Heegaard unions, generalized Property R, and the Akbulut-Gompf example

Dublin

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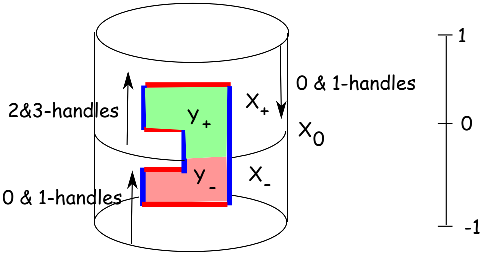
Let

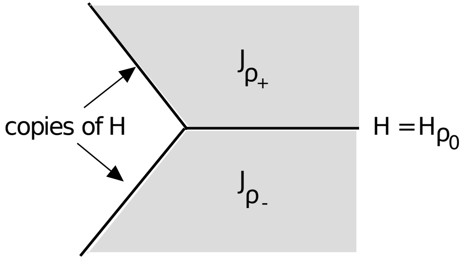
- $X_0 = X \cap S^3_0$
- $X_- = X \cap S^3 \times [-1, 0]$
- $X_+ = X \cap S^3 \times [0, 1]$



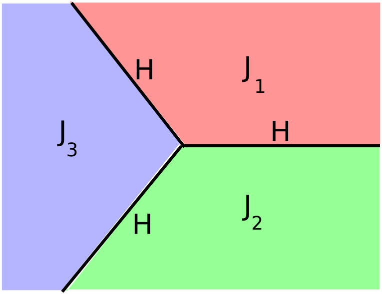
Is reimbedding Y so $X_0 =$ handlebody useful?
(Recall: can be done for genus ≤ 3.)

Suppose X_0, is 3-dim. handlebody H_{ρ_0}.
Then $X = X_+ \cap_{X_0} X_-$ is a Heegaard union:

- Union of two 4-dimensional handlebodies J_{ρ_-}, J_{ρ_+}
- along 3-dimensional handlebody $H_{\rho_0} \subset \partial J_{\rho_\pm}$.
- such that ∂H_{ρ_0} Heegaard splits both ∂J_{ρ_+} and ∂J_{ρ_-}



Connection: Gay-Kirby show any closed connected orientable 4-manifold can be "trisected": divided by Heegaard splittings into three 4-dimensional handlebodies.
'Heegaard union' is just 2 sectors of a trisection



Since $P \cong S^3$ can set $J_3 = D^4$ to get trisection of a homotopy 4-sphere. Then $X = D^4 \iff$ homotopy sphere $= S^4$.

Proposition

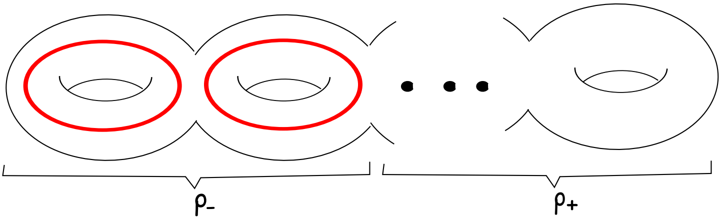
Suppose a 4-dimensional homology ball X is the Heegaard union of J_{ρ_-}, J_{ρ_+} along H_{ρ_0}, and $\rho_- \leq \rho_+$. Then X has a handle decomposition consisting of ρ_- 1-handles and ρ_- 2-handles.

Proof:

- X a homology ball $\implies$ $\rho_0 = \rho_+ + \rho_-$ (Mayer-Vietoris)
- Standard splitting of $\partial J_{\rho_+} = \partial(H_{\rho_+} \times I)$ is $H_{\rho_+} \times \{0\} \cup_{\partial H_{\rho_+}} H_{\rho_+} \times \{1\}$
- Our splitting is $(\rho_0 - \rho_+) = \rho_-$-stabilization (Waldhausen)

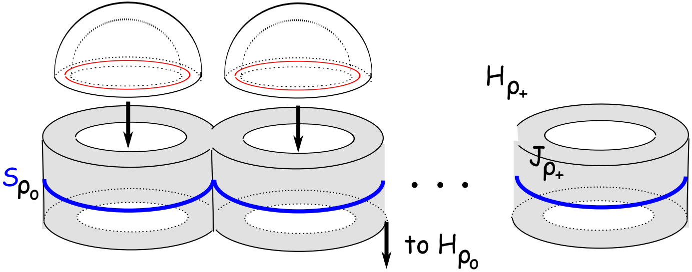
Alternate picture of this stabilization:

- Begin with standard splitting on ∂J_{ρ_0}:
  $H_{\rho_0} \times \{0\} \cup_{\partial H_{\rho_0}} H_{\rho_0} \times \{1\}$
- 0-surger core of ρ_- copies of $S^1 \times B^2$ in $H_{\rho_0} \times \{1\}$.



- Surgery turns each $S^1 \times S^2$-summand to S^3 (i. e. deletes it)
- Splitting becomes genus 1-split S^3 (i. e. stabilization)

Alternate picture of J_{ρ_+}:
4-dimensional 'trace' of each 0-surgery is attachment of 2-handle.



So Heegaard union same as attaching ρ_- 2-handles to J_{ρ_-}.
(Along link in ∂J_{ρ_-} with tunnel # $\rho_0 - 1$.)

Corollary

Schoenflies Conjecture true for genus 3 case.

Proof:

- Reimbedding (via sutured manifold theory) makes X genus 3 Heegaard union.
- Since $\rho_- + \rho_+ = 3, \rho_- \leq \rho_+ \implies \rho_- = 1$.
- Single 1-handle and 2-handle is Gabai's famed Property R

Leads to general Property R question: If

- X is a homology ball
- $\partial X = S^3$
- X has only 1- and 2-handles,

Is $X = B^4$?

Versions:

- Strict version (generalized Property R Conjecture:) Yes, and can do this without adding any other handles. Usually stated: If surgery on n-component link $L \subset S^3$ gives $\sharp_n(S^1 \times S^2)$, can handle-slides turn L into the unlink?
- Weak version 1: Yes, and can do this by adding only canceling 1 and 2 handle pairs.
- Weakest version: Yes, if also use canceling 2- & 3-handles.

Latest betting: Strict version false.

Weakest version plus success at reimbedding X_0 to handlebody would suffice for Schoenflies.

On to genus 4?

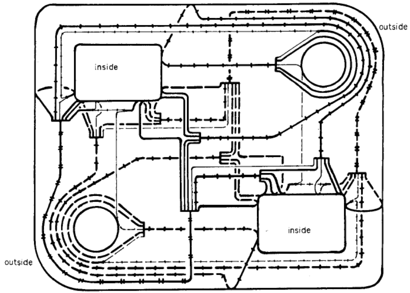
Theorem (Akbulut-Kirby 1985)

There is a homotopy 4-sphere Σ whose handle structure naturally presents $\pi_1(\Sigma) =< a, b \mid aba = bab, a^5 = b^4 >$.

Theorem (Gompf 1991)

- *AK example extends to family with* $\pi_1(\Sigma_n) \cong< a, b \mid aba = bab, a^{n+1} = b^n >$.
- *Each is S^4, via canceling 2-handle 3-handle pair*
- $\exists$ *Kearton-Lickorish Heegaard embedding $\Sigma_4 - B^4 \subset S^4$.*
- *Presentation $\pi_1(X) =< a, b \mid aba = bab, a^5 = b^4 >$ probably cannot be Andrews-Curtis trivialized, e.g. via handle-slides.*

Gompf example very complicated:



Three questions:

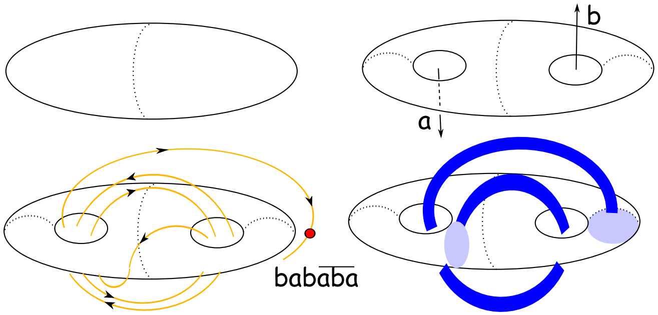
- Can the example be simplified?
- Can embedding be extended to other n (even $n \leq 3$ unknown)
- Can a side be trivialized (shown to be B^4) without stabilization via reimbedding?

Can reimbedding again save the day?

Goals:

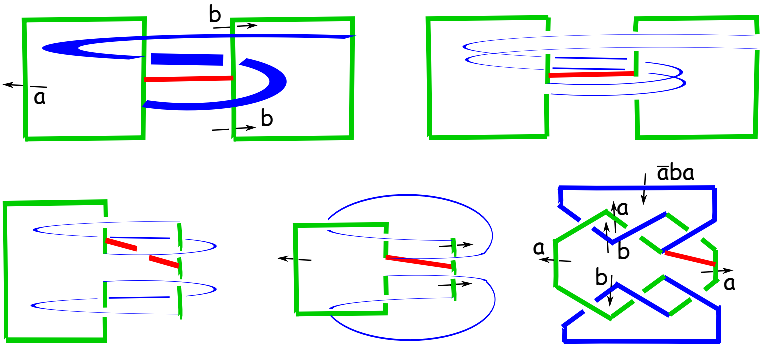
- Simplify Akbulut-Gompf example
- so that 'natural' picture suggests simpler reimbedding of Y
- perhaps extend to all n

Example: The first three 1-handles.



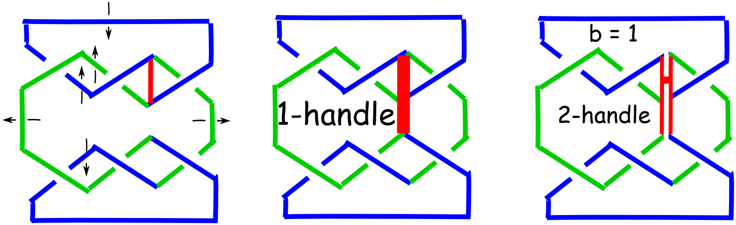
X_{-} unaffected by last 1-handle $\implies \pi_1(X_{-}) \cong \langle a, b \mid \rangle = \mathbb{Z} * \mathbb{Z}$.

The first 2-handle:



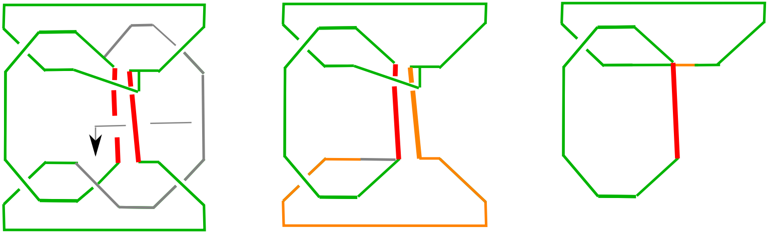
Now $\pi_1(X_-) \cong < a, b \mid aba = bab >= \pi_1(S^3 - \textit{trefoil knot})$.

The last 1 handle & second 2-handles? If 2-handle in Y then relator for $\pi_1(X_-)$



Computation shows: $\pi_1(X) \cong < a, b \mid aba = bab, b = 1 >$.

Now need exactly 2 outside 2-handles:



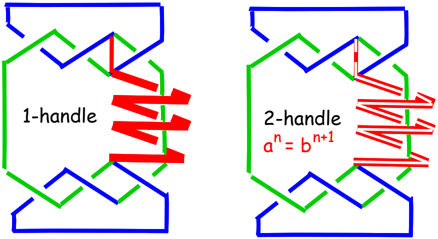
We have them!

Note:

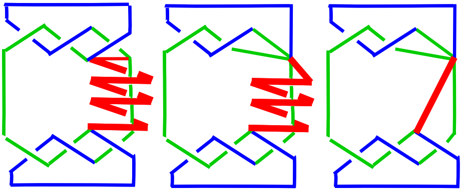
- Since these are in X_0, no further effect on $\pi_1(X)$
- In any case $\pi_1(X) = 0$. Let's pretend we know $X = B^4$.

So what?

Reimbed last 1-handle, then second 2-handle:



Topology of exterior the same: just unwind.

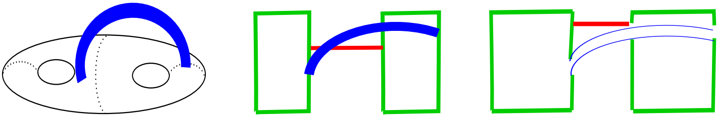


So can cap off with 2-handles in X. (But are they the same 2-handles? Or is Y now just a Mazur manifold?)

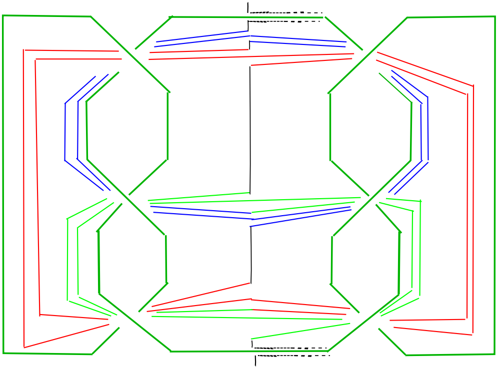
Hope

Y is a reimbedded B^4, indeed the Akbulut-Gompf example.

Additionally: could try reimbedding first 1-handle, also suggested by thin position:



Example above inspired by a true event. In Akbulut-Gompf, crucial 2-handle comes from Dehn twisting multiple times along specific embedded torus:



Seems beyond the range of graphics or computer check.

Last half take-away:

Perhaps in general reimbedding can substitute for stabilization, but

- So far only one interesting example and
- it's really difficult to work with this example.