Small Triangulations of Projective Spaces

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What do we mean by Triangulations?

Simplicial Complexes

An *abstract* simplicial complex is a collection of sets closed under inclusion.
We think of this as a subset of $\mathcal{P}([n])$.

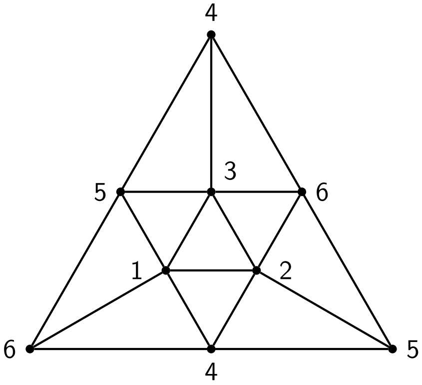
- Let K be a simplicial complex with vertex-set $[n]$. The *geometric carrier* of K is constructed by extending the map $i \mapsto e_i$ in $\mathbb{R}^n$.
- Let X be a (compact, connected,...) manifold. A *triangulation* of X is a simplicial complex K, whose geometric carrier is (PL-)homeomorphic to X.
- The boundary of the n-dimensional simplex ($\mathcal{P}([n+1])$) triangulates S^{n-1}.
- A *combinatorial d-manifold* is a simplicial complex in which the link of every vertex is PL-homeomorphic to a $(d-1)$-sphere.

Small triangulations are *interesting*!

$\mathbb{RP}^2_6$

We can triangulate $\mathbb{RP}^2$ with six vertices.

$$[123], [124], [135], [146], [156], [236], [245], [256], [345], [346]$$



Manifolds like a Projective Plane

Eells and Kuiper classified "manifolds of Morse number 3", compactifications of $\mathbb{R}^{2d}$ by an d-sphere.

They only exist in five different dimensions d.

- $d = 0$: Three points
- $d = 2$: Homeomorphic to real projective plane, $\mathbb{R}\mathrm{P}^2$
- $d = 4, 8, 16$: Is simply connected, and has the same integral cohomology as the complex projective plane $\mathbb{C}\mathrm{P}^2$, quaternionic projective plane $\mathbb{H}\mathrm{P}^2$, or octonionic projective plane $\mathbb{O}\mathrm{P}^2$

Triangulations with few vertices

Brehm, Kühnel (1987):

Let M be a combinatorial d-manifold with n-vertices. Then,

1. if $n < 3\lceil \frac{d}{2} \rceil + 3$, then M is homoeomorphic to S^d
2. if $n = 3\lceil \frac{d}{2} \rceil + 3$, then either M is homeomorphic to S^d, or M is a manifold like a projective plane.

If such a combinatorial manifold exists, we call it a BK-manifold.

Existence results: BK-manifolds

$d = 2, n = 6$ Classical

$\mathbb{R}\mathrm{P}^2$ has a unique 6-vertex triangulation.

$d = 4, n = 9$ Kühnel, Lassman (1983)

There exists a unique such combinatorial manifold, and it triangulates $\mathbb{C}\mathrm{P}^2$.

$d = 8, n = 15$ Brehm, Kühnel (1992)

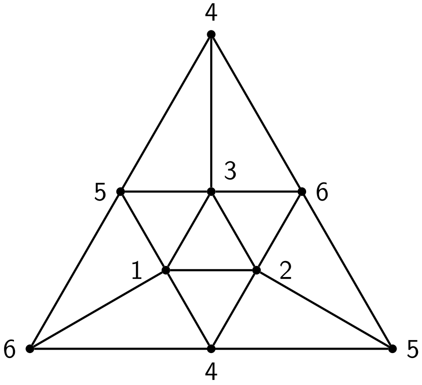
There are six such combinatorial manifolds all triangulating the same cohomology $\mathbb{H}\mathrm{P}^2$. Discovered by computer search!
Gorodkov (2016) proved that these are homeomorphic to $\mathbb{H}\mathrm{P}^2$.

$d = 16, n = 27$

Open!

How were these objects discovered?

These triangulations have a large amount of symmetry, and satisfy very restrictive combinatorial properties, such as complementarity.



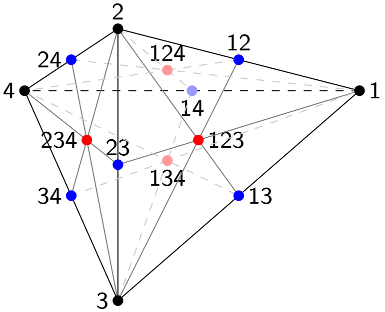
This helps in narrowing down the search space somewhat.

Real Projective Spaces

Think of Real projective space of dimension d, $\mathbb{RP}^d$ as the sphere S^d with antipodal points identified. $(x \sim -x)$.
$\mathbb{RP}^d$ can be triangulated.

Barycentric subdivision of the simplex

Let Δ^d denote the d-dimensional simplex with vertices $1, 2, \ldots, d+1$. The (first) barycentric subdivision of Δ^d is the complex obtained by subdividing $\partial\Delta^d$ at each of its proper faces.



- Antipodal d-sphere
- Each vertex a nonempty, proper subset of $[d+2]$
- $2^{d+2} - 2$ vertices
- Each facet $\sim$ an ordering of $[d+2]$
- $(d+2)!$ facets
- Quotient is $\mathbb{RP}^d$

State of the art on triangulations of $\mathbb{RP}^d$

- The above construction uses $n = 2^{d+1} - 1$ points to triangulate $\mathbb{RP}^d$.
- All known infinite families require $O(2^n)$ vertices.
- We know better constructions for $d = 4, 5$
- Best known lower bound for $\mathbb{RP}^d$ is $n = \binom{d+2}{2} + 1,\ d \geq 3$, due to Arnoux, Marin (1991).

Trying to bridge the gap

Bistellar Flips

Given a combinatorial manifold M, let Δ_1 be a simplex in M. Say the link of Δ_1 in M is boundary $\partial\Delta_2$ of another simplex Δ_2. Further suppose that $\Delta_2 \notin M$. Then replacing $\Delta_1 * \partial\Delta_2$ in M with $\partial\Delta_1 * \Delta_2$ gives a new combinatorial manifold M' that is PL-homeomorphic to M.

Pachner's Theorem

Two combinatorial manifolds M and M' are *PL*-homeomorphic if and only if M can be obtained from M' through a sequence of bistellar flips.

The `BISTELLAR` program

Lutz (1999) applied a simulated annealing algorithm to start with large triangulations of manifolds and search for smaller ones.

Geometric constructions are also possible with `Polymake`, `SimpComp` etc. But these need significant inspiration!

Open Problems

Can we construct a 27-vertex $\mathbb{O}\mathrm{P}^2$?

Can we do better than `BISTELLAR`?

Can $\mathbb{R}\mathrm{P}^d$ be triangulated with $O(d^2)$ vertices?