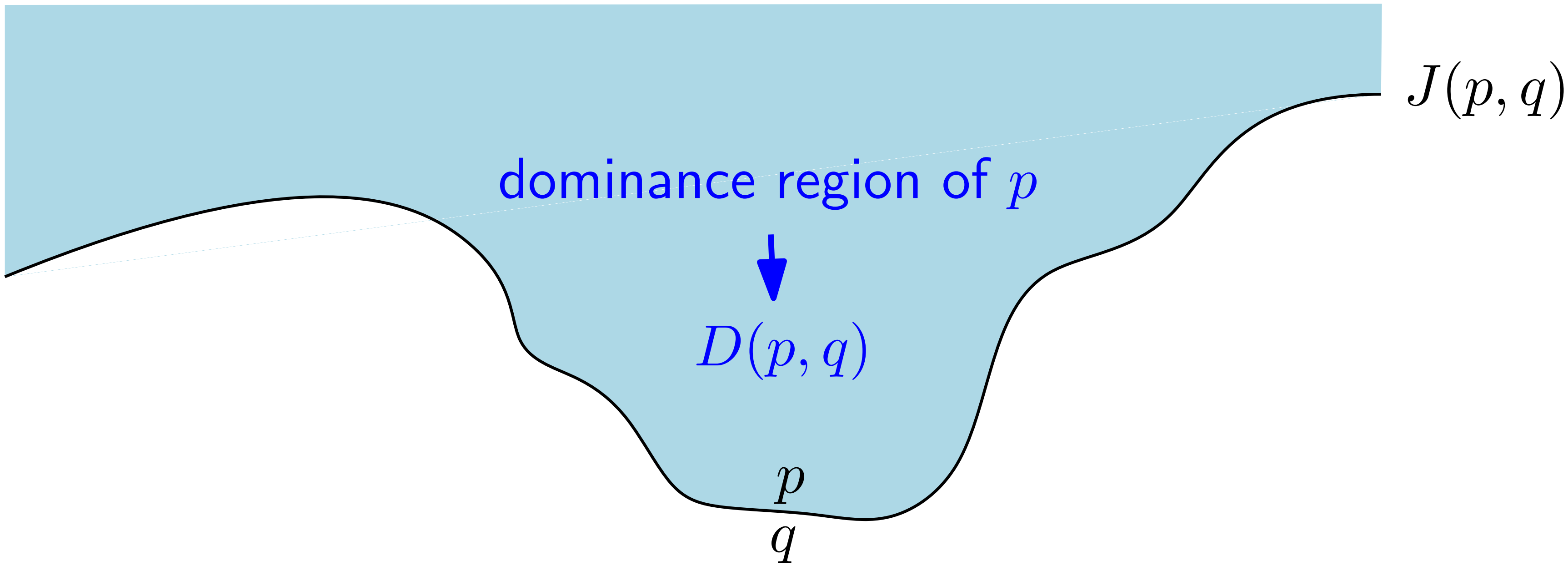

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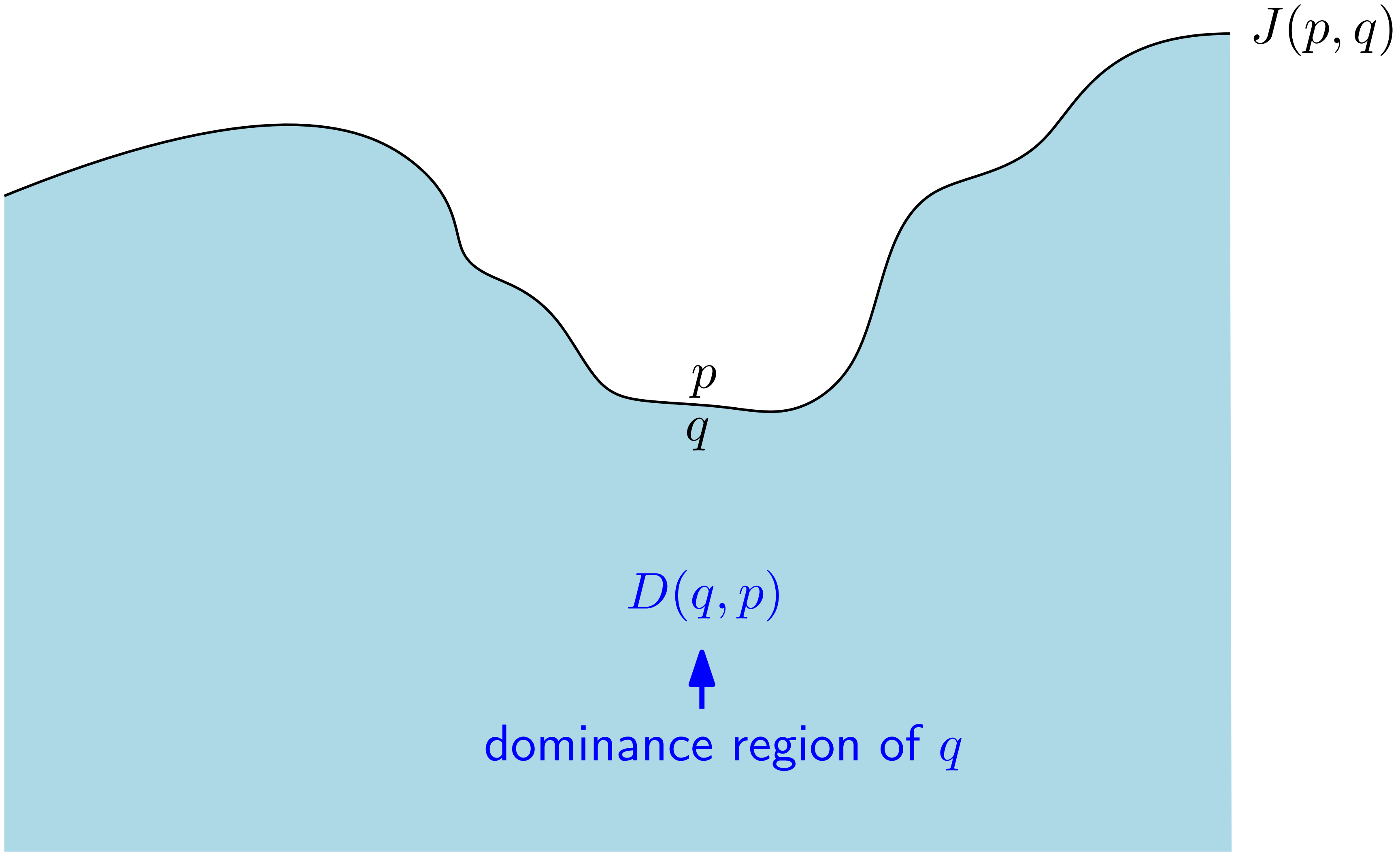

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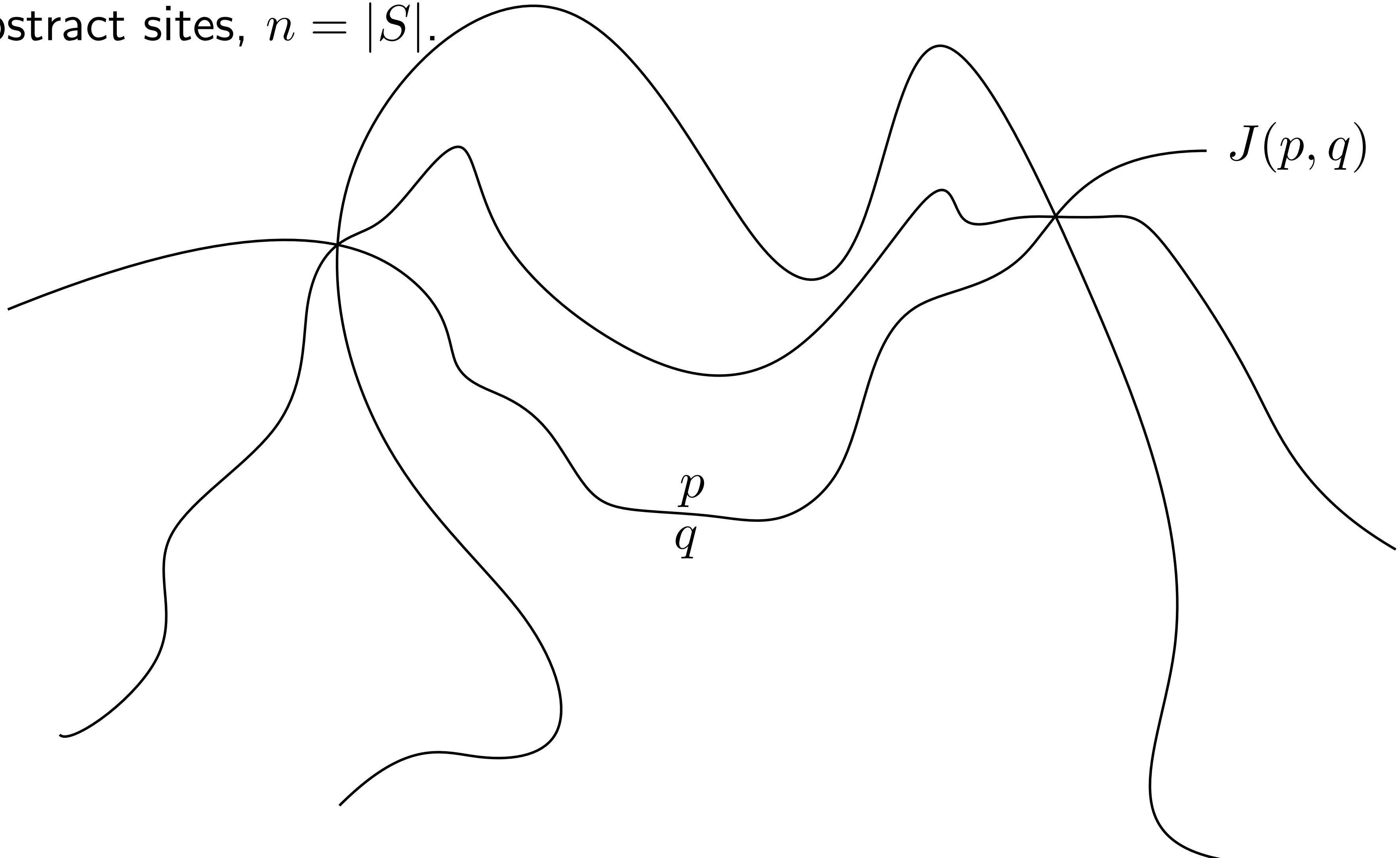

Given a set of bisectors $\mathcal{J} := \{J(p,q) : p \neq q \in S\}$.

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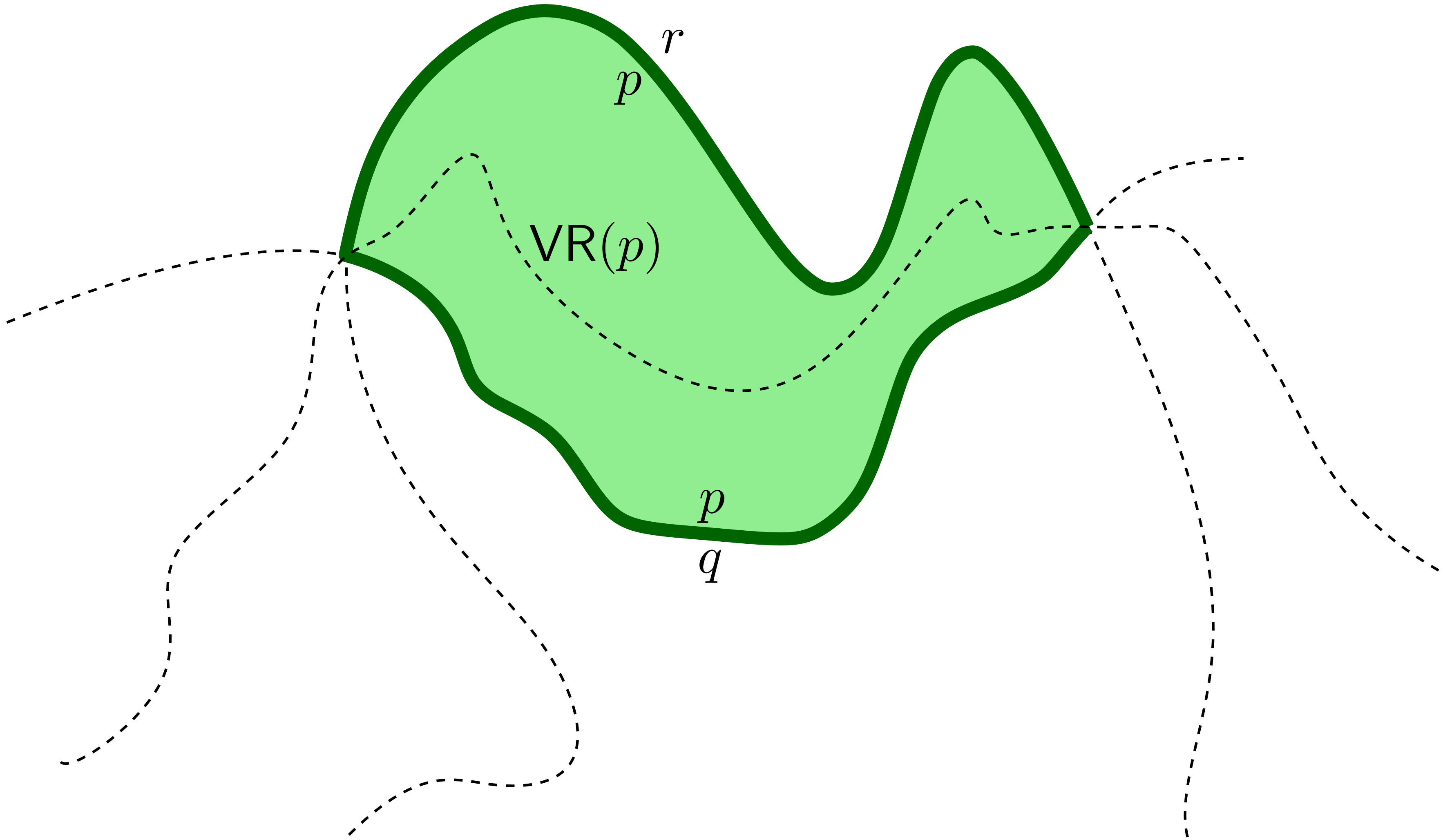

Voronoi region: $\boxed{\mathrm{VR}(p) = \bigcap_{q \in S \setminus \{p\}} D(p,q)}$

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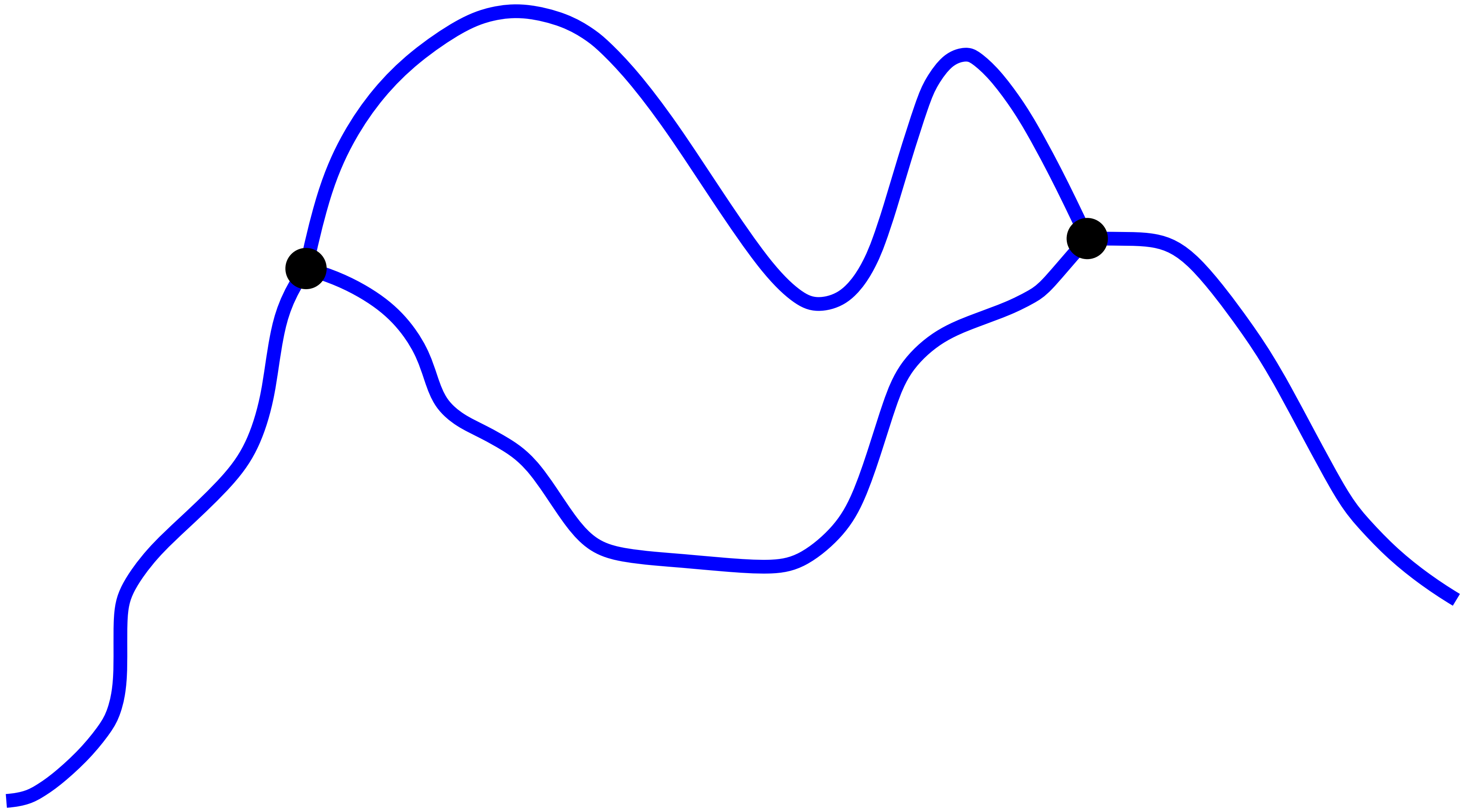

Voronoi diagram: $\boxed{\mathcal{V}(S) = \mathbb{R}^2 \setminus \bigcup_{p \in S} \mathsf{VR}(p, S)}$

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Admissible bisector system

Given $\mathcal{J} := \{J(p,q) : p \neq q \in S\}$.

For every $S' \subseteq S$:

- (A1) Voronoi regions are non-empty and connected.
- (A2) Voronoi regions cover the plane.
- (A3) Bisectors are unbounded Jordan curves.
- (A4) Transversal and finite # intersections.

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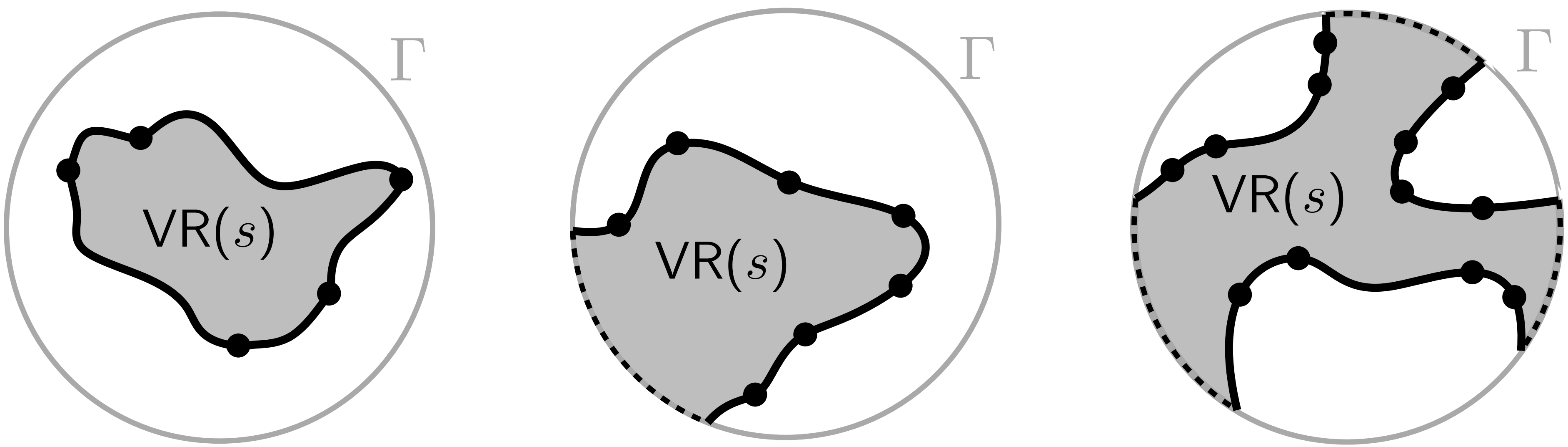
Admissible bisector system

- For simplicity we always assume a big circle Γ, containing all intersections.
  We restrict all computations in the interior of Γ.

- VR(s) can be bounded, unbounded, and have several openings to infinity (Γ-arcs).



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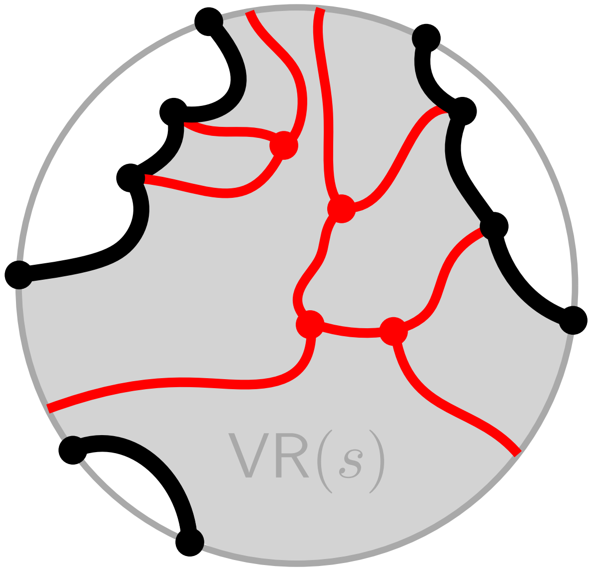
Site deletion

Problem: Compute $\mathcal{V}(S \setminus s) \cap \mathsf{VR}(s)$ (within $\mathsf{VR}(s)$).



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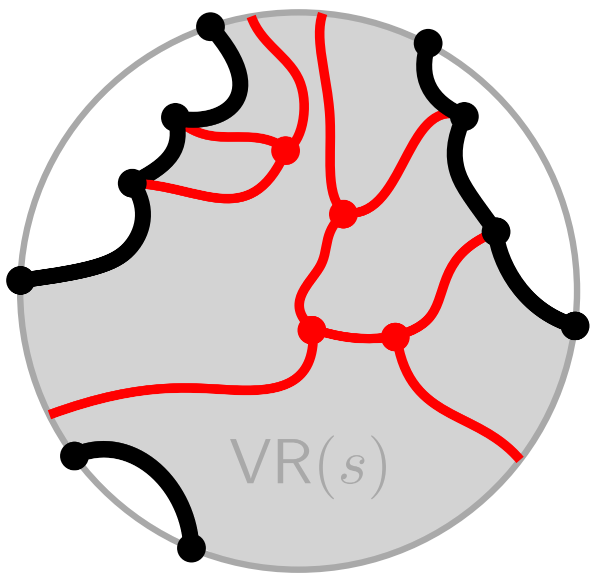
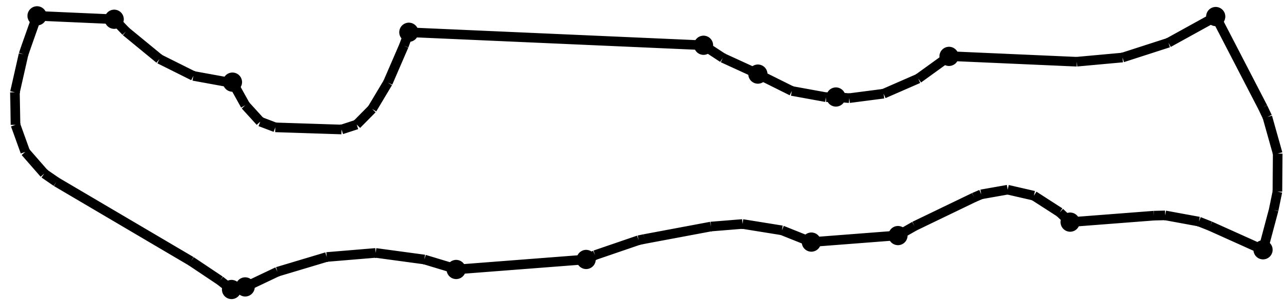


Lemma: $\mathcal{V}(S \setminus s) \cap \mathsf{VR}(s)$ is a **forest** with **one face per Voronoi edge** of $\partial\mathsf{VR}(s)$.

Voronoi regions of arcs

Idea: Treat the boundary arcs (Voronoi edges) of $\mathrm{VR}(s)$ as **sites**.



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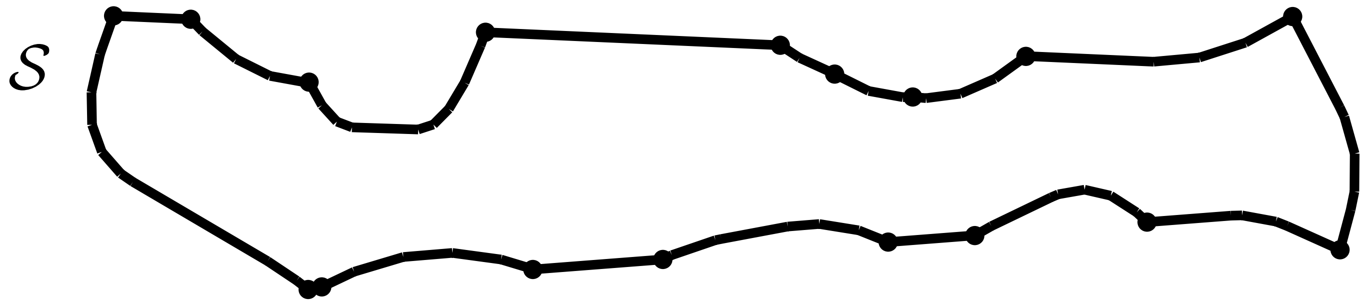

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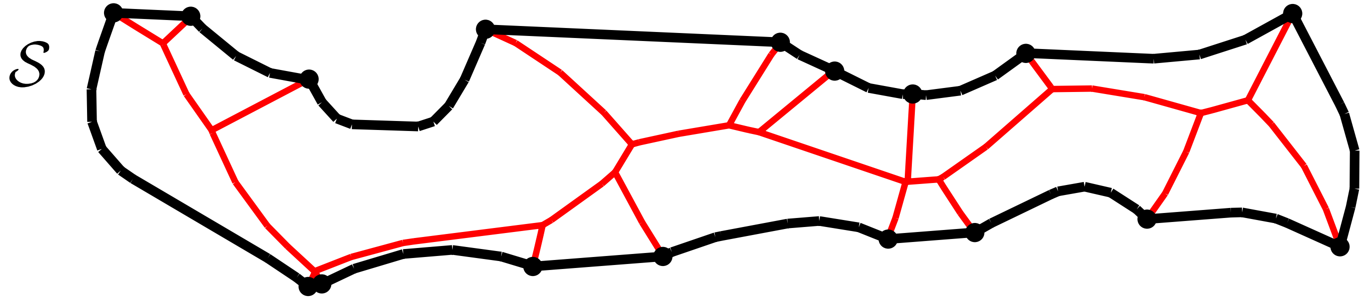

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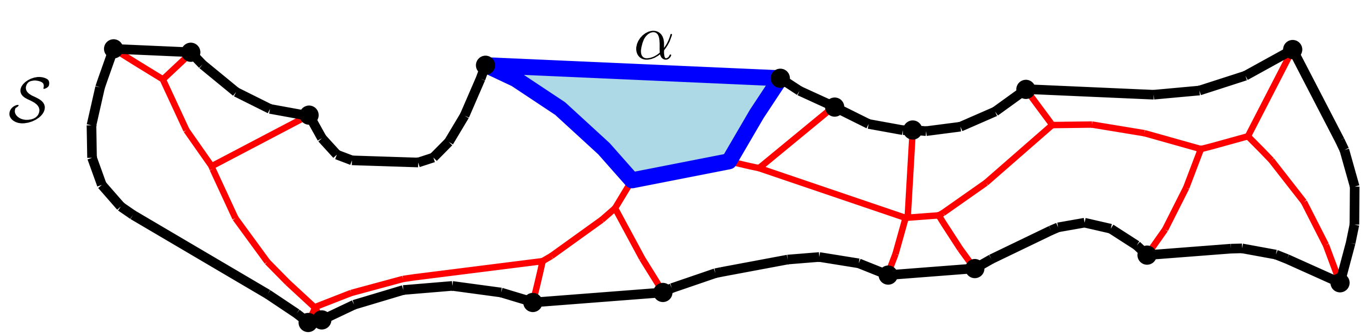

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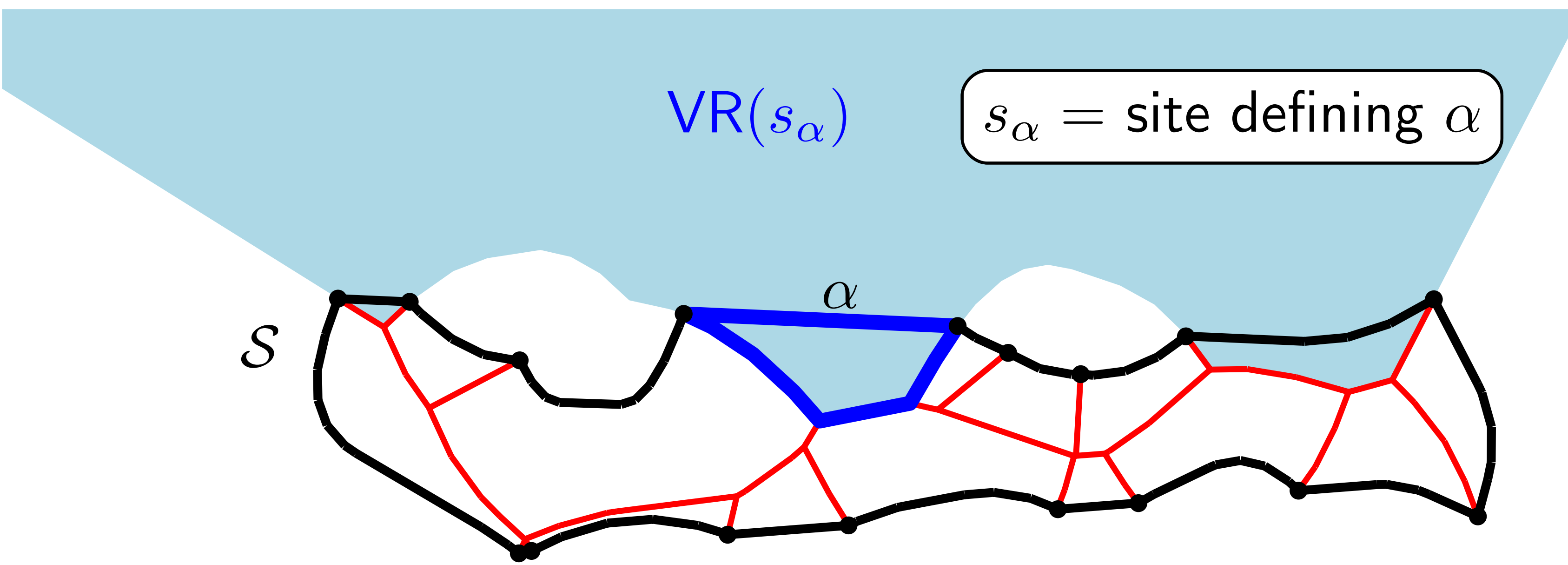
Site s_α can have $\Theta(n)$ faces within VR(s).
Treat each face independently (different arc).



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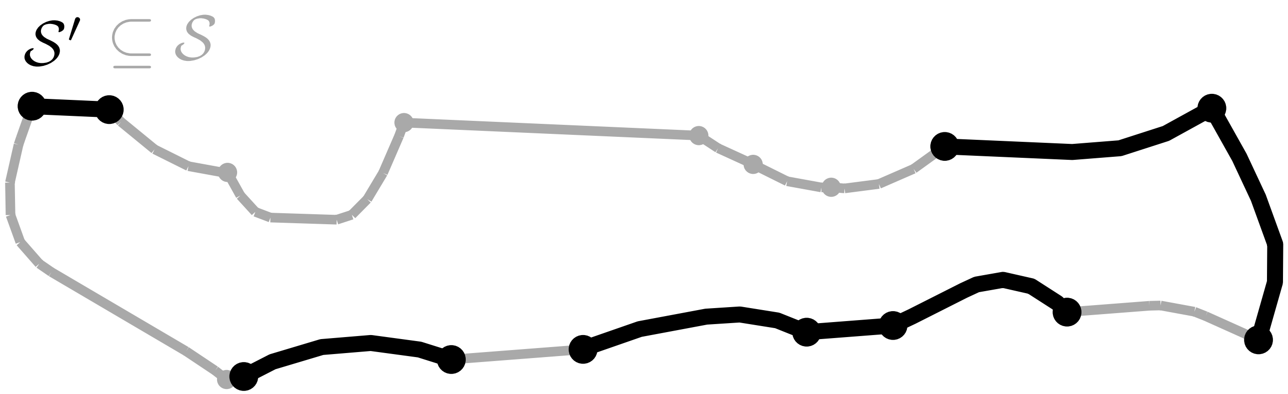

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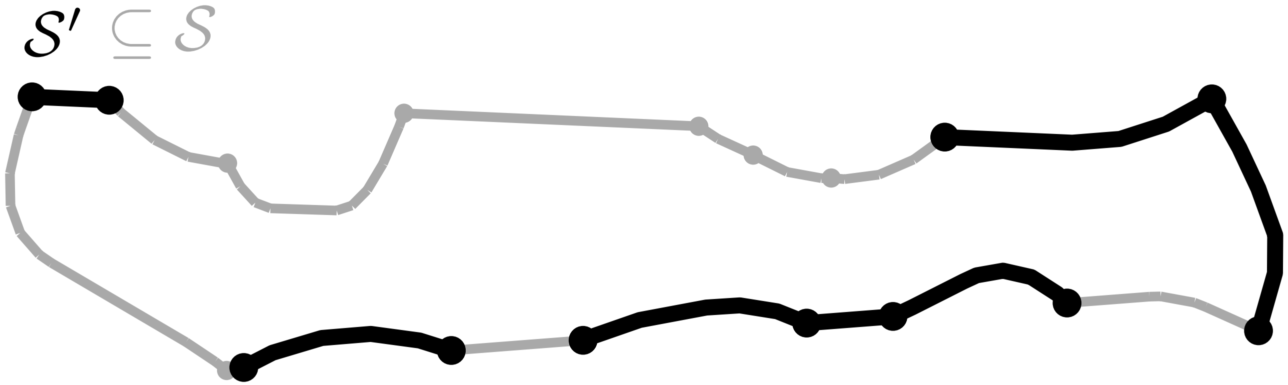
Instead we define a **Voronoi-like diagram**
for a subset of arcs $\mathcal{S}' \subseteq \mathcal{S}$.



Wish: Voronoi diagram of a **subset of arcs** $\mathcal{S}' \subseteq \mathcal{S}$.
But that does not exist. ↯

Instead we define a **Voronoi-like diagram**
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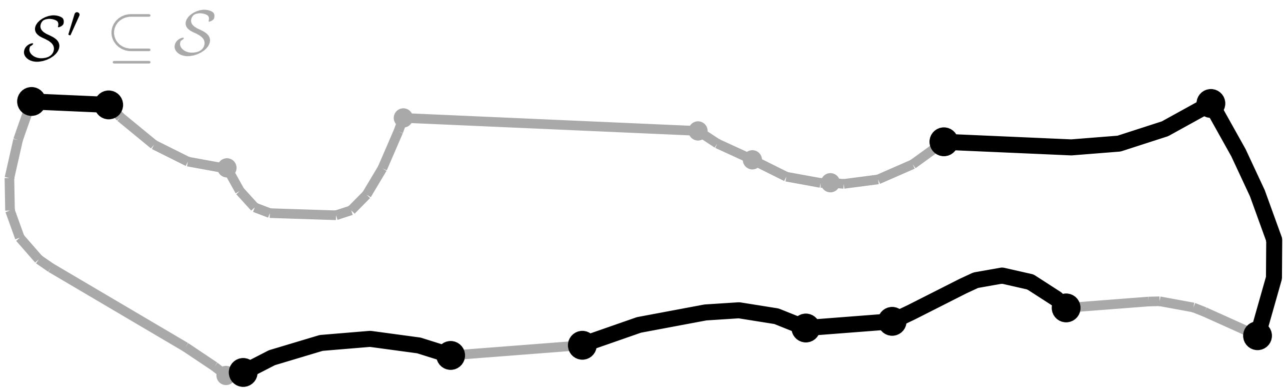
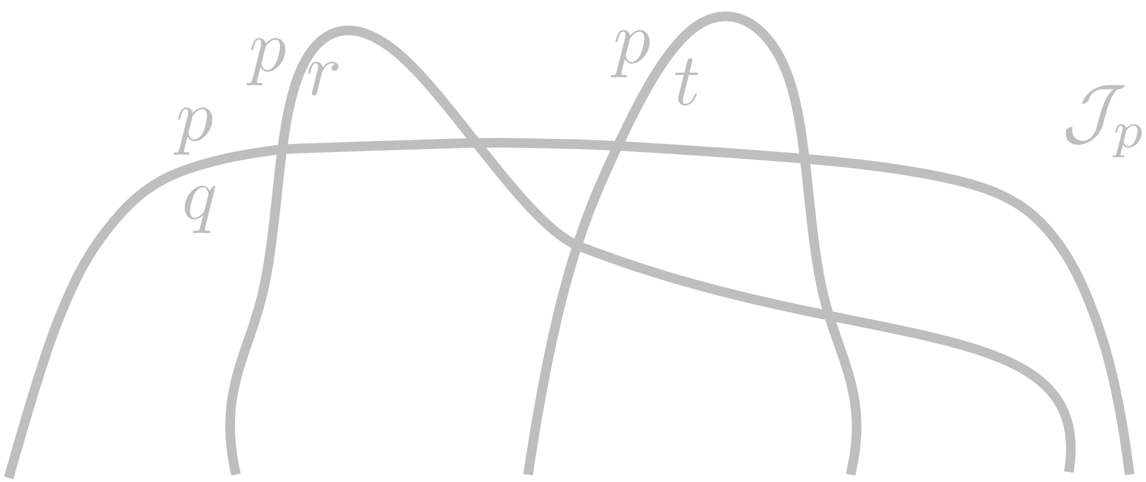
Next: Definitions...



p-monotone paths

Let $p \in S$ be a site. Let $\mathcal{J}_p$ be the arrangement of all p-related bisectors.



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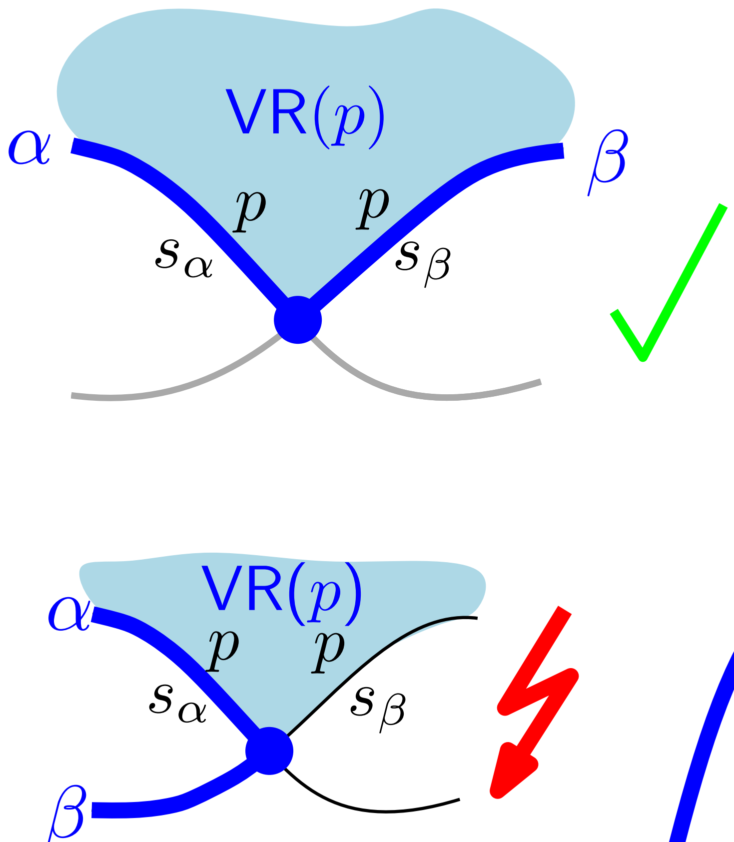

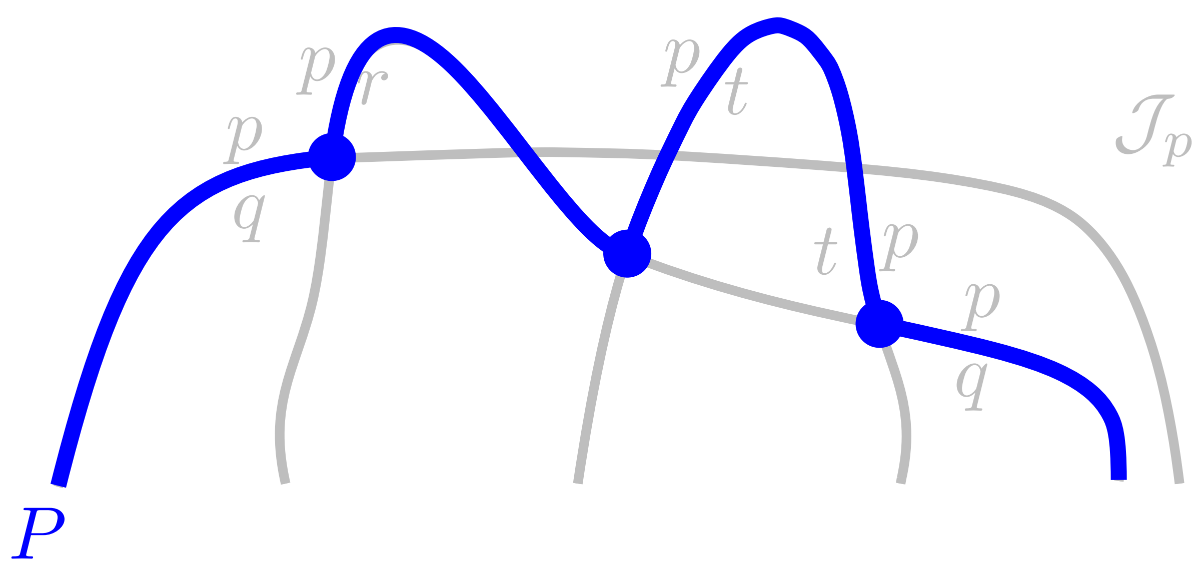

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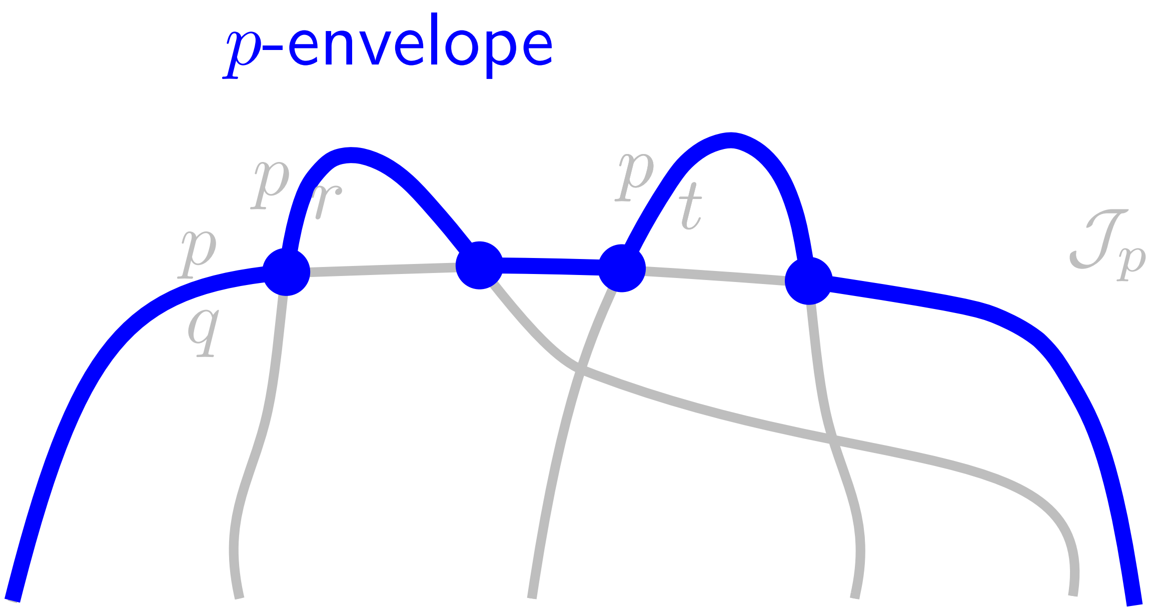
A path in $\mathcal{J}_p$ is the p-**envelope**, if it is the boundary of $\mathrm{VR}(p)$



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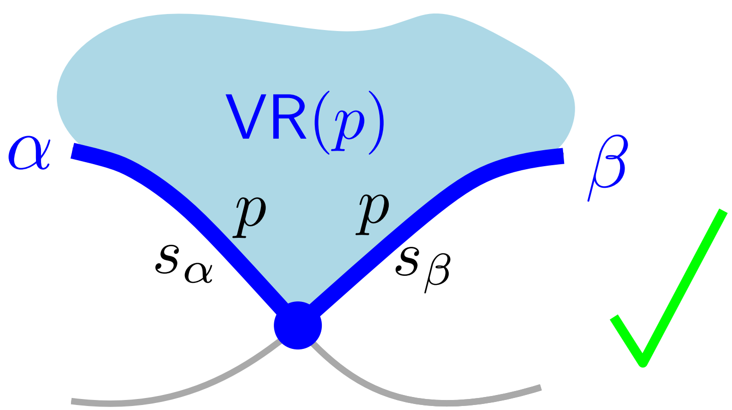
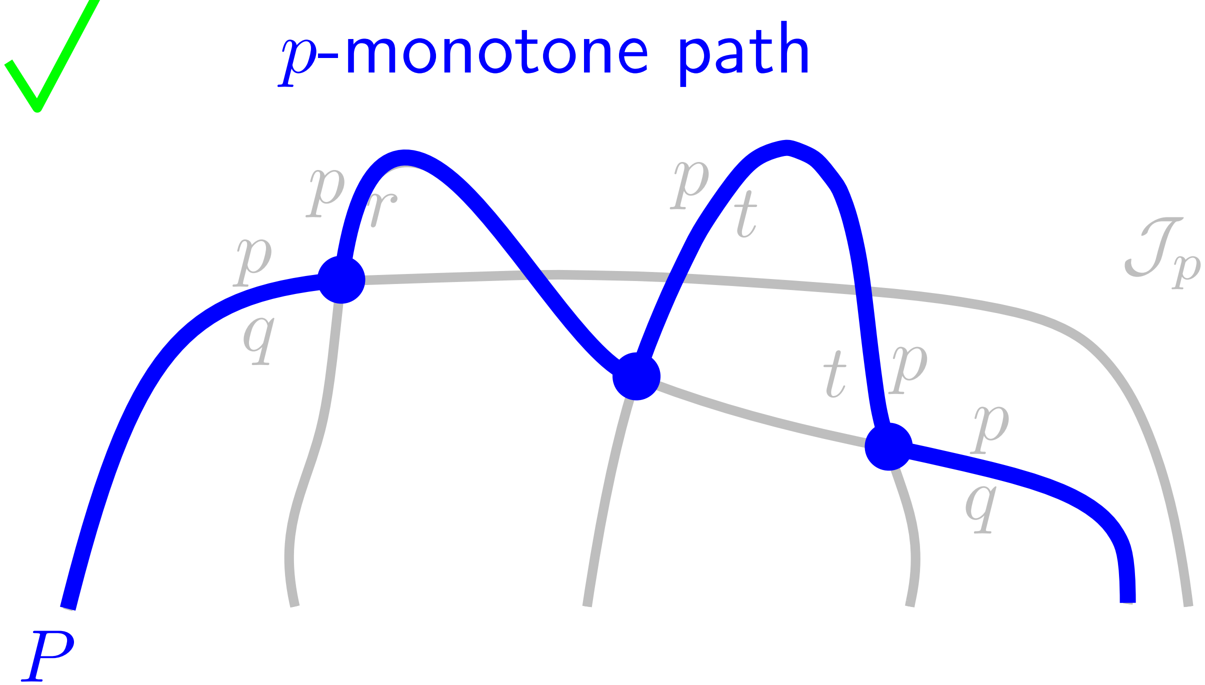
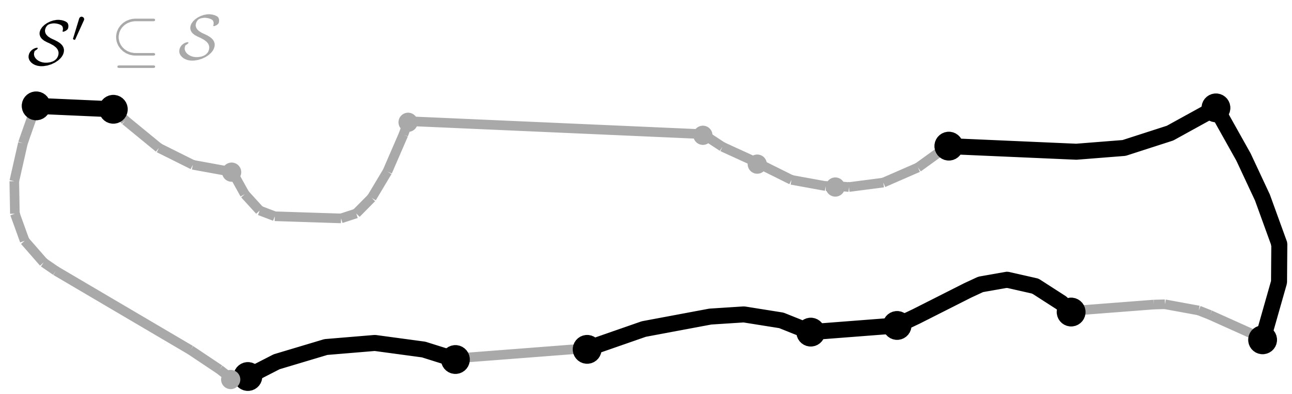




Boundary curve

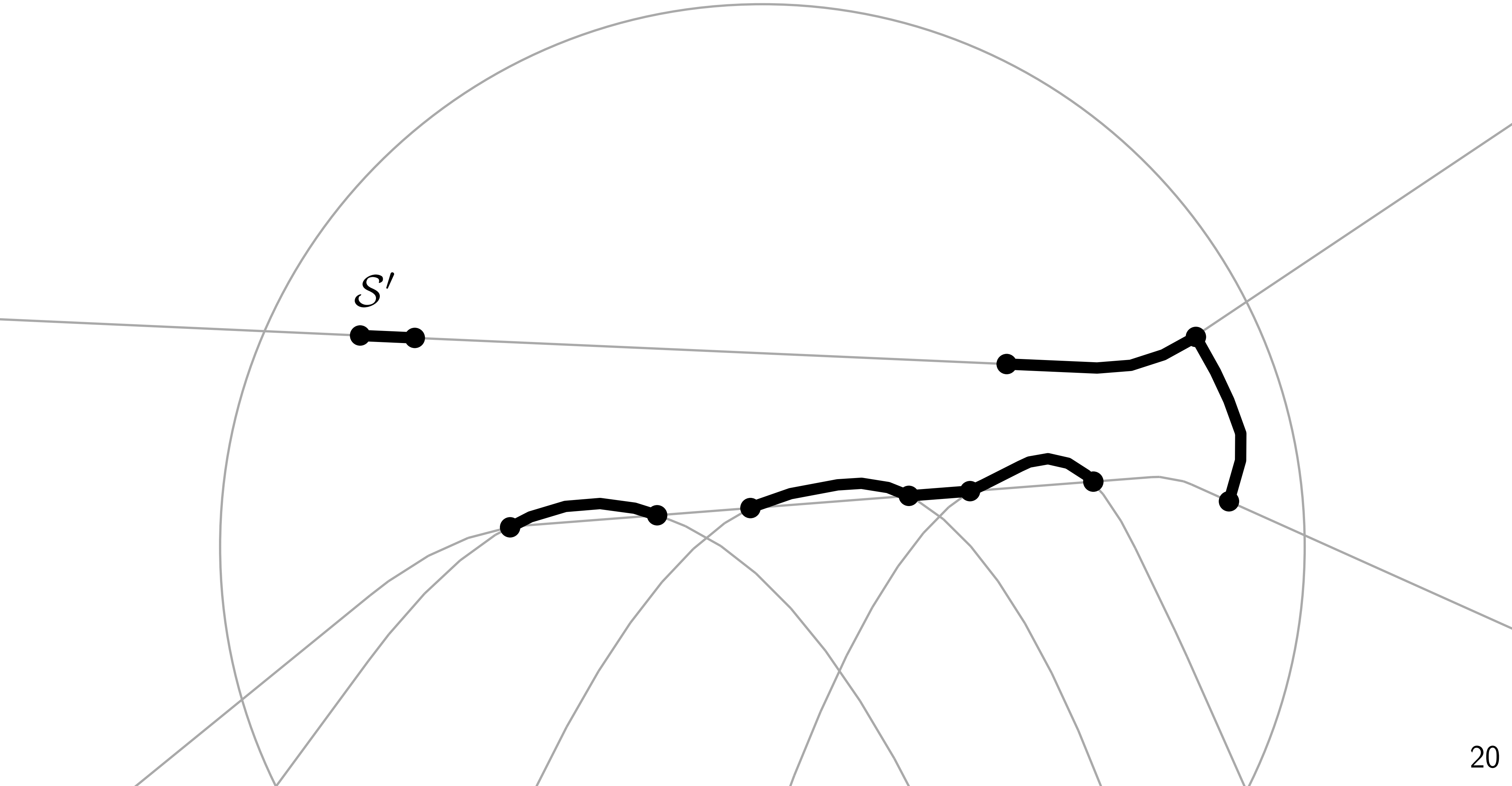
Let $\mathcal{S}' \subseteq \mathcal{S} =$ boundary arcs (Voronoi edges) along $\partial\mathrm{VR}(s)$.



Boundary curve

Let $\mathcal{S}' \subseteq \mathcal{S}$ = boundary arcs (Voronoi edges) along ∂VR(s).

Consider the arrangement of all s-related bisectors of arcs in $\mathcal{S}'$.



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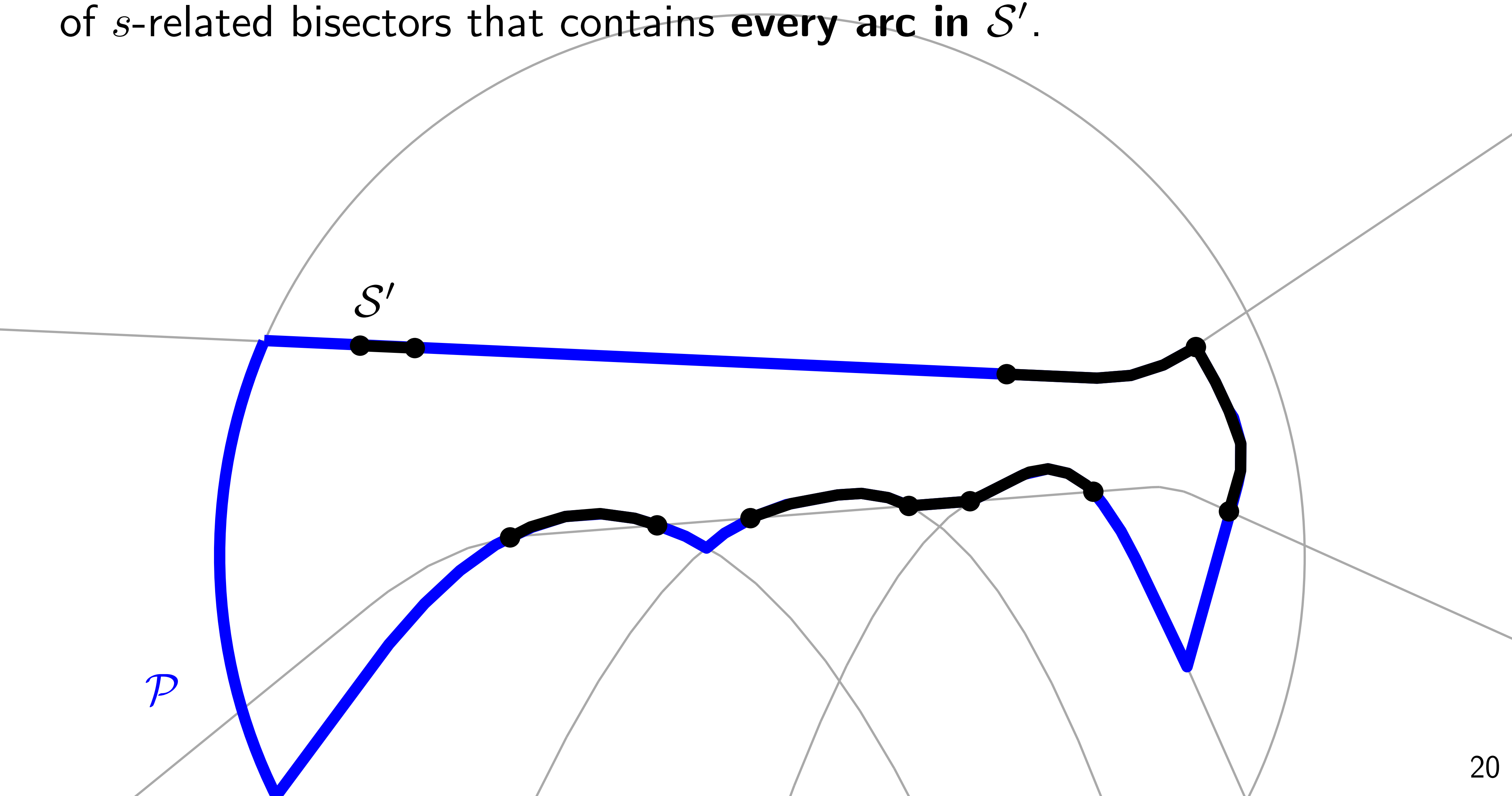
A **boundary curve** $\mathcal{P}$ for $\mathcal{S}'$ is an s-**monotone** path in the arrangement of s-related bisectors that contains **every arc in** $\mathcal{S}'$.



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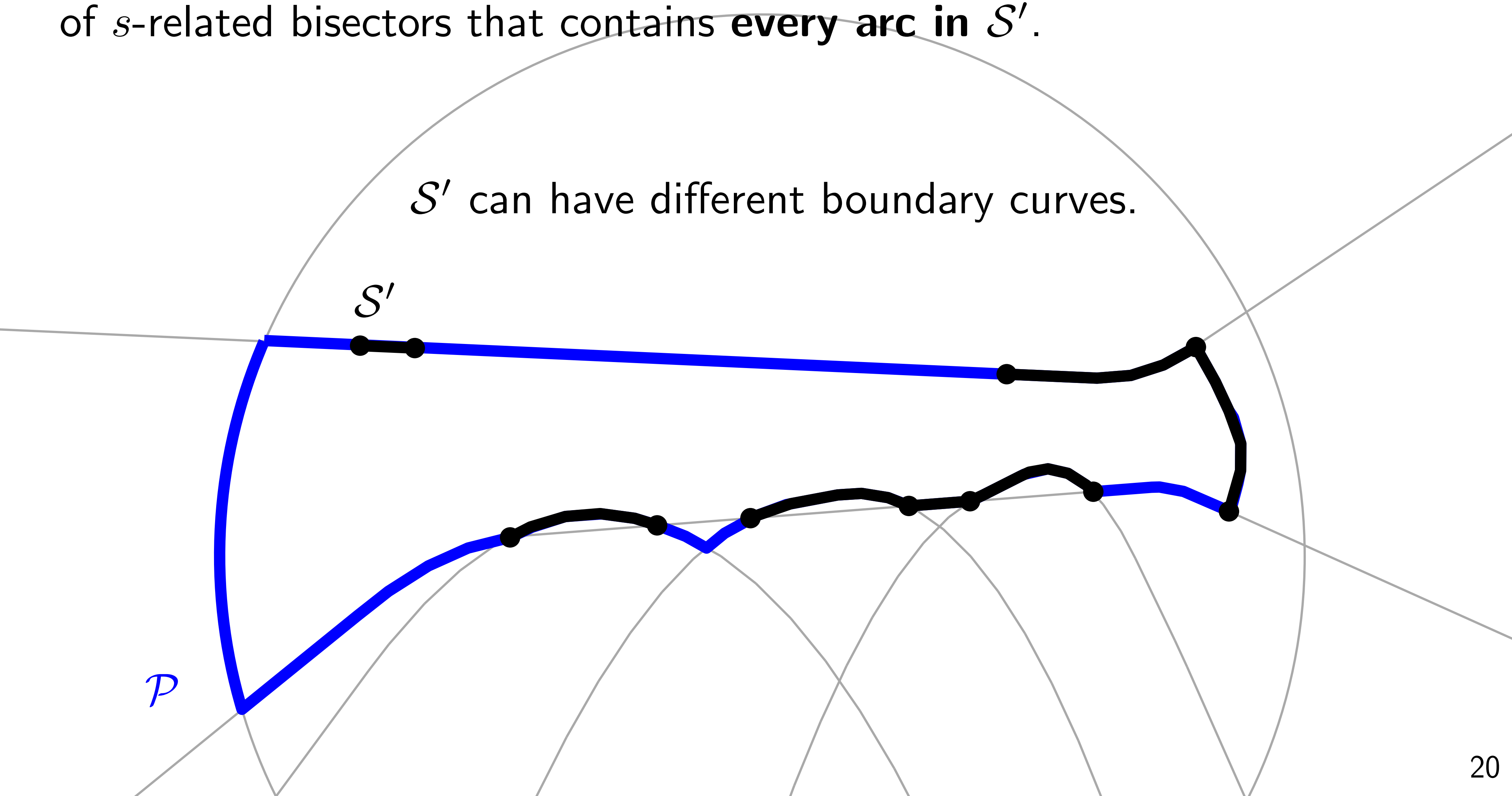
$\mathcal{S}'$ can have different boundary curves.



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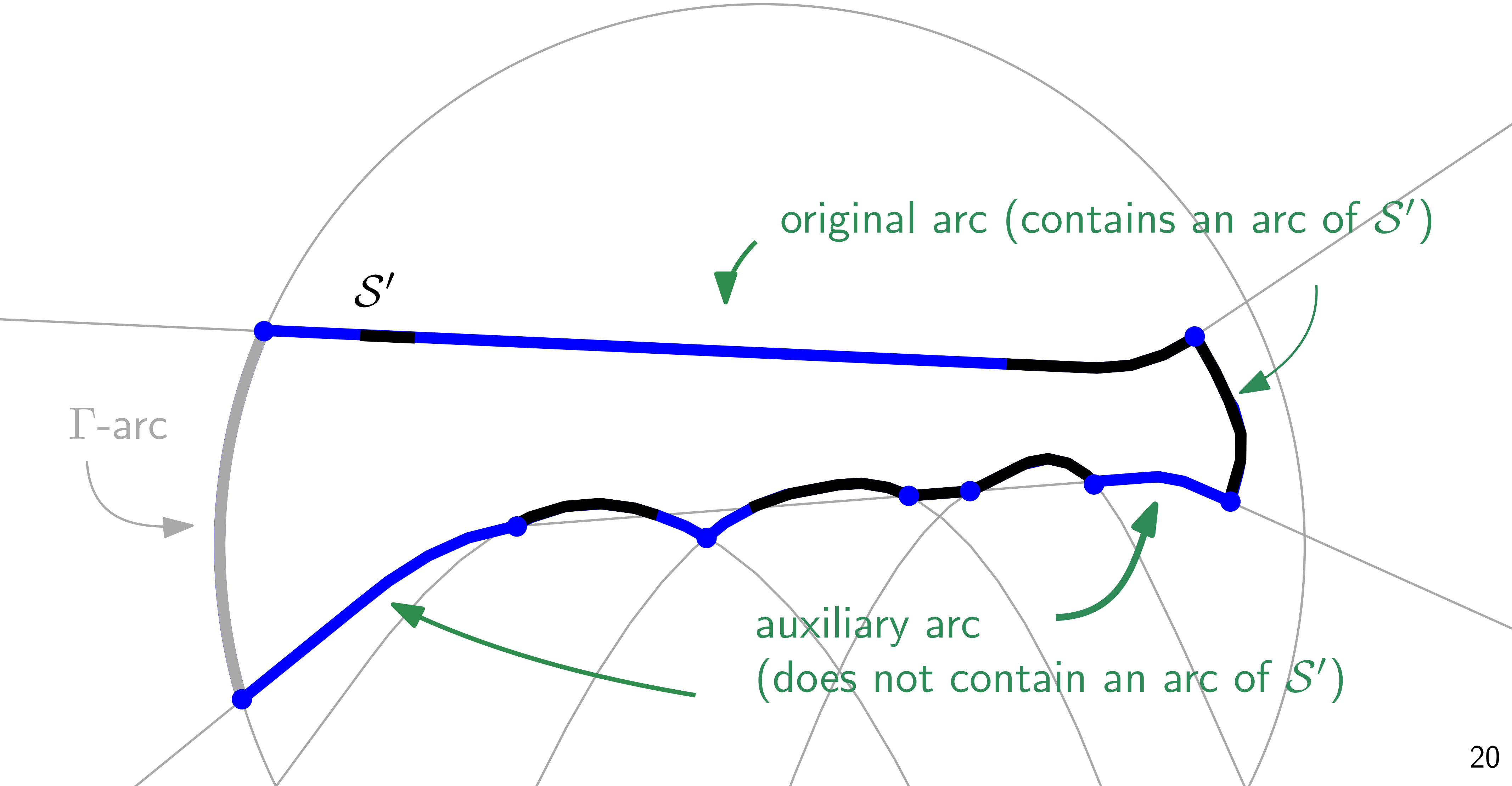

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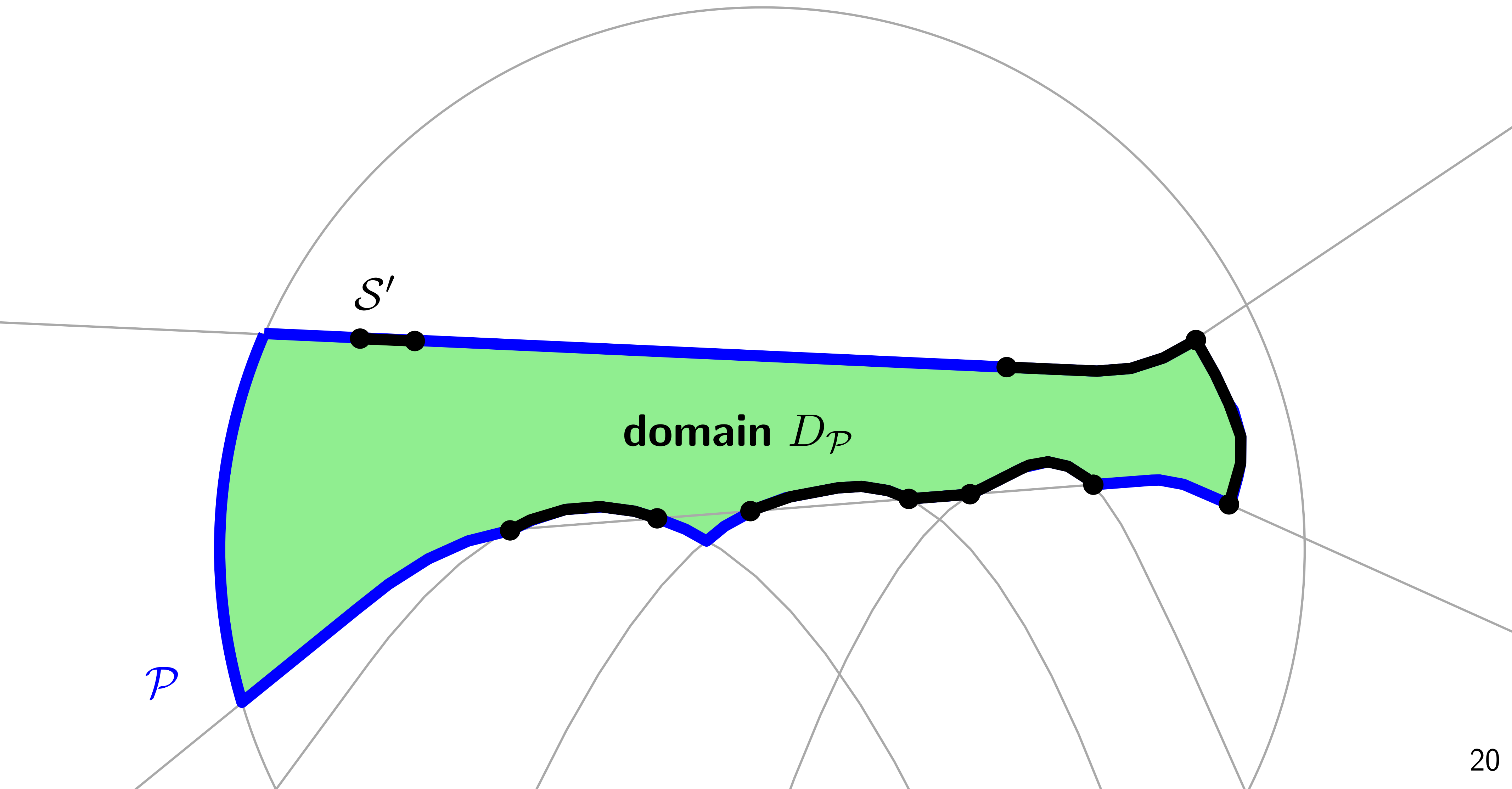
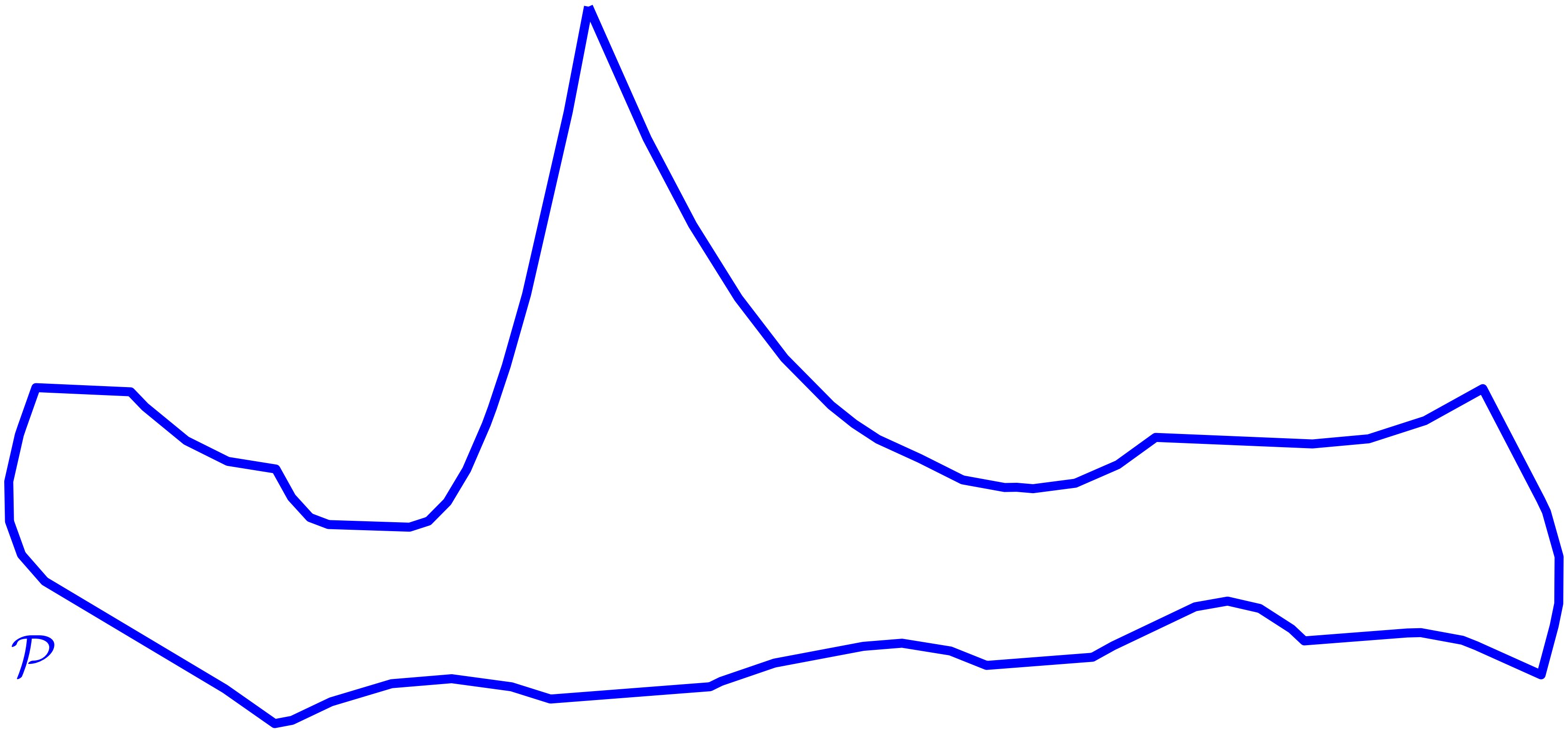


Voronoi-like diagram

Definition: Given a boundary curve $\mathcal{P}$, the Voronoi-like diagram $\mathcal{V}_l(\mathcal{P})$ is a subdivision of the domain $D_{\mathcal{P}}$ such that:



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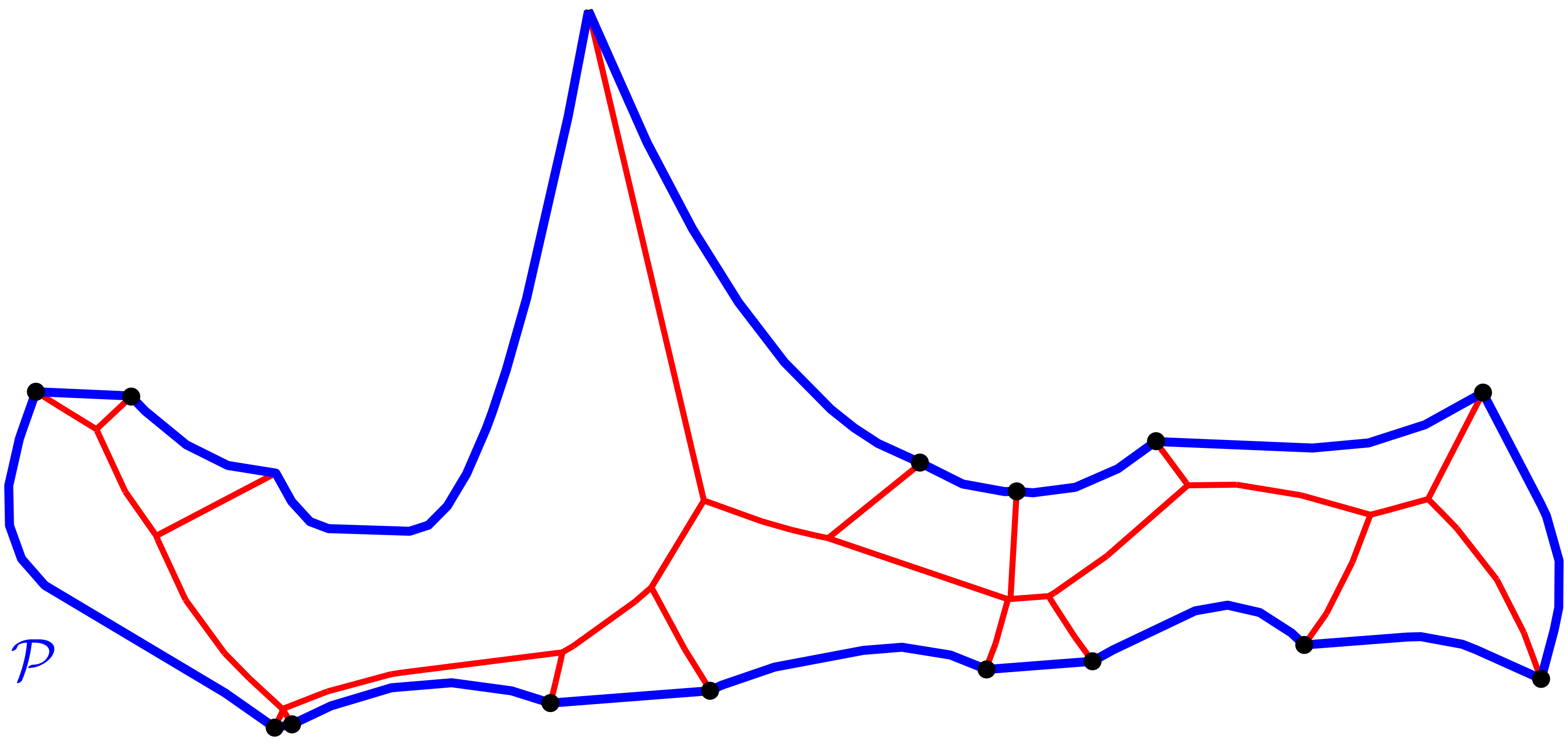

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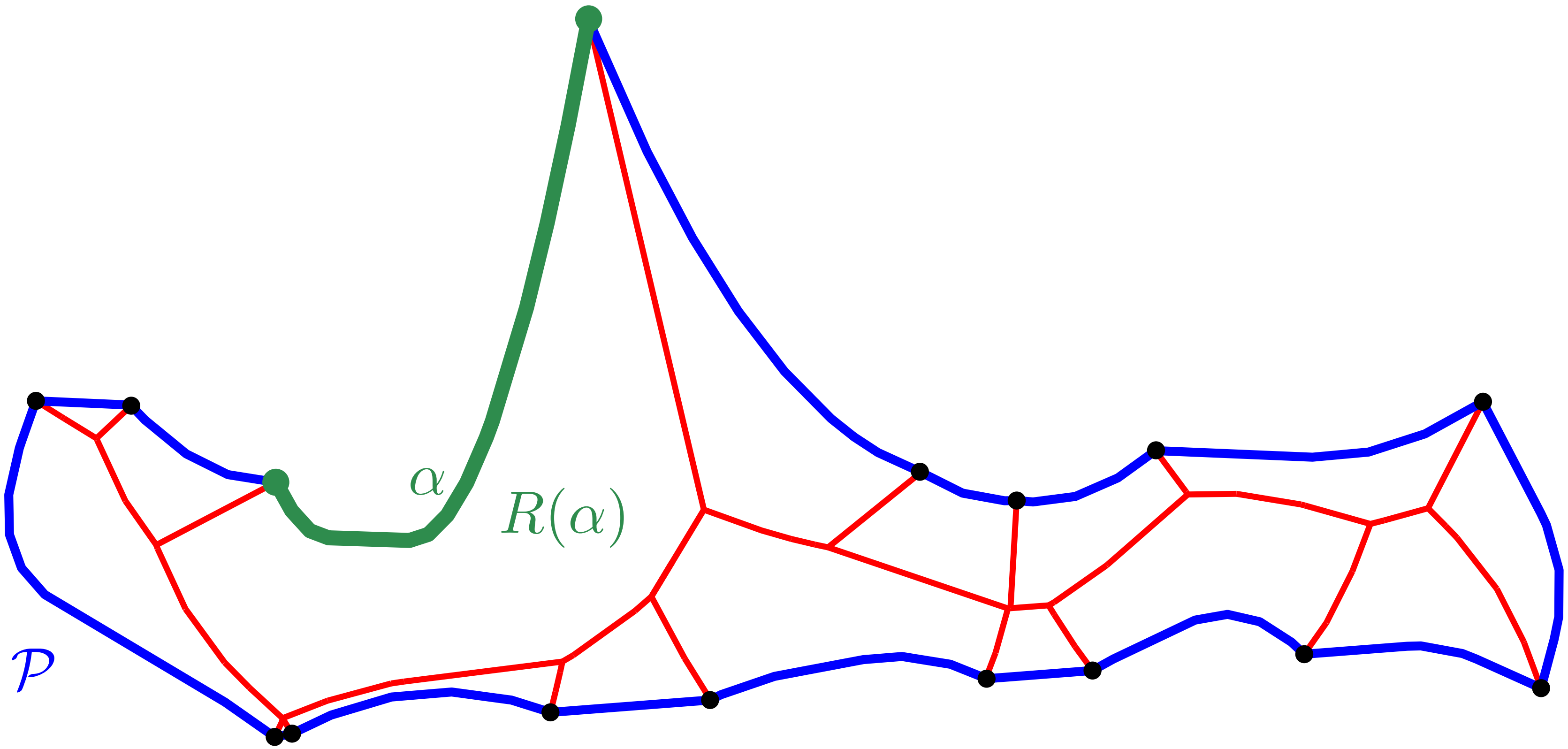

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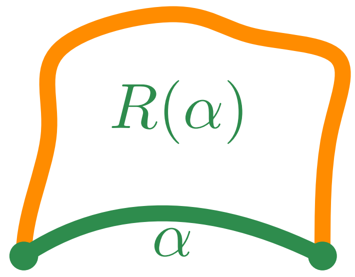
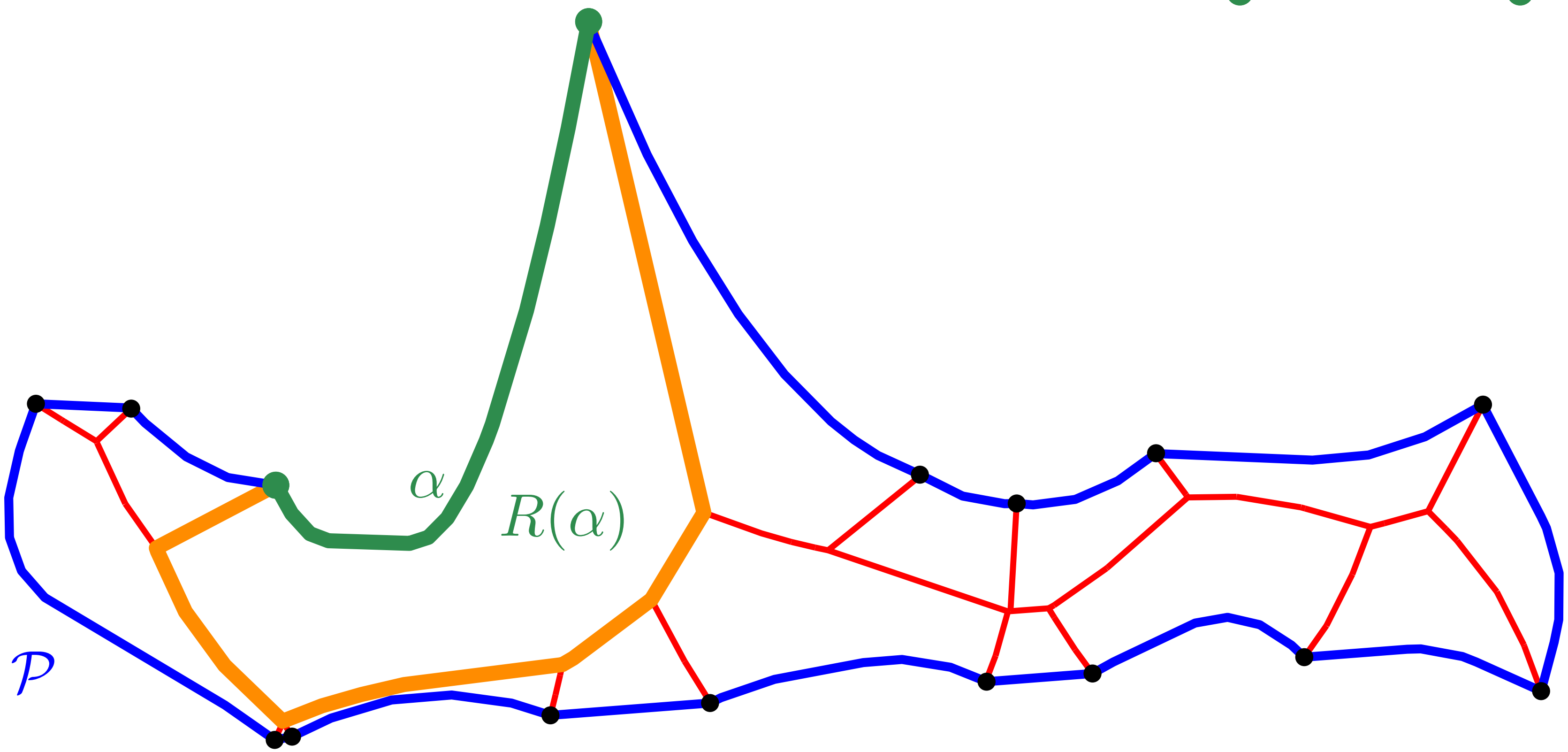
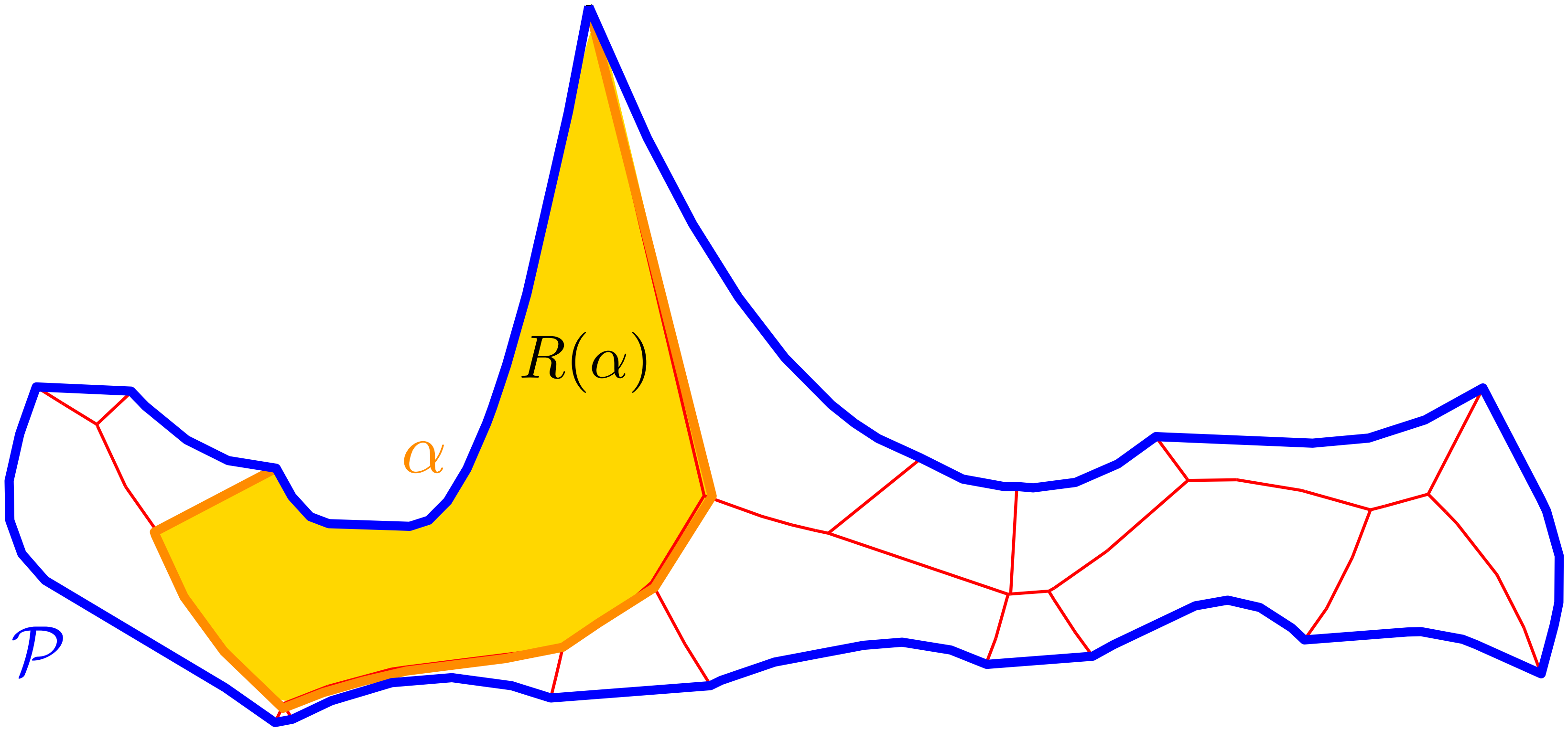
- $\partial R(\alpha)$ is an s_α-monotone path plus α.





Properties of Voronoi-like diagrams

- Voronoi-like regions are supersets of the real Voronoi regions.



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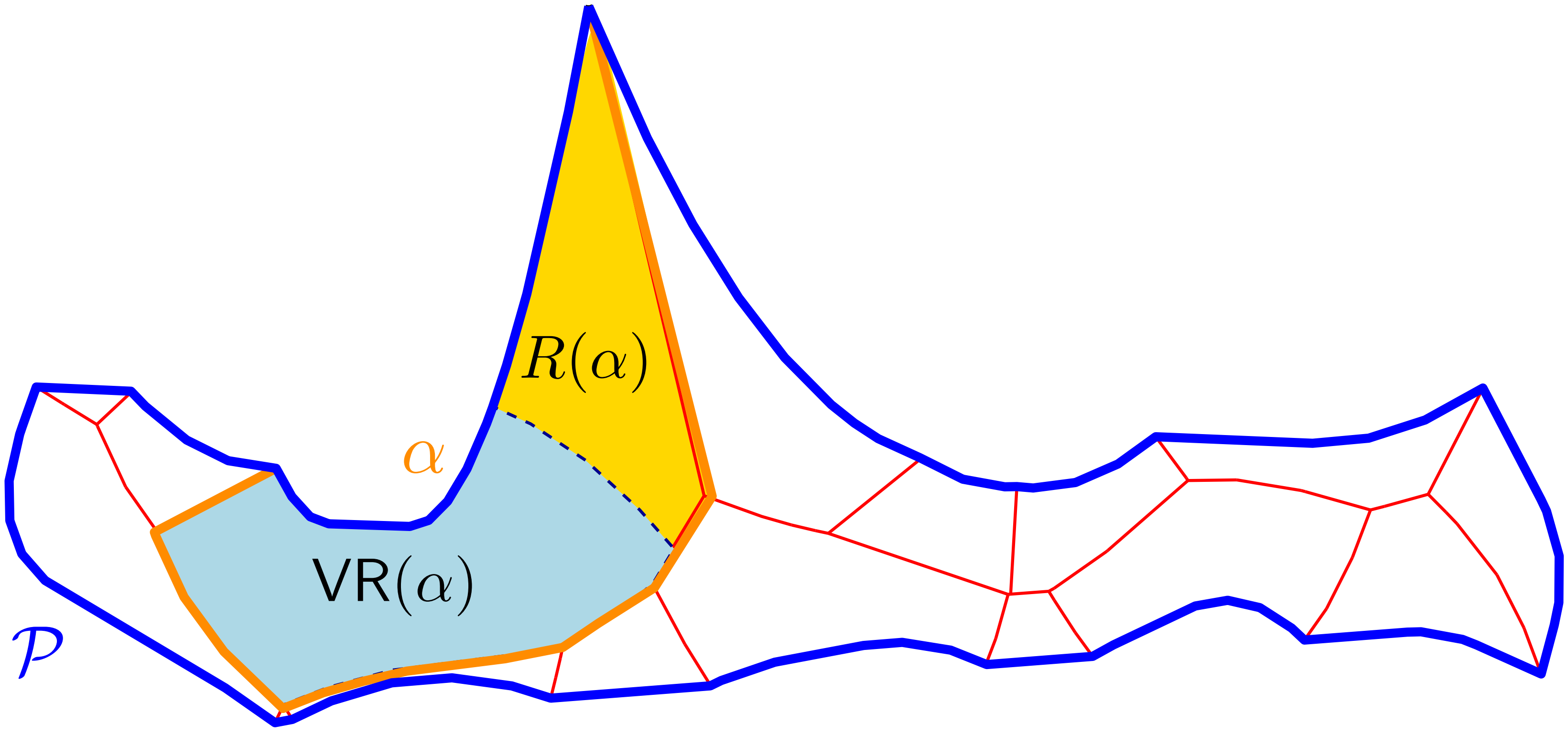

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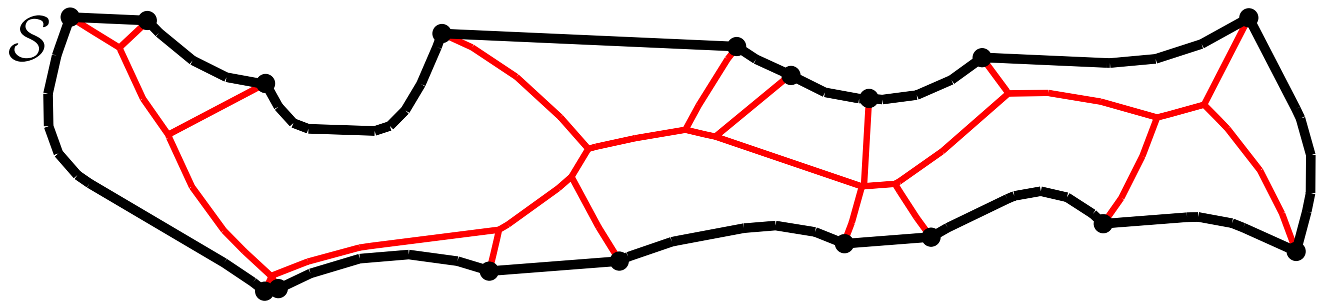

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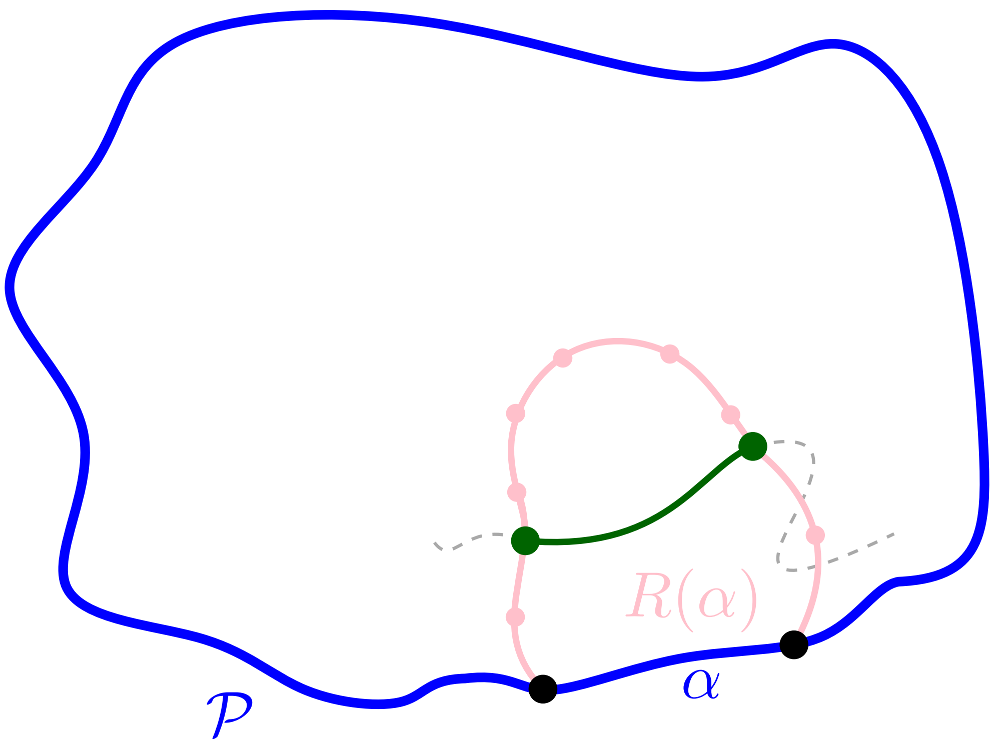
  Suppose an α-related bisector appears within $R(\alpha)$. Then there is an arc β "missing" from $\mathcal{P}$.



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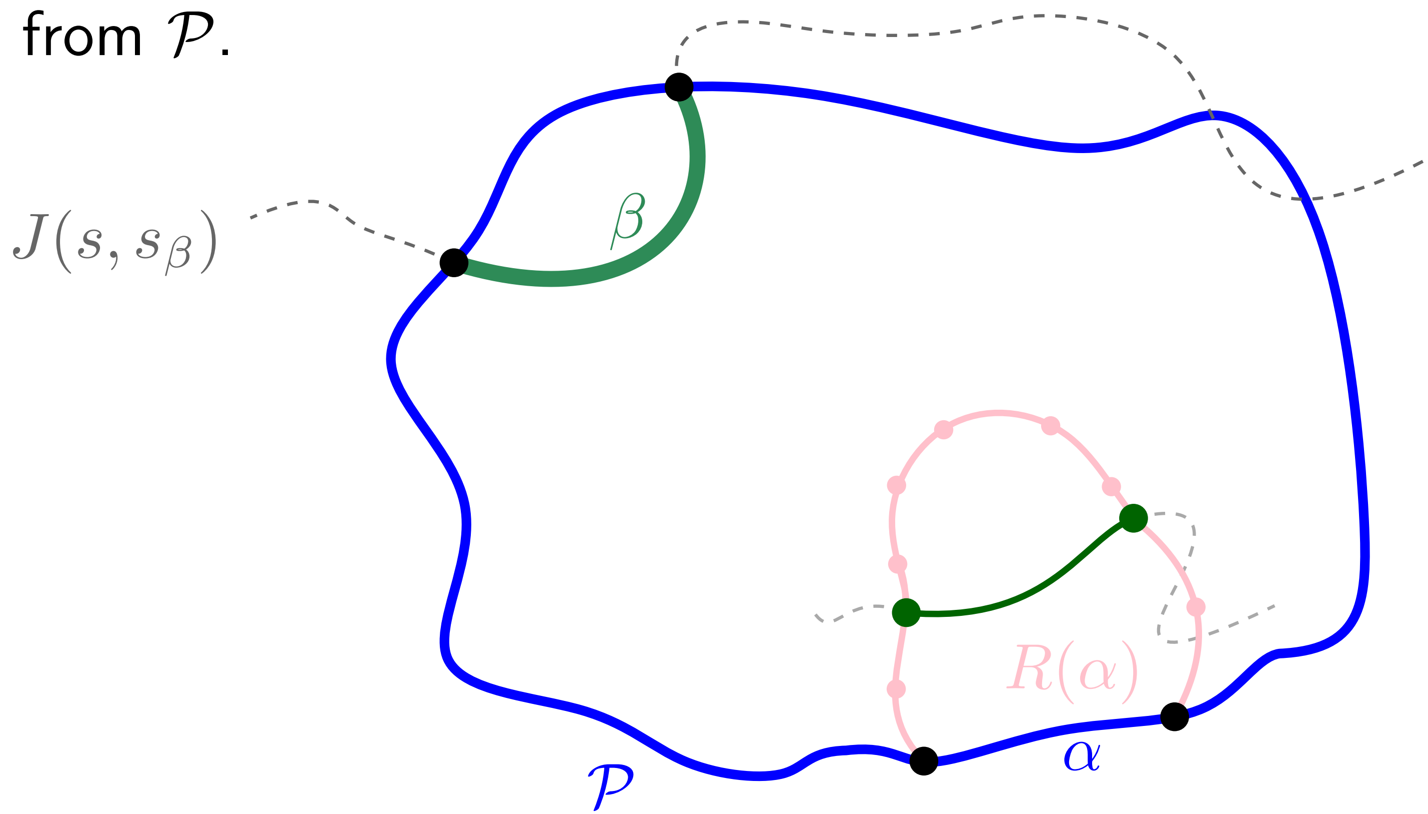
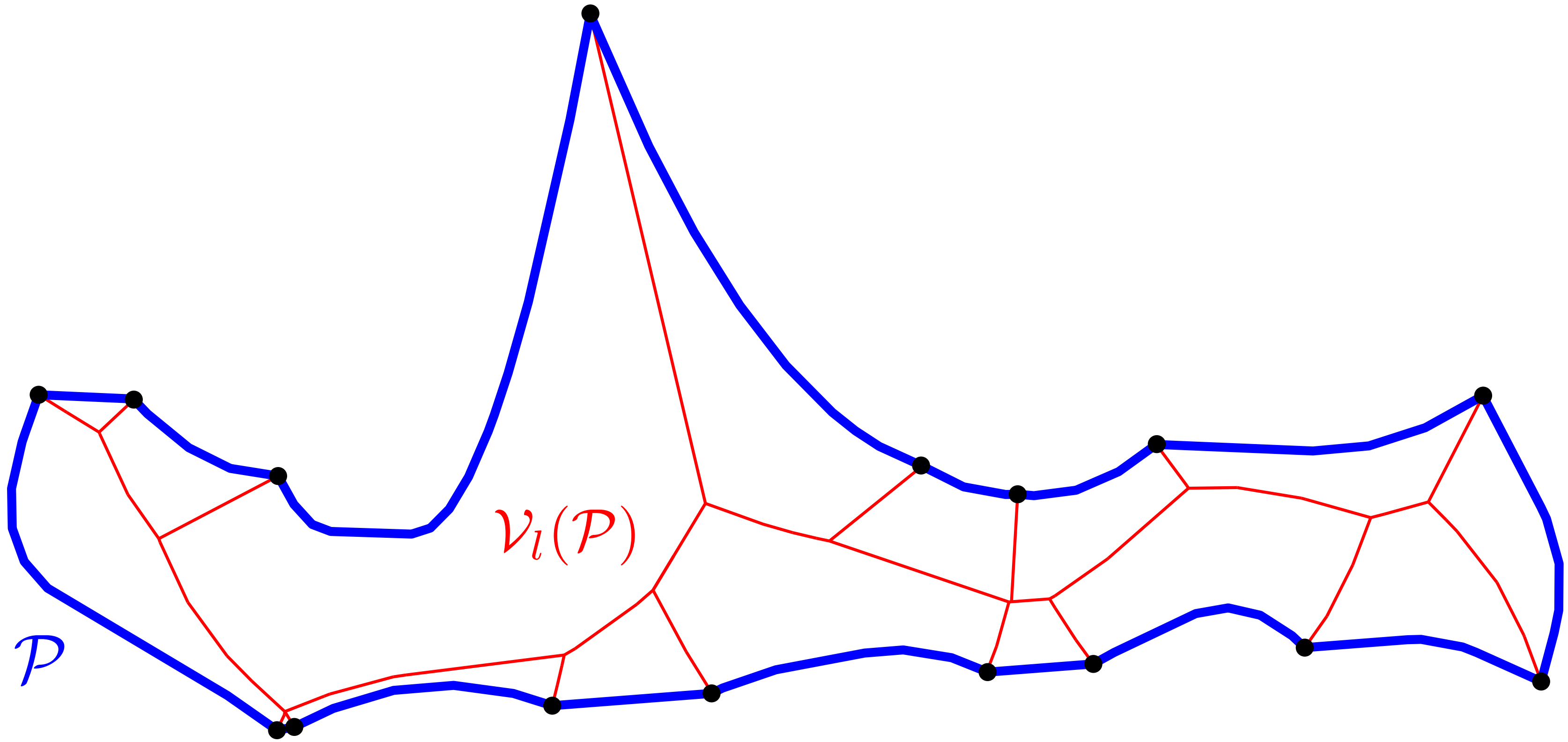


Uniqueness of Voronoi-like diagrams

Theorem:
The Voronoi-like diagram $\mathcal{V}_l(\mathcal{P})$ of a boundary curve $\mathcal{P}$ is **unique**.



No monotonicity property

Voronoi-like regions do not have the standard monotonicity property of real Voronoi regions:

$$\text{Voronoi diagram: } S' \subseteq S \Rightarrow \mathsf{VR}(p, S) \subseteq \mathsf{VR}(p, S')$$

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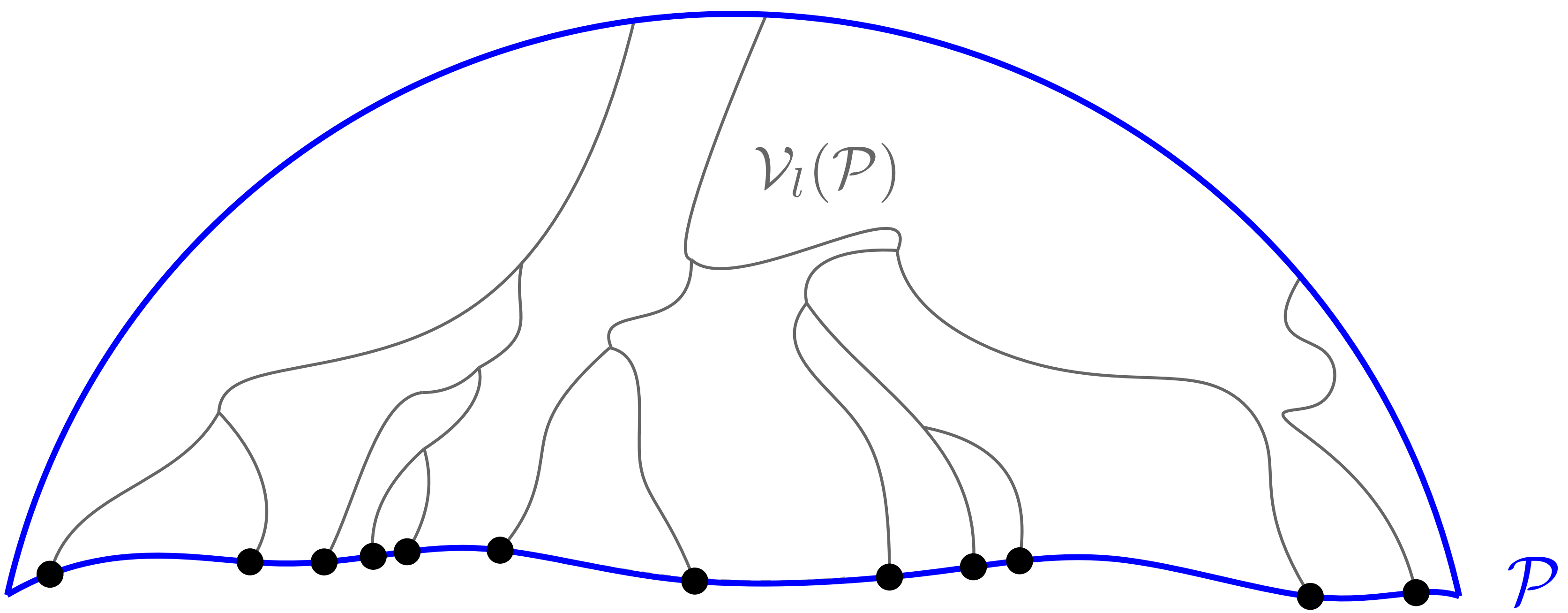
In proofs, use missing-arc lemma instead

Arc insertion

Arc insertion

Problem: Given a boundary curve $\mathcal{P}$ for $\mathcal{S}' \subset \mathcal{S}$ and its Voronoi-like diagram $\mathcal{V}_l(\mathcal{P})$, **insert** arc $\beta^* \in \mathcal{S} \setminus \mathcal{S}'$, prolong $\beta^* \subseteq \beta$, compute $R(\beta)$, and update the diagram to $\mathcal{V}_l(\mathcal{P}) \oplus \beta$.



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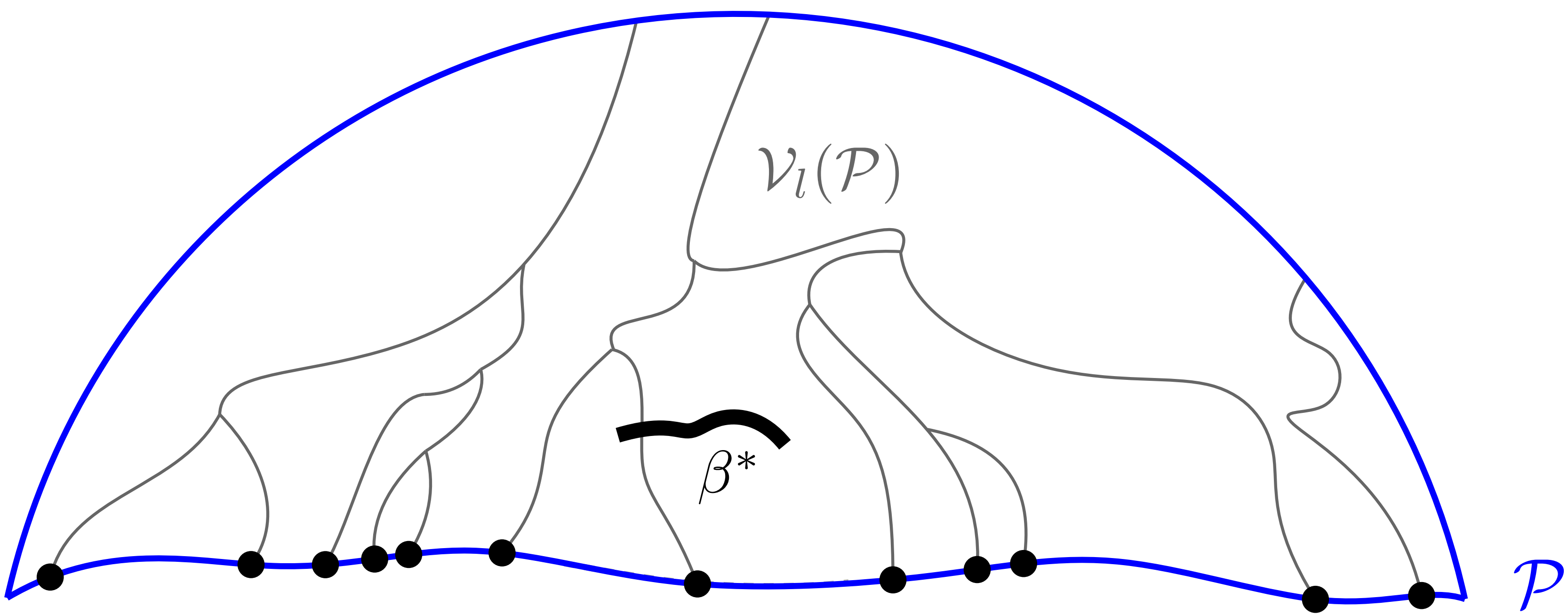

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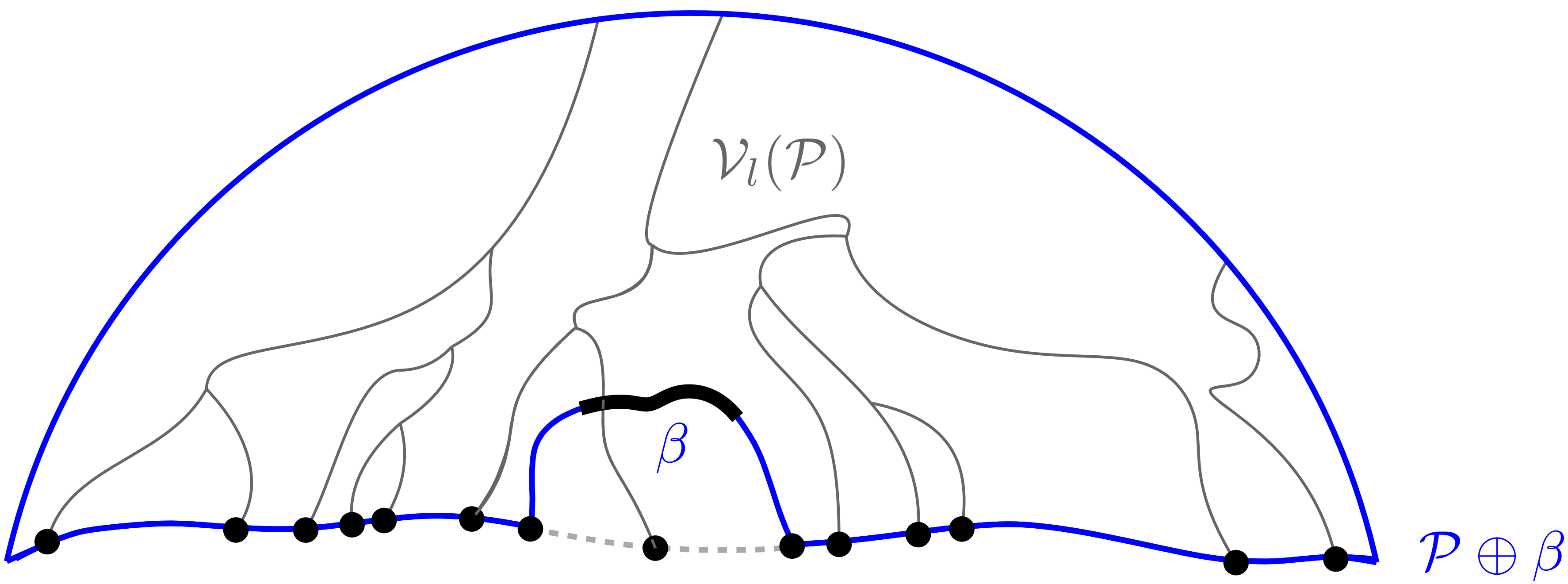

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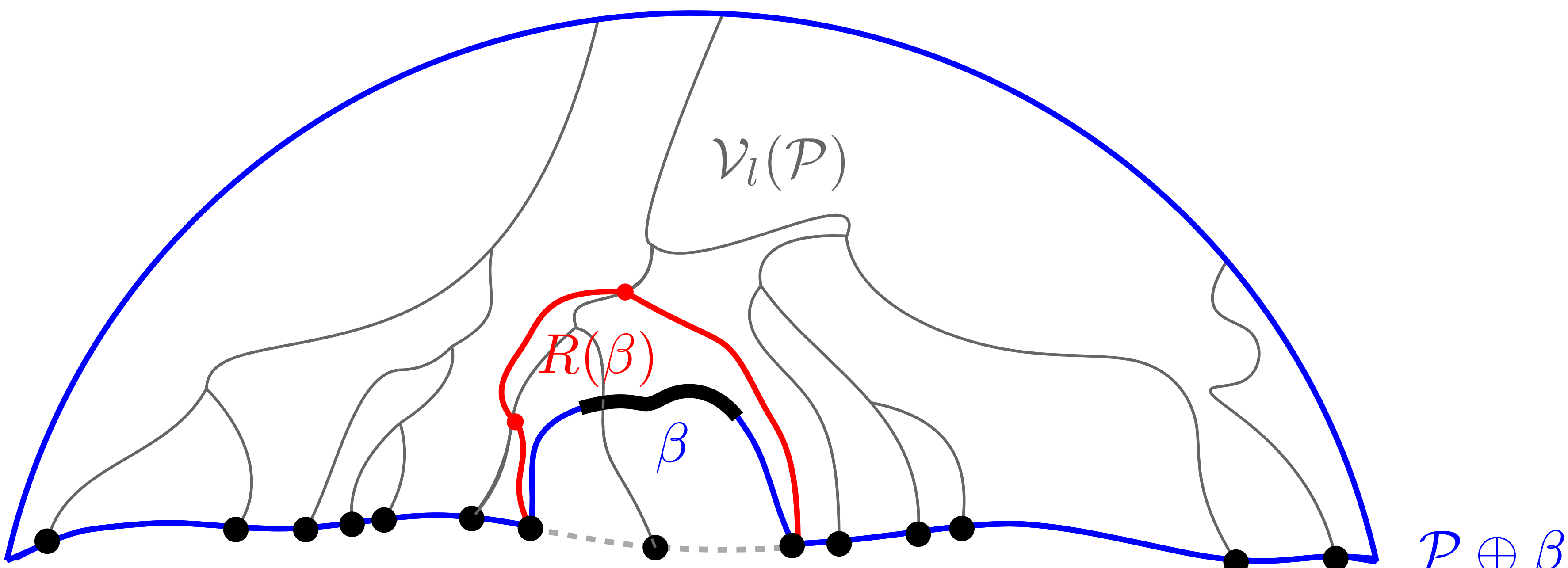

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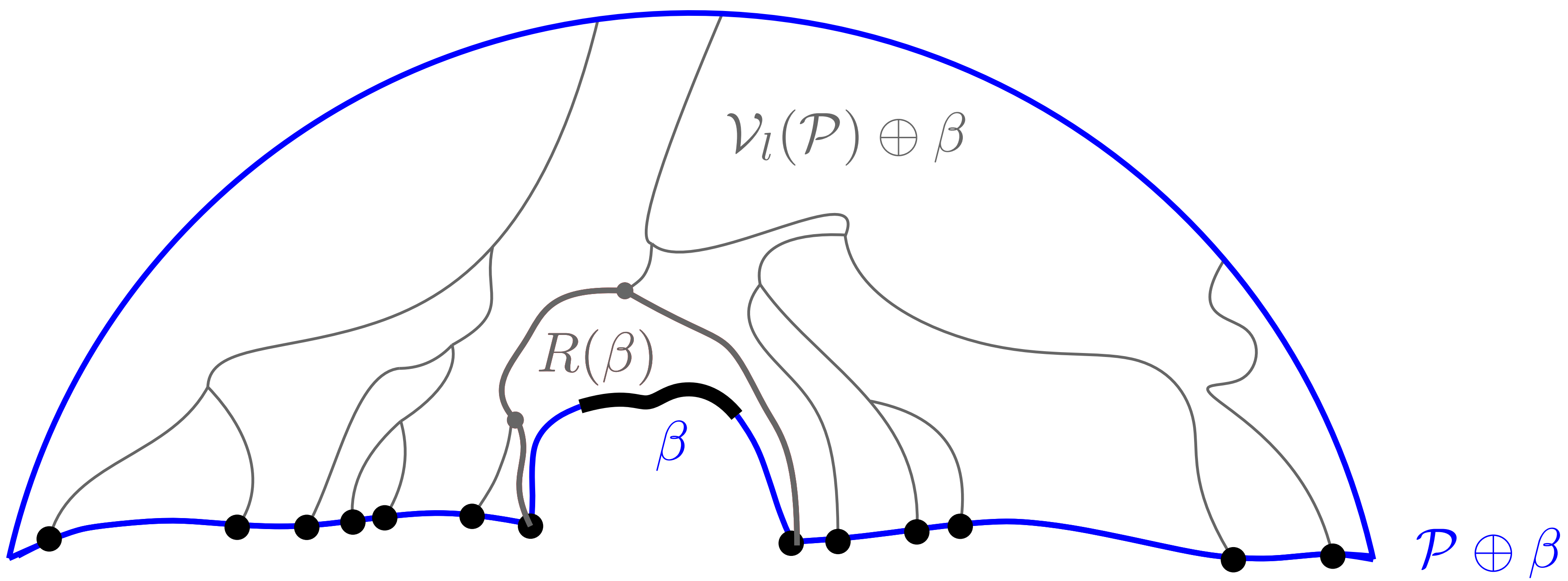

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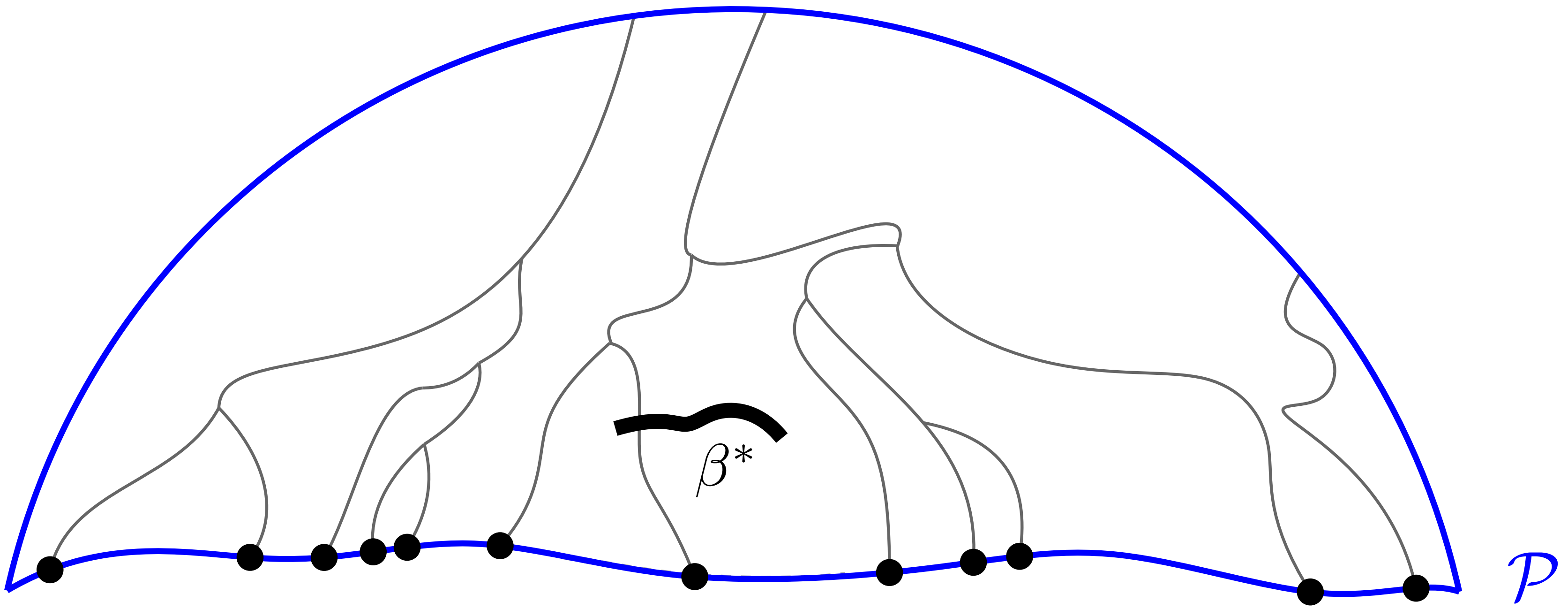

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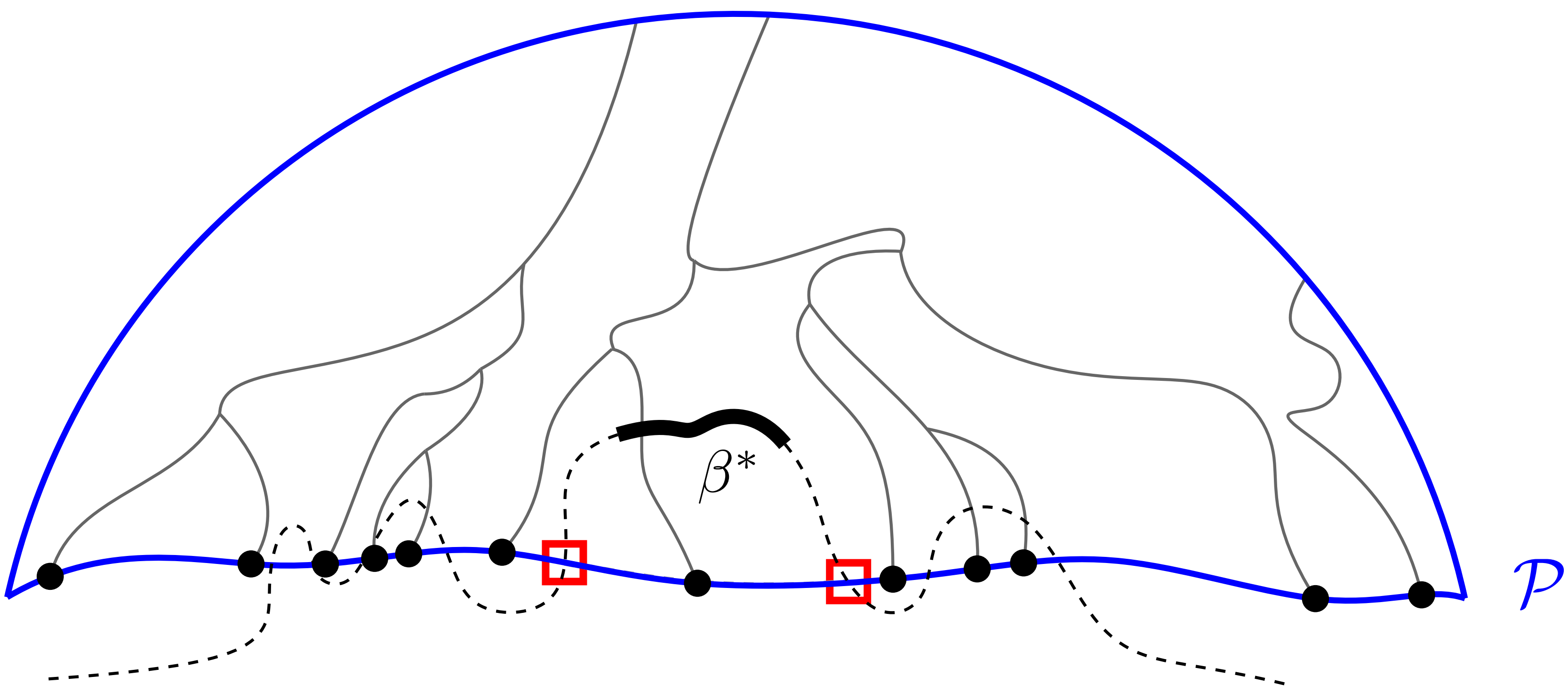

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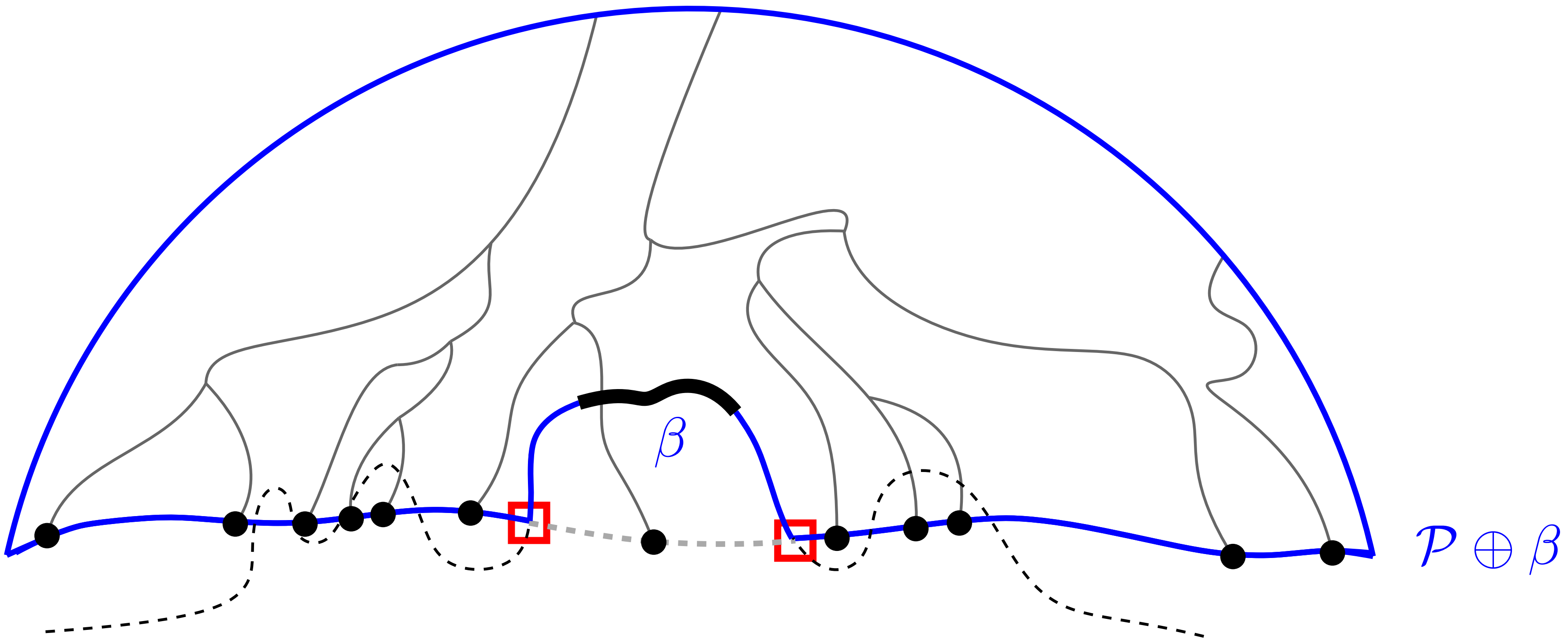

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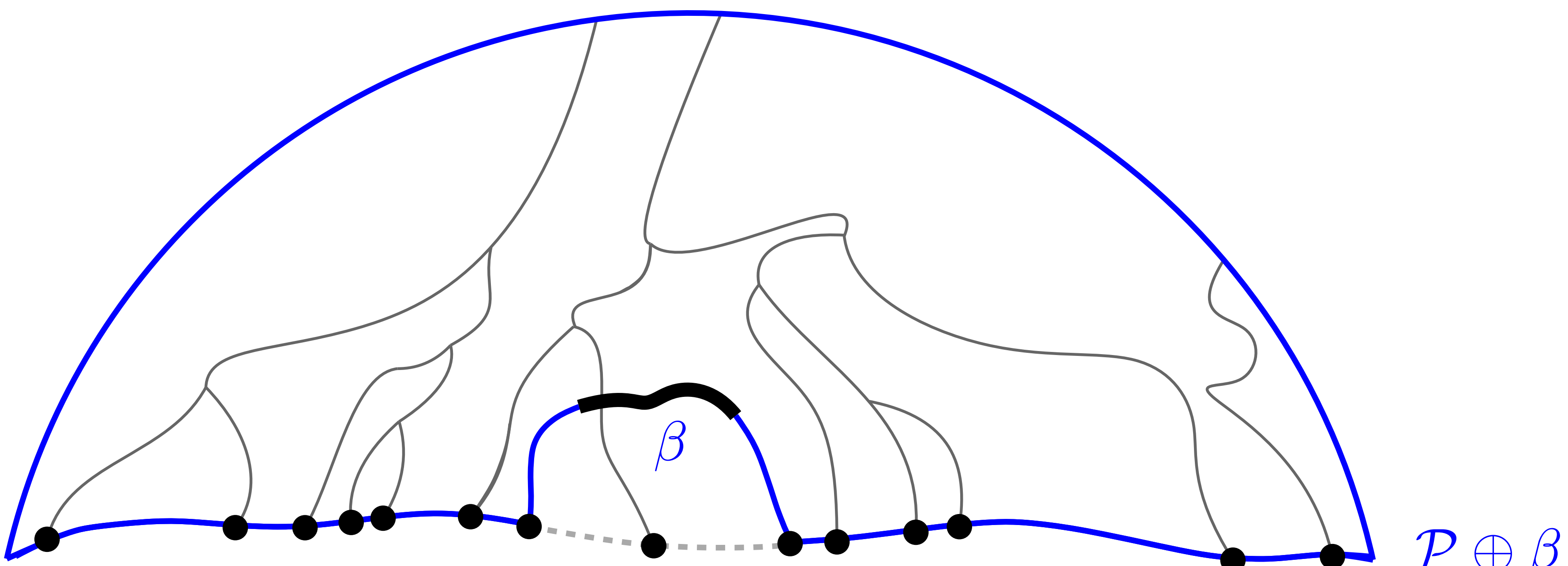

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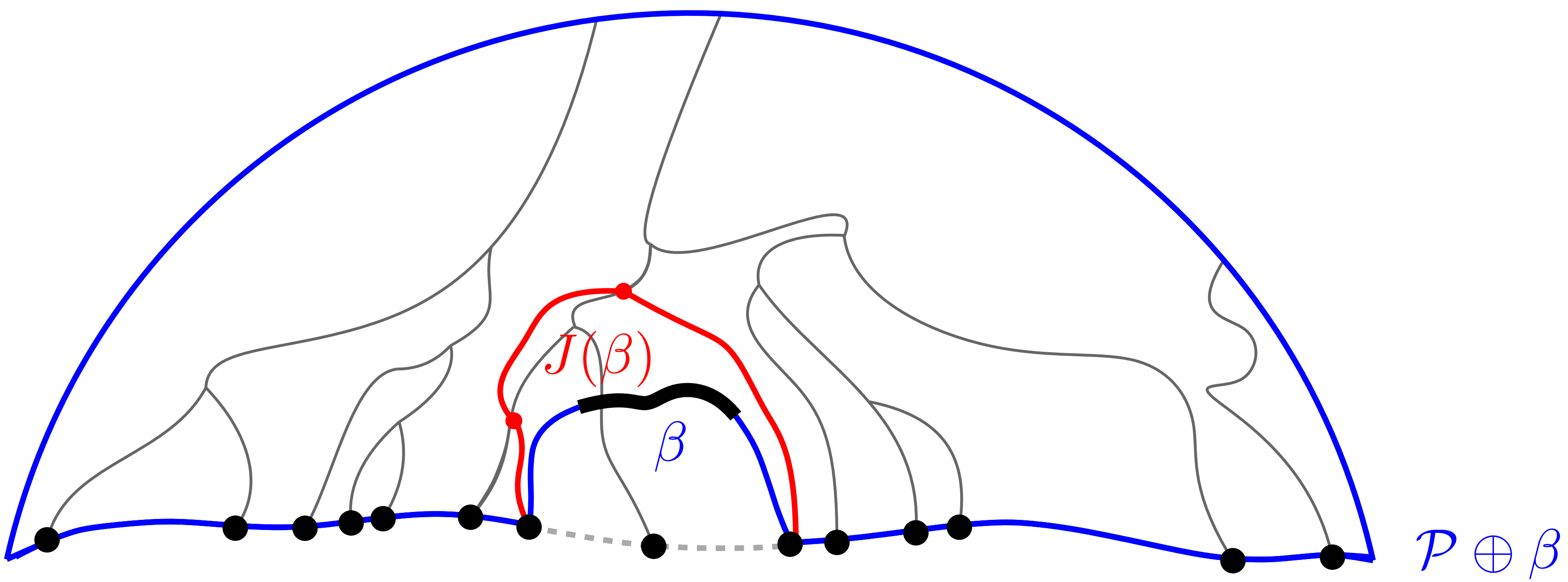

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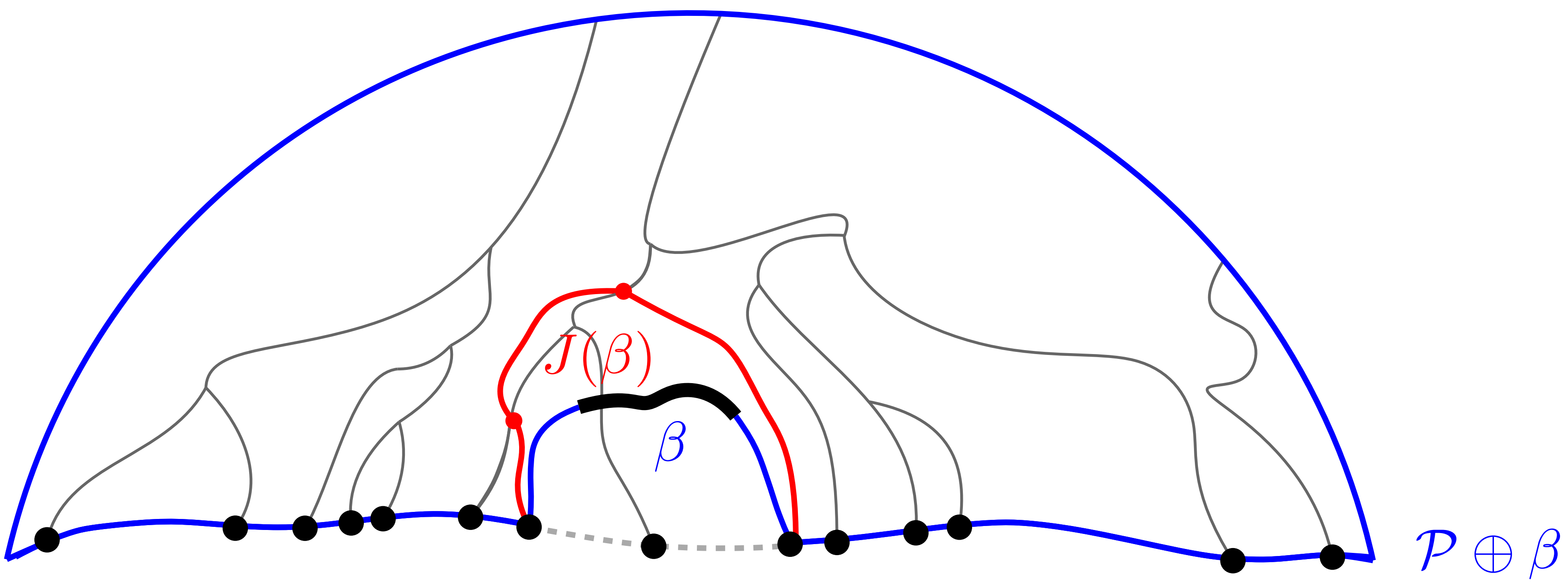

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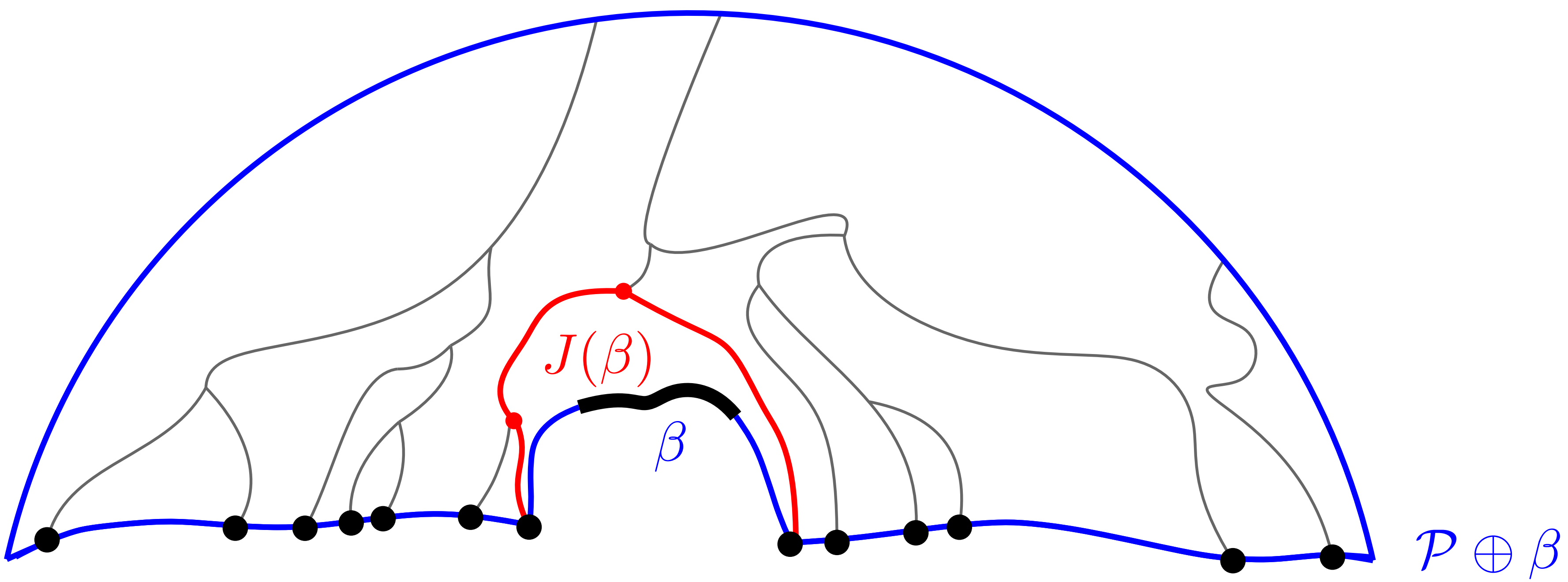
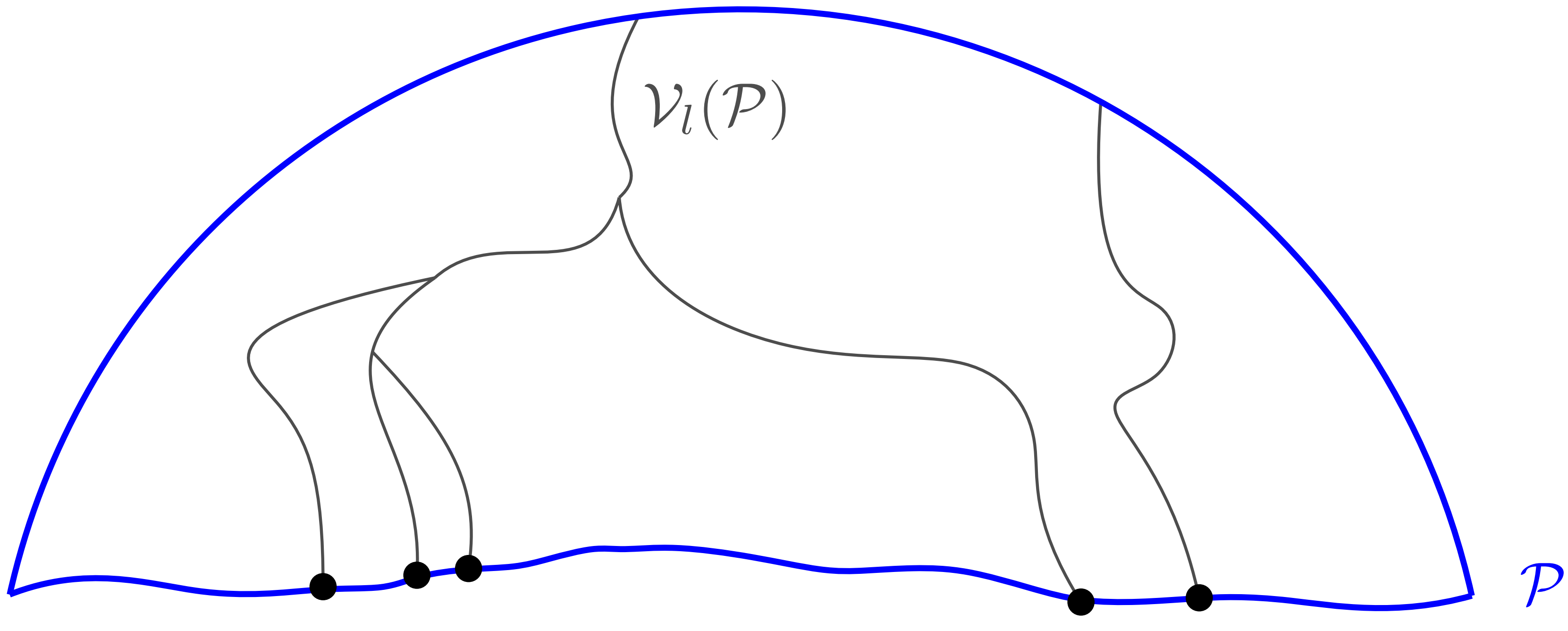


Insertion of β splits an arc

When inserting β, a face (and its arc) may split in two, creating a new auxiliary arc (γ') that was not in $\mathcal{S}$.



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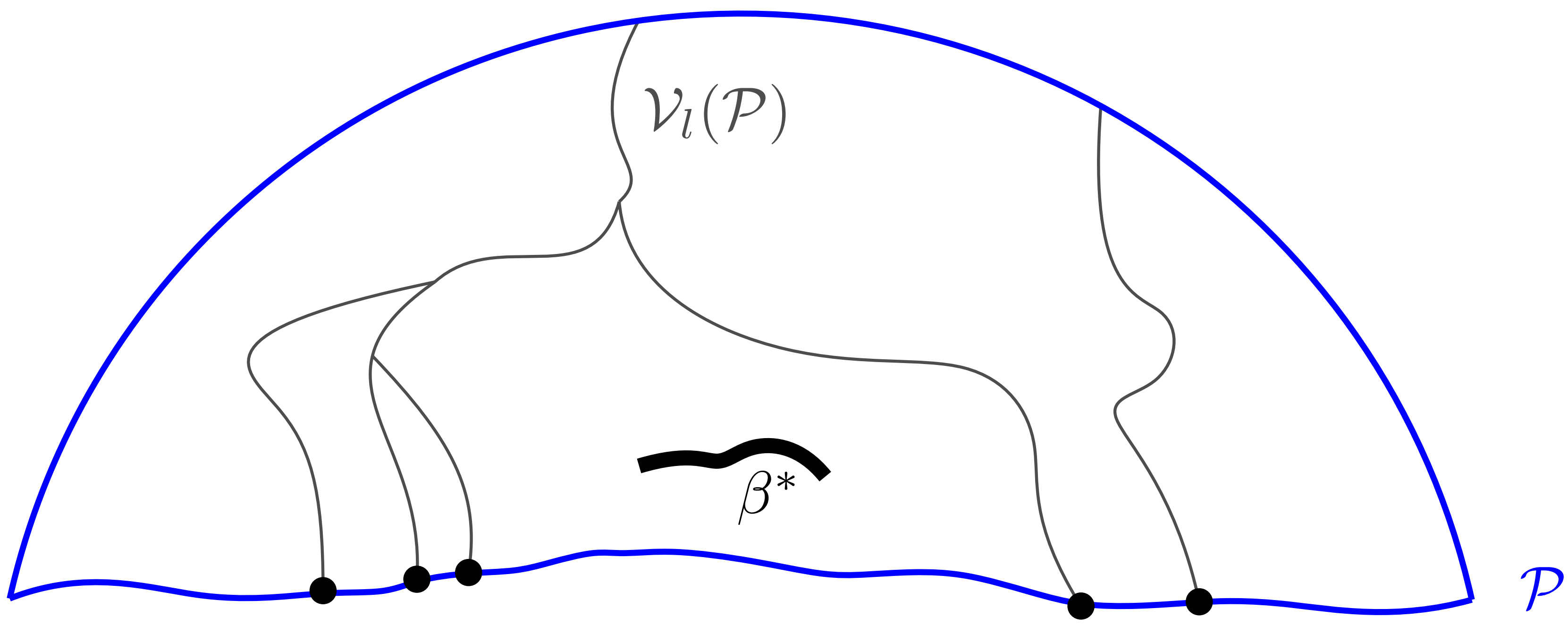

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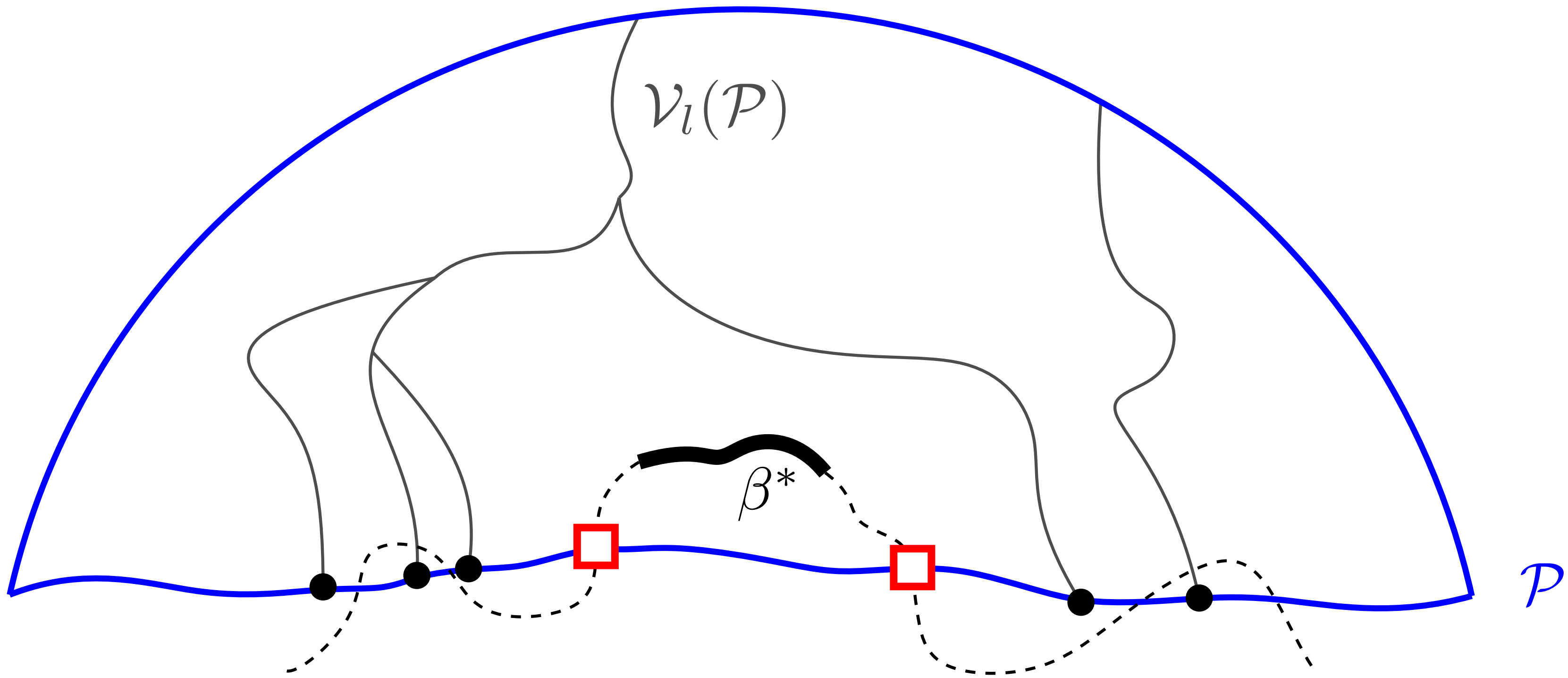

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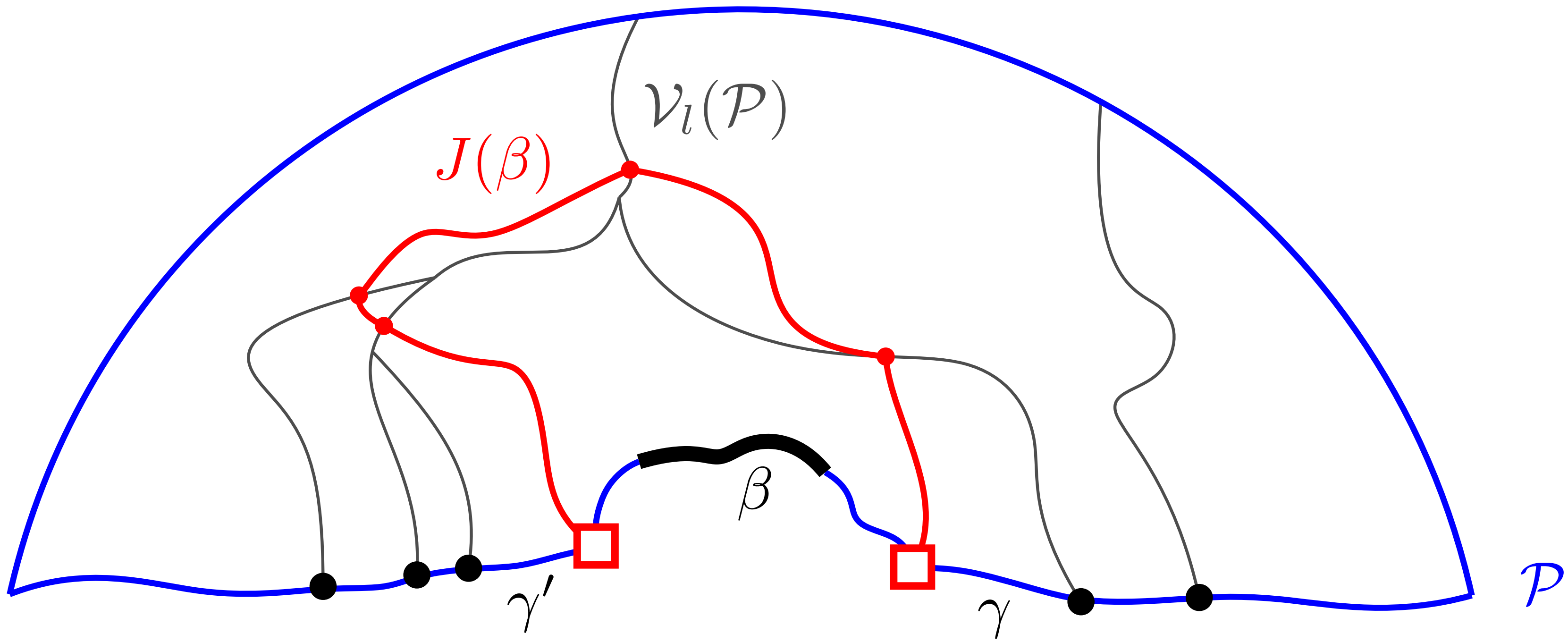

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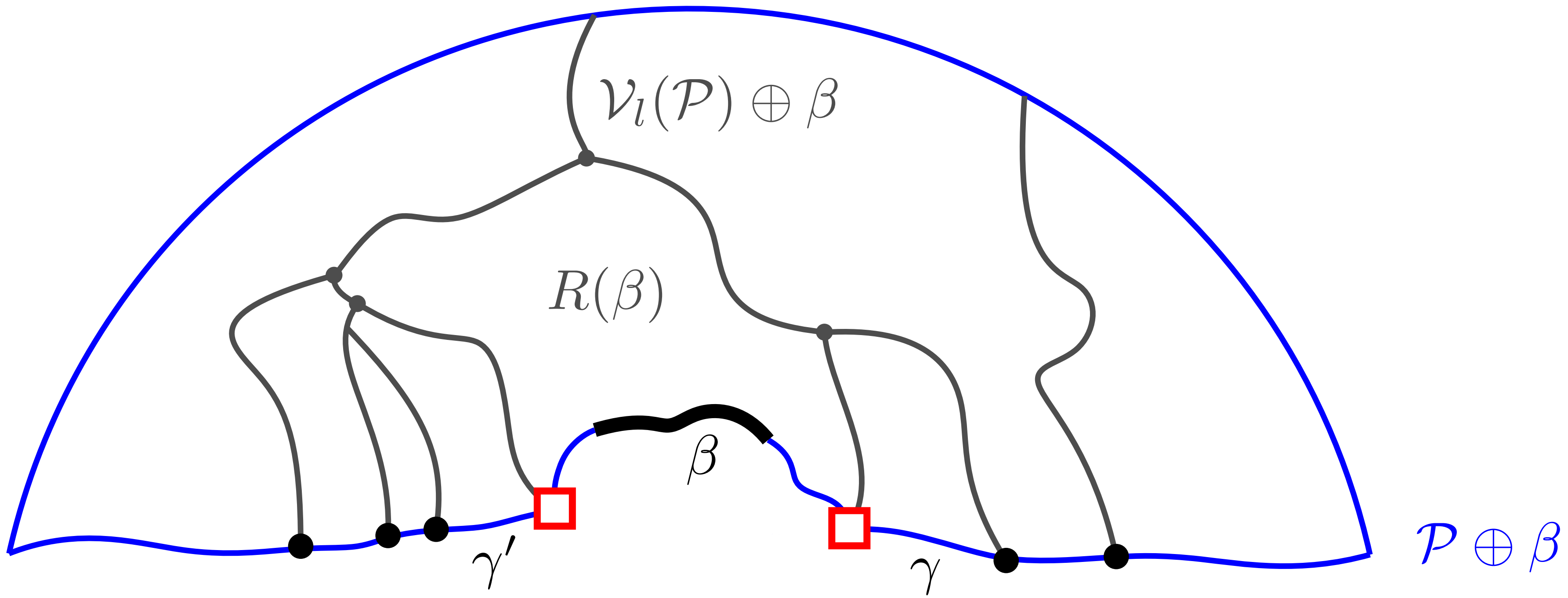
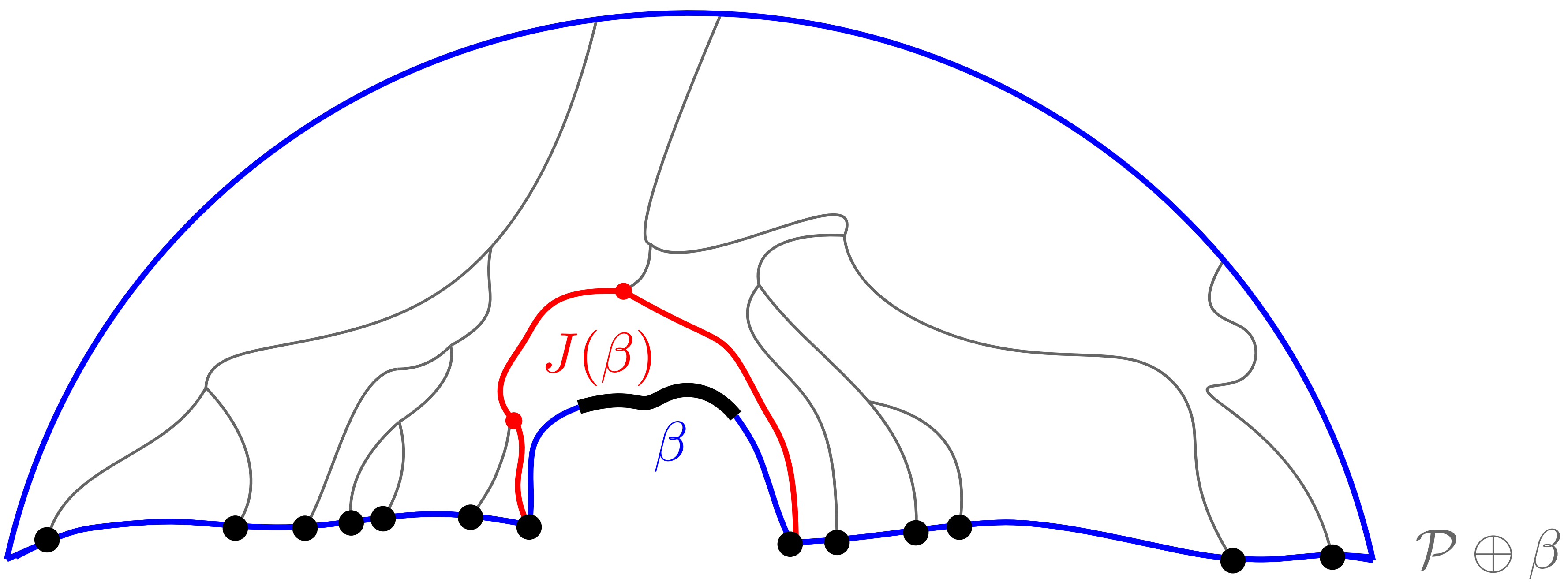


Arc insertion

Theorem: The merge curve $J(\beta)$ is an s_β-monotone path.



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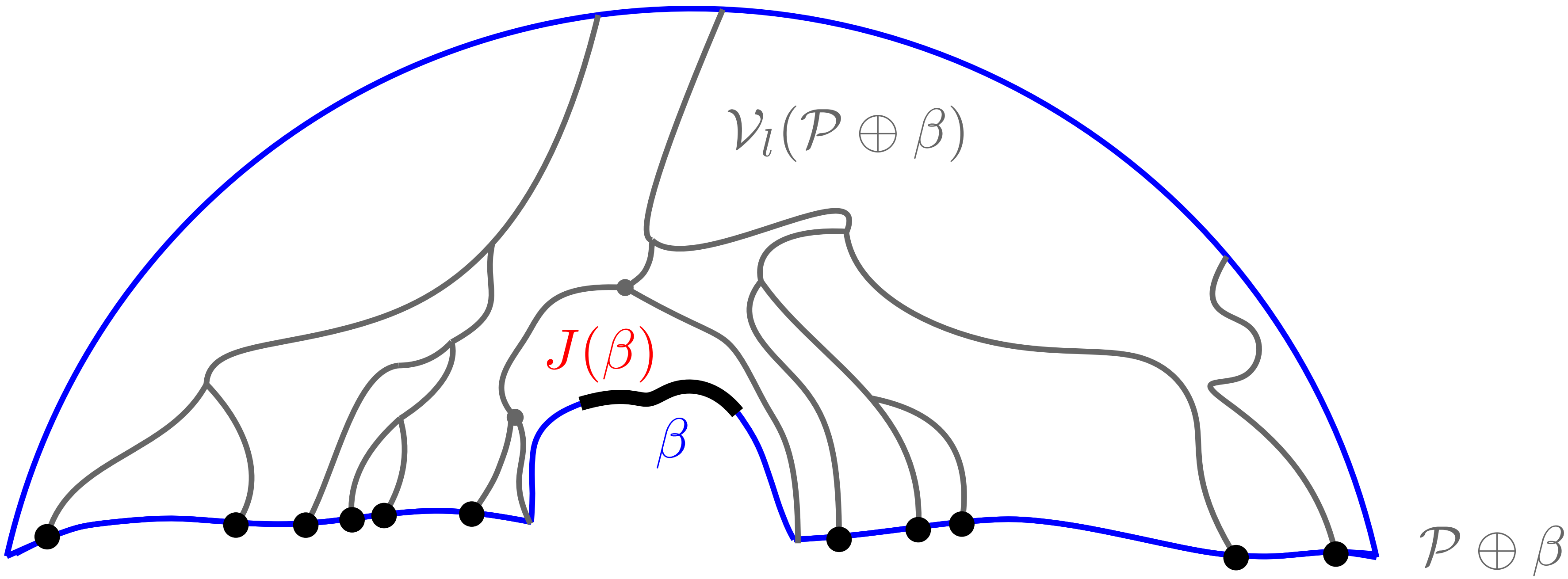
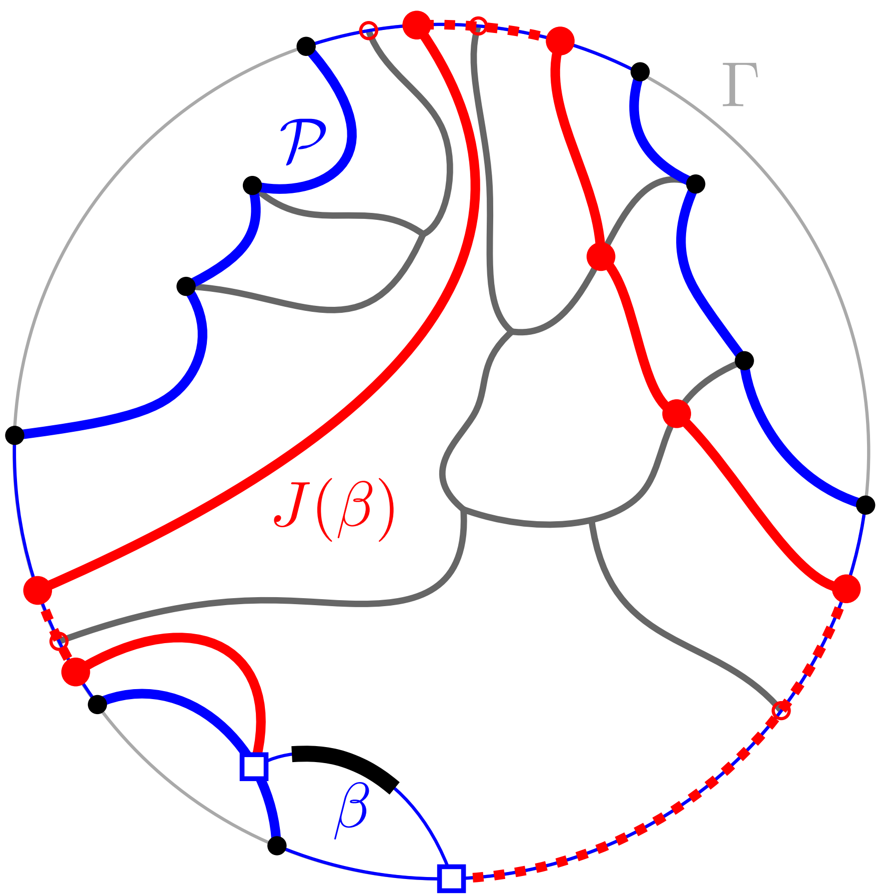
Theorem: $\mathcal{V}_l(\mathcal{P}) \oplus \beta$ is the Voronoi-like diagram, $\mathcal{V}_l(\mathcal{P} \oplus \beta)$.



Proof sketch

Theorem: The merge curve $J(\beta)$ is an s_β-monotone path.

Use a bi-directional induction starting at the two endpoints of β. Show:



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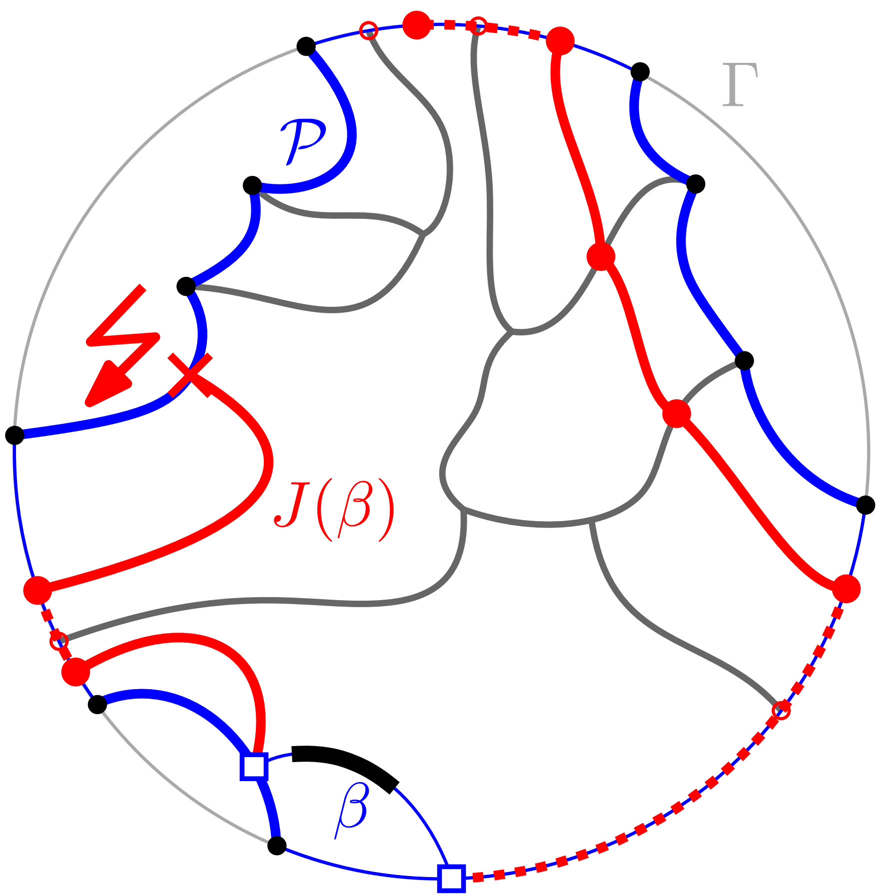

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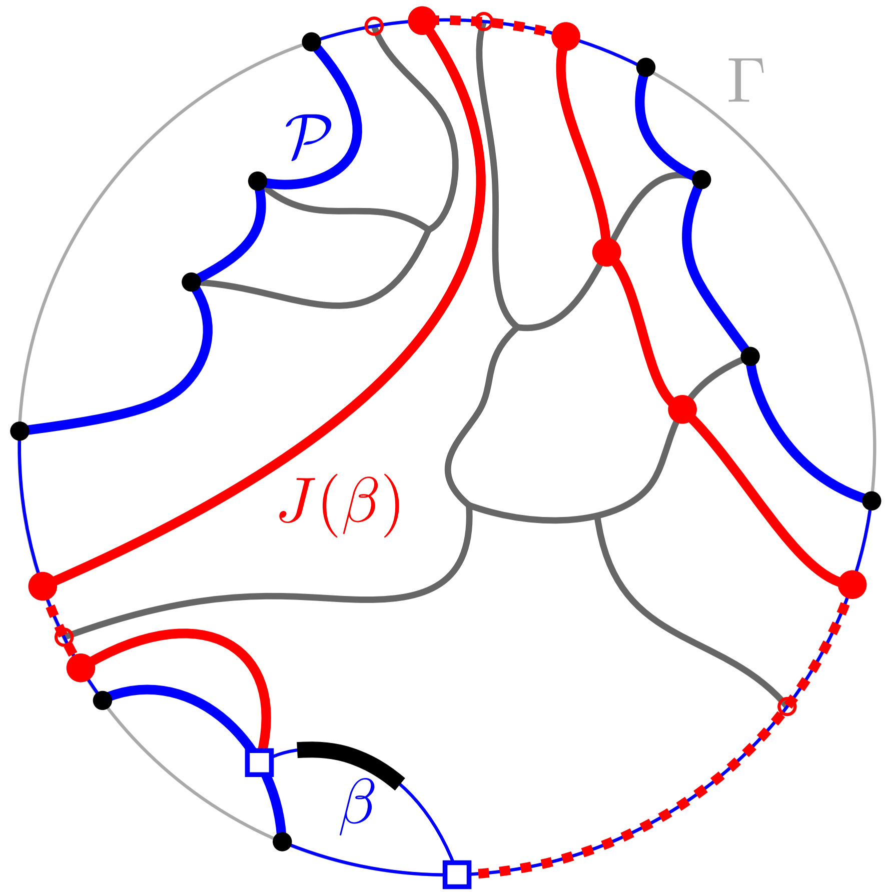

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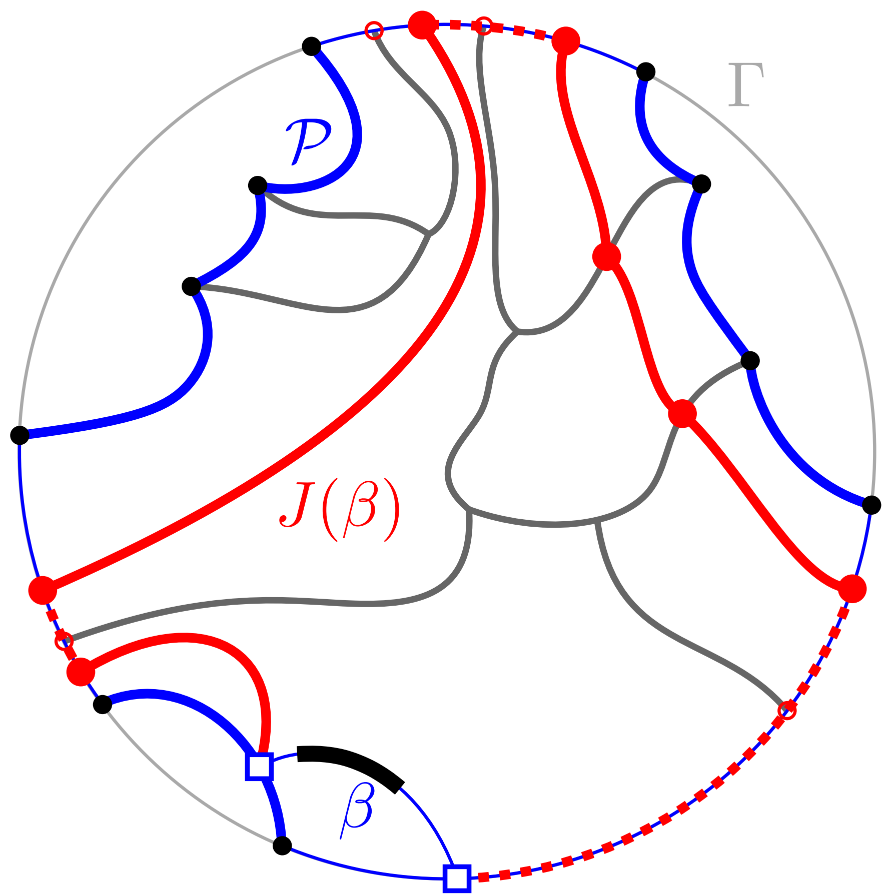
- $J(\beta)$ cannot hit a boundary arc.
- $J(\beta)$ cannot get stuck on Γ.
- $J(\beta)$ can visit a Voronoi-like region at most once.

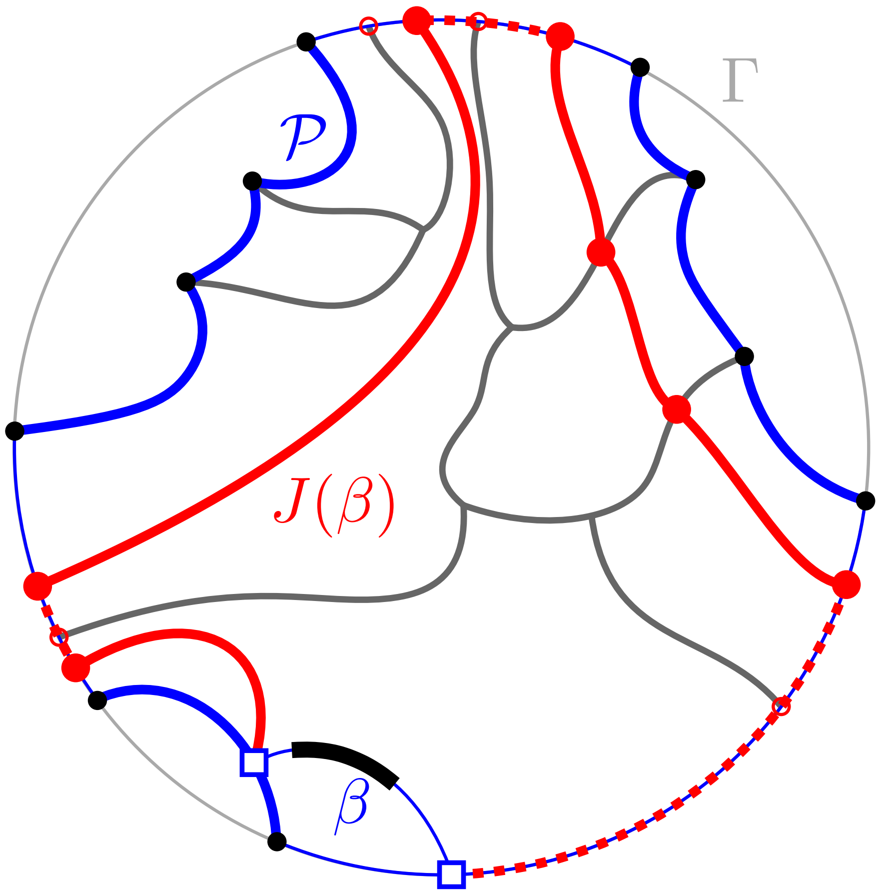


Proof sketch

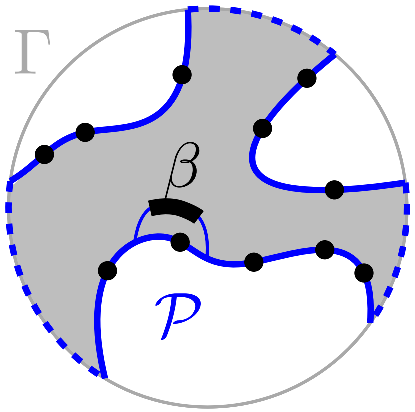
Theorem: The merge curve $J(\beta)$ is an s_β-monotone path.

Use a bi-directional induction starting at the two endpoints of β. Show:

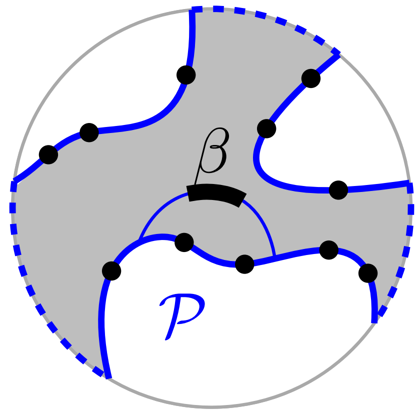
- $J(\beta)$ cannot hit a boundary arc.
- $J(\beta)$ cannot get stuck on Γ.
- $J(\beta)$ can visit a Voronoi-like region at most once.
- At the end, the 2 brunches of $J(\beta)$ meet in the same region



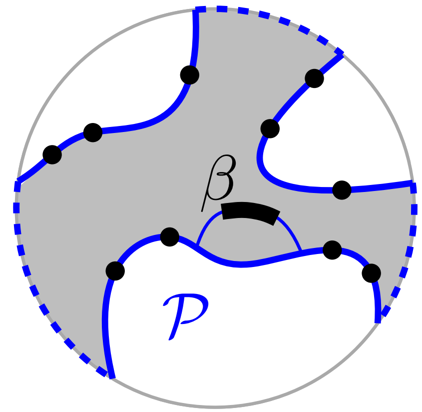
Possibilities for inserting β in $\mathcal{P}$



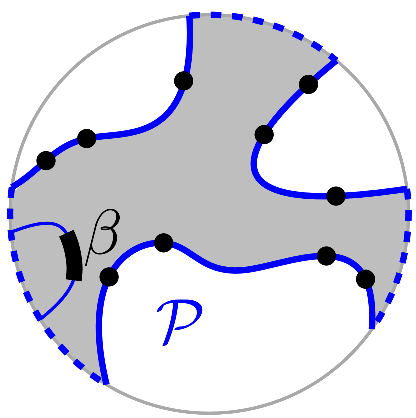
(a) Ordinary.



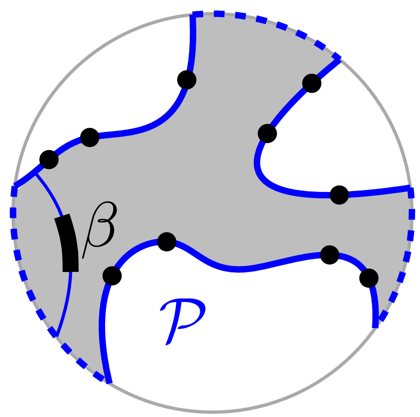
(b) Delete arc.



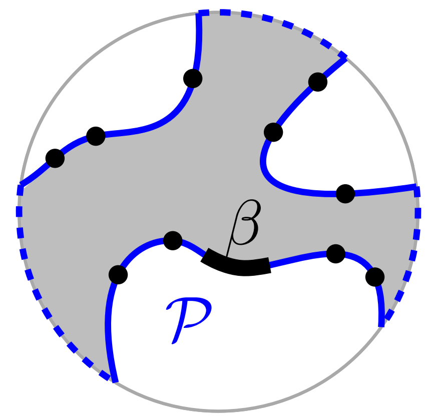
(c) Split arc.



(d) Split Γ-arc.



(e) Shrink Γ-arc.



(f) Trivial.

A randomized incremental algorithm

Consider a random permutation of the arcs $\mathcal{S}$.

Phase 1: Delete arcs from $\mathcal{S}$, recording their neighbors at time of deletion (inspired by Chew).

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...constructing Voronoi-like diagrams within a series of shrinking domains.

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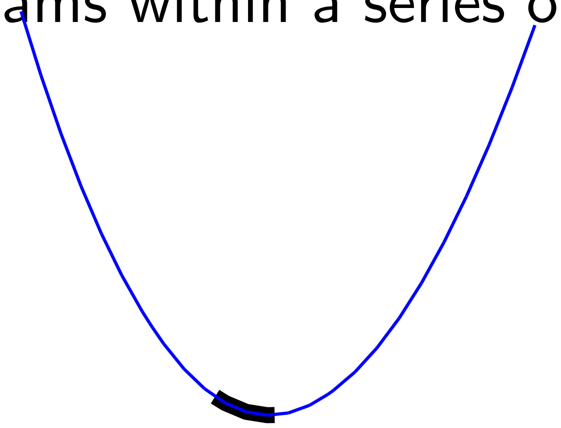

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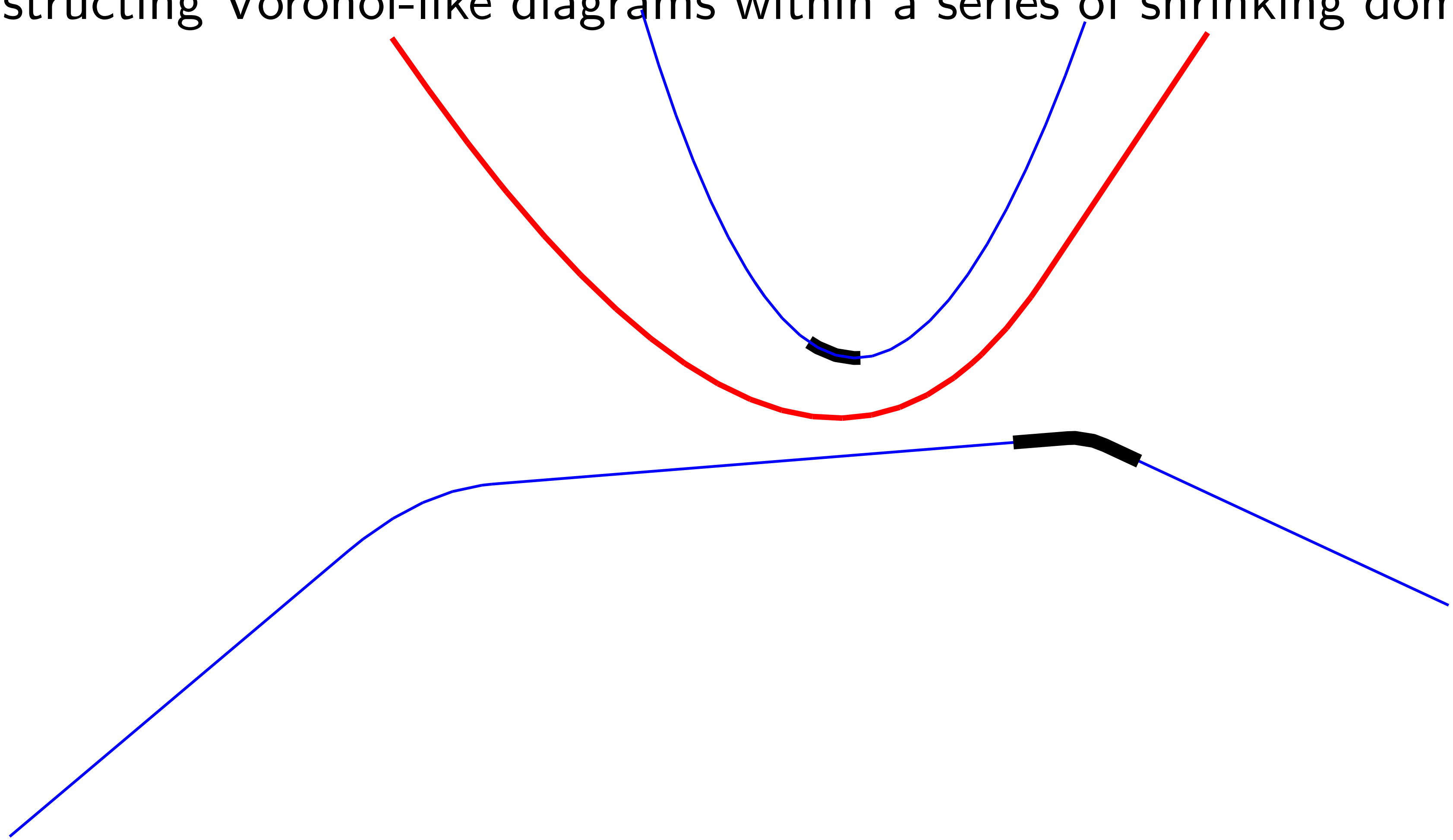

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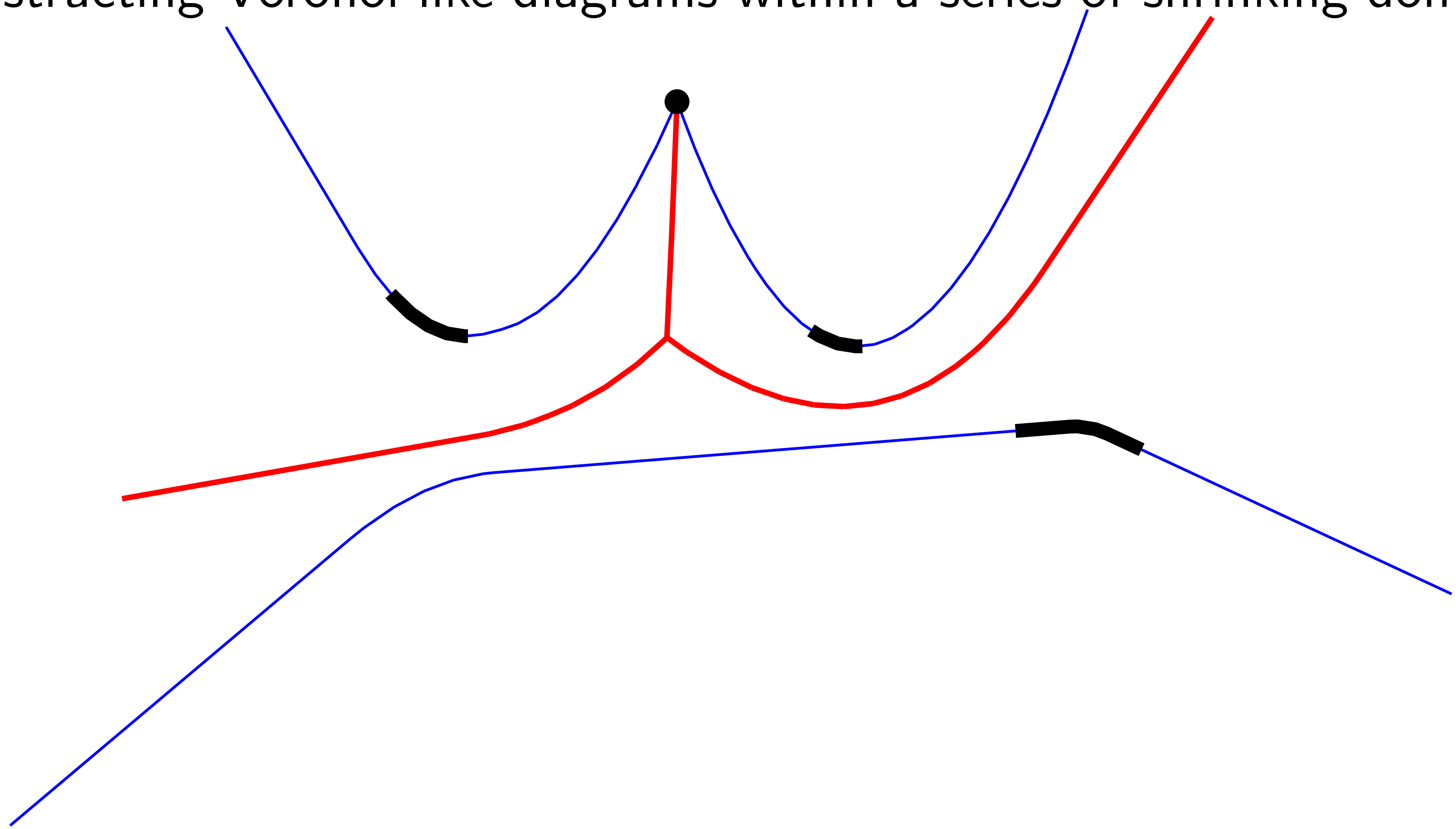

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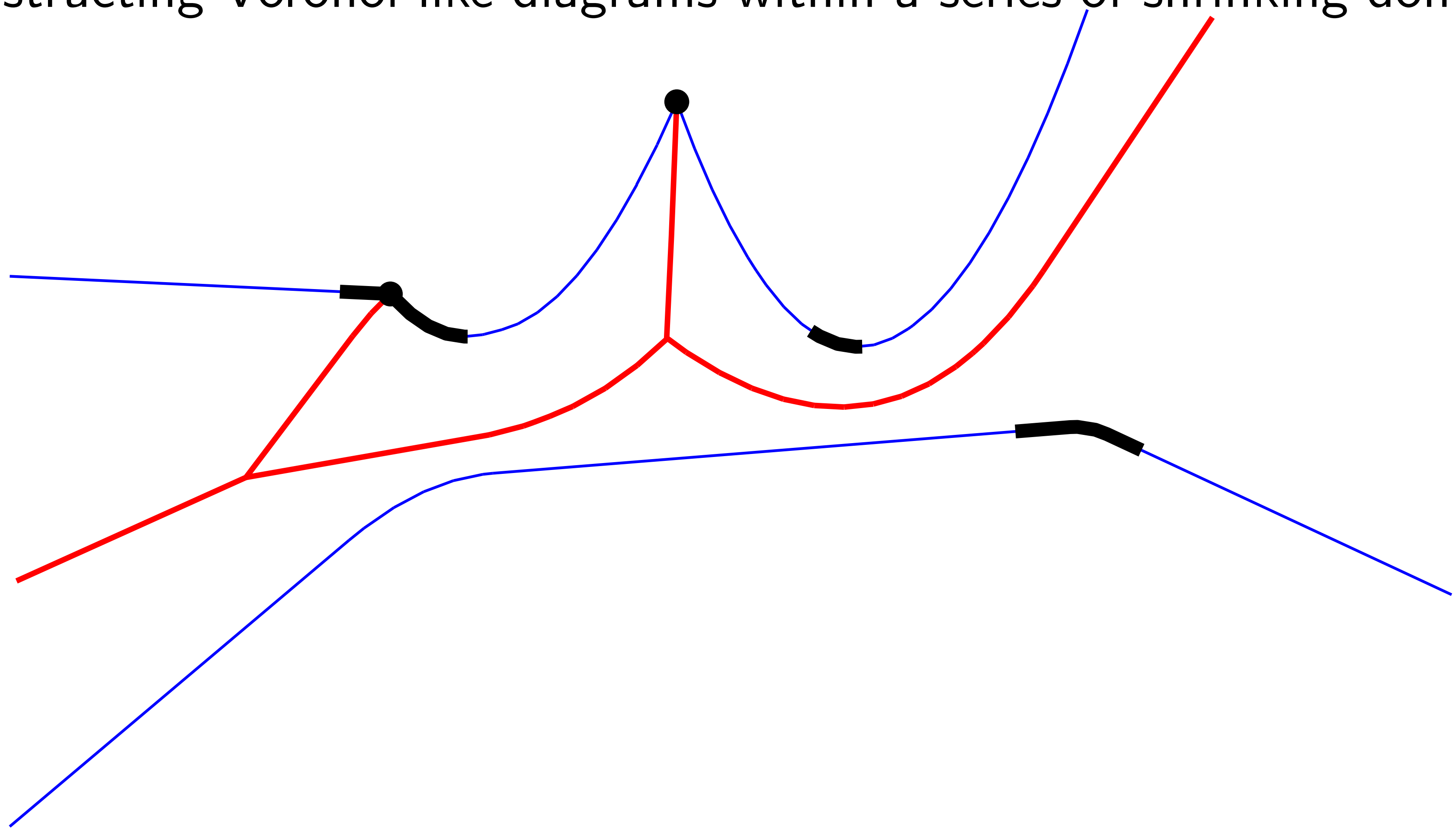

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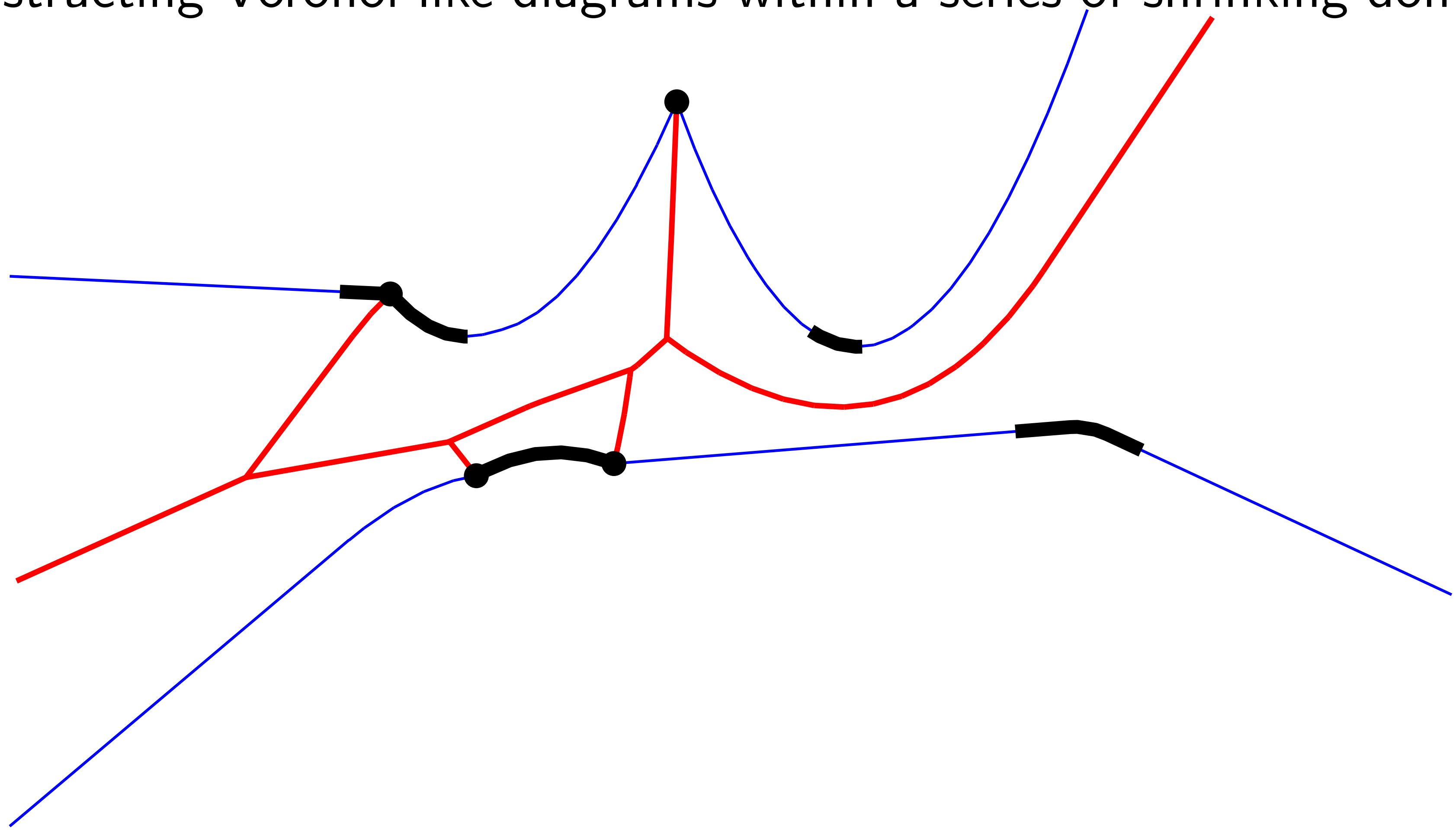

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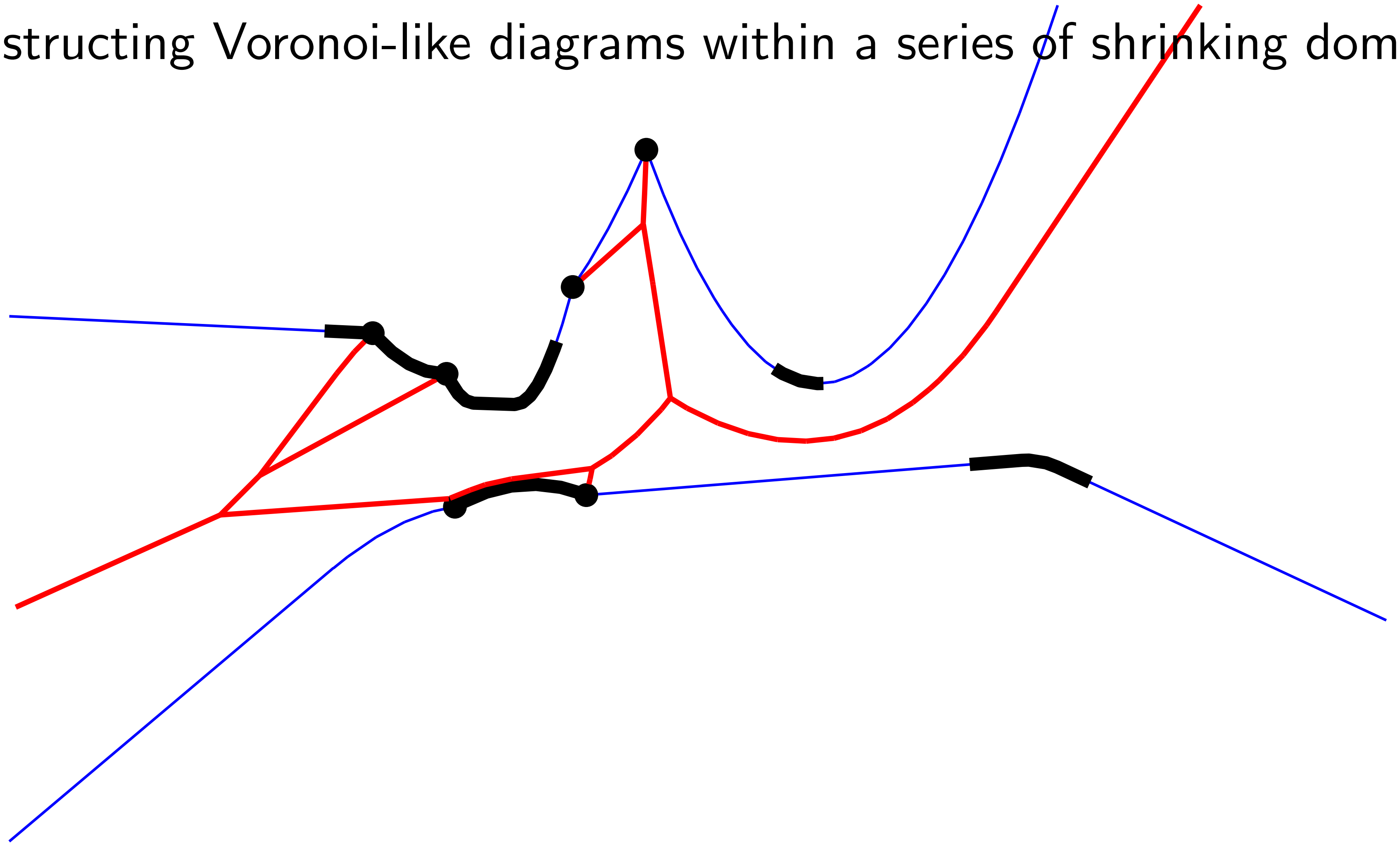

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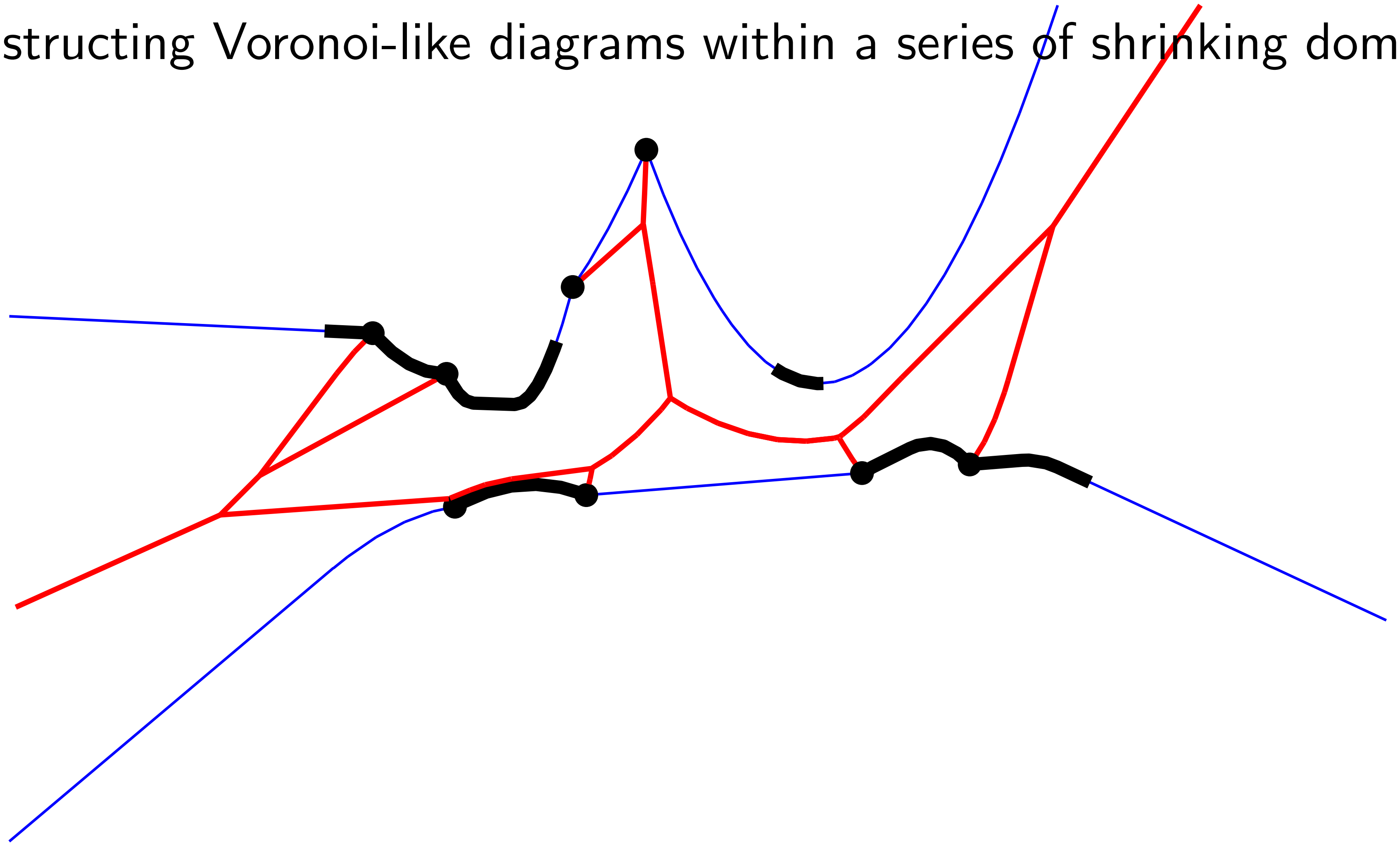

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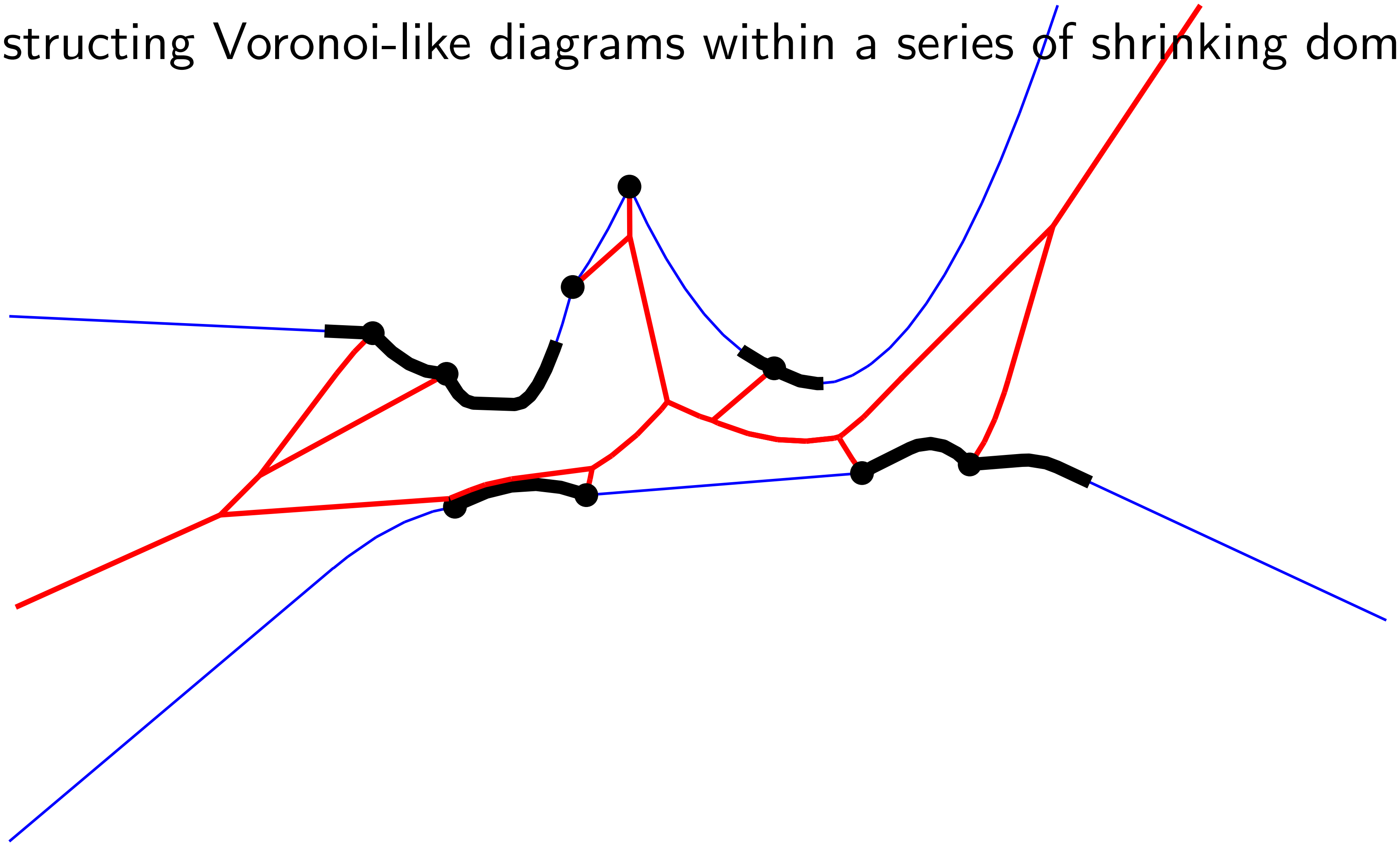

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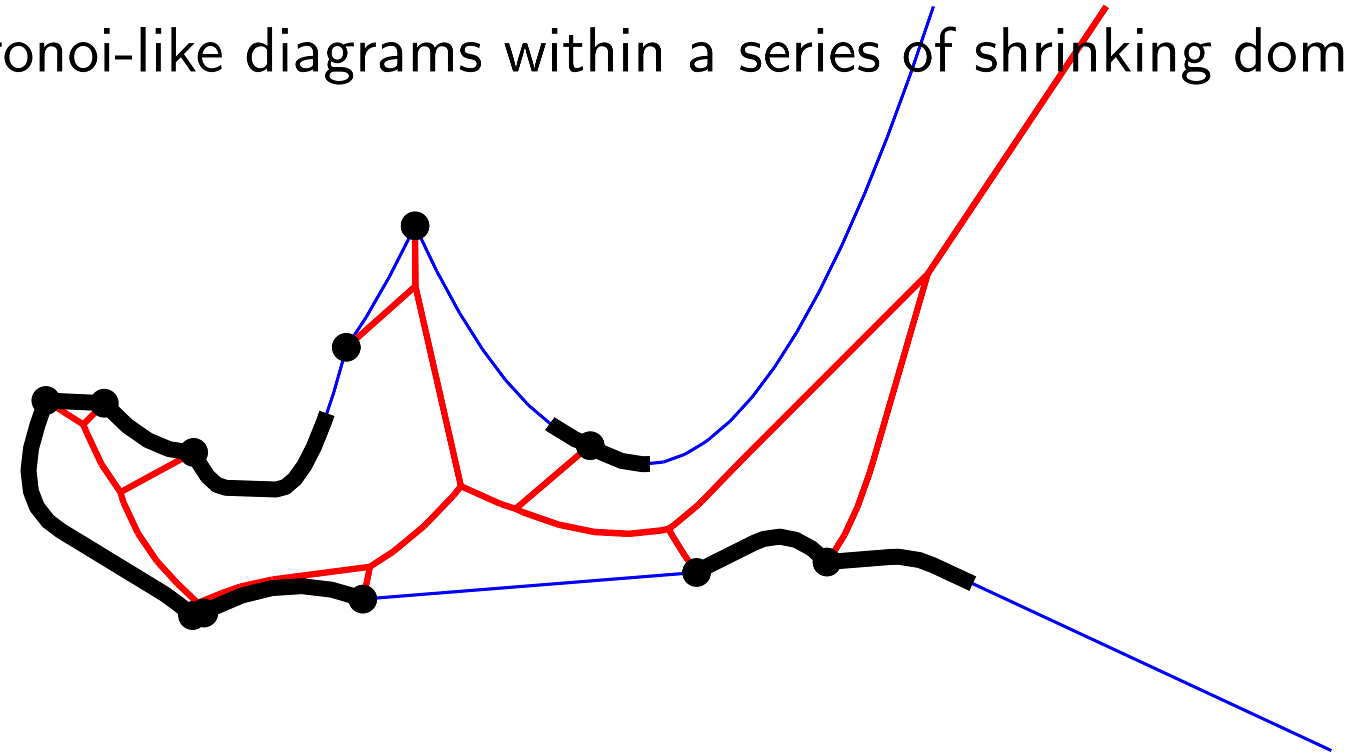

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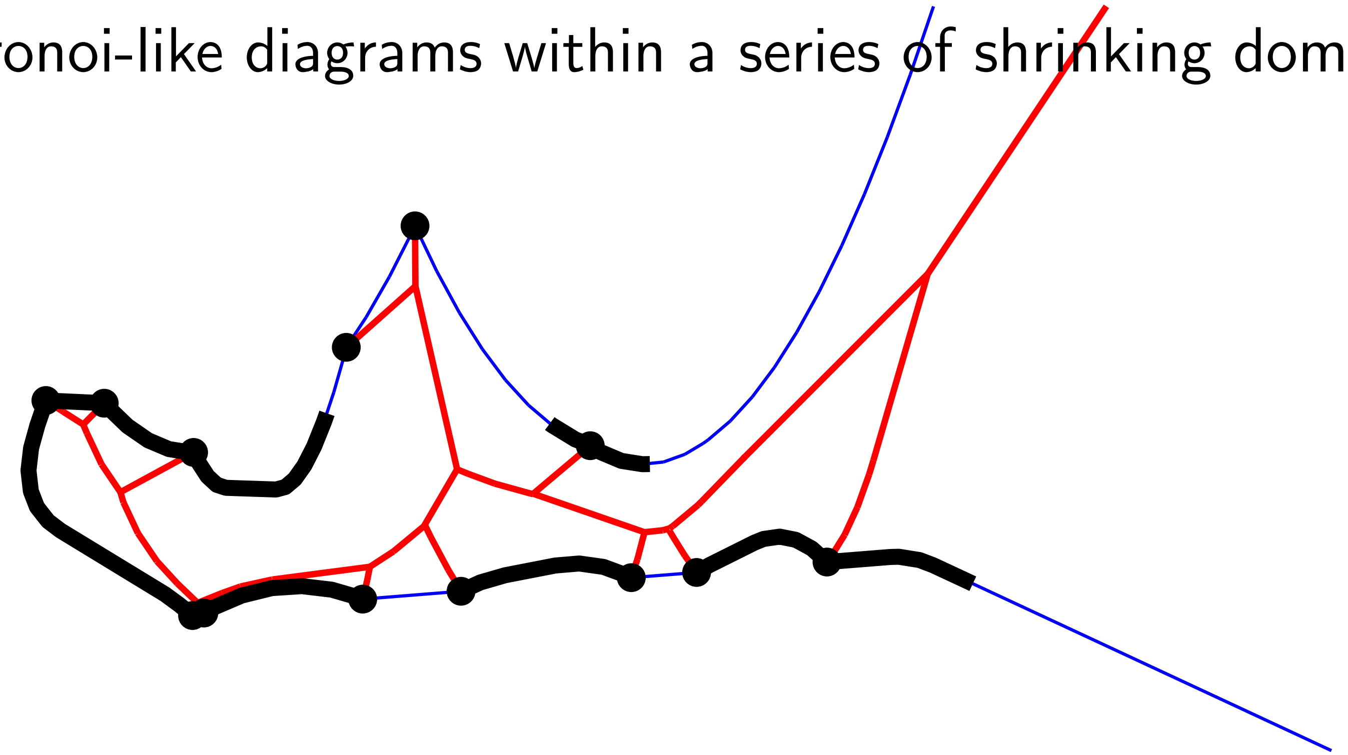

Because of auxiliary arcs, when we insert an arc, its neighbors need not be the ones of Phase 1. Need to trace auxiliary arcs (expected constant).

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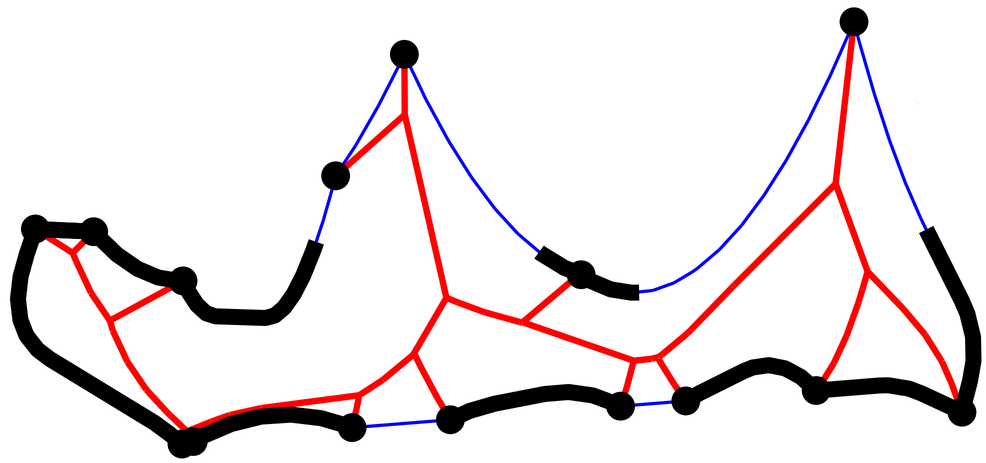

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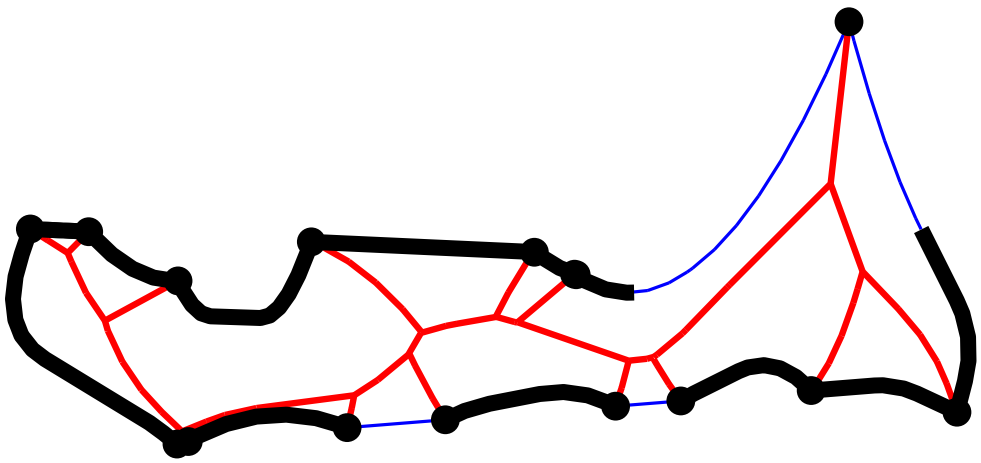

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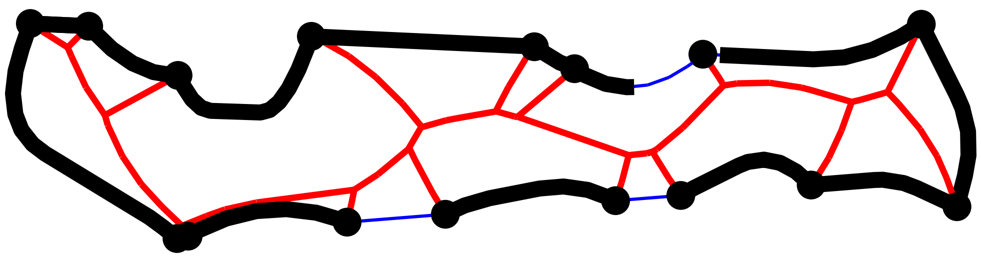

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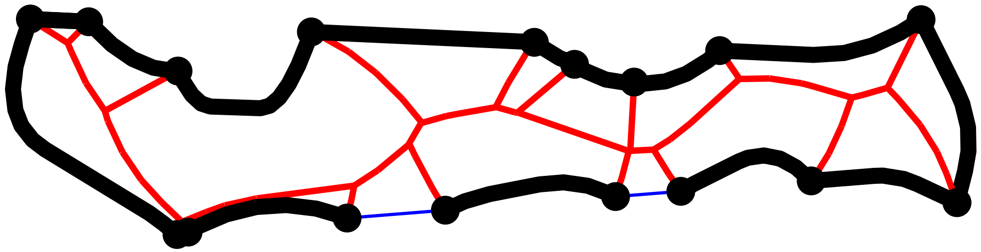

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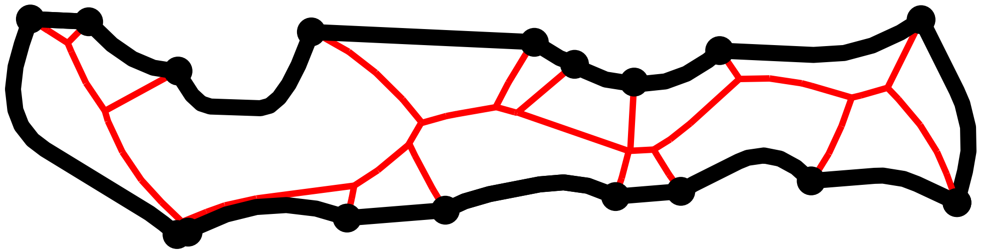

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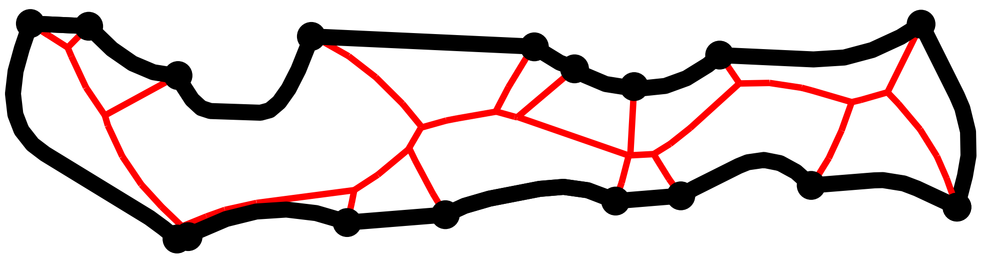

In the end we obtain $\mathcal{V}_l(\mathcal{S}) = \mathcal{V}(\mathcal{S}) = \mathcal{V}(S \setminus s) \cap \mathrm{VR}(s)$.

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Our main result

Theorem:
Given $\mathcal{V}(S)$ and a site $s \in S$, the Voronoi diagram $\mathcal{V}(S \setminus \{s\})$ can be computed in expected time linear in the complexity of $\partial\mathsf{VR}(s)$.

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In the **proof** we use:

- **Lemma:** The boundary curve at step i consists of at most $2i$ arcs.
- **Lemma:** The expected number of arcs that are visited during one insertion is constant.
- Backward analysis, similar to line segments [Khramtcova, Papadopoulou, ISAAC 2015]

Future work / Open problems

- Deterministic linear-time algorithm for deletion in abstract Voronoi diagrams ?

  Currently, we are exploring using Voronoi-like diagrams in the framework of [Aggarwal, Guibas, Saxe and Shor, DCG 1989].

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Thank you for your attention!