Reimbedding and the Schoenflies Conjecture

Dublin

August 2015

Setting: $S^3 \cong_{PL/DIFF} P \subset S^4$ divides S^4 into X and Y.

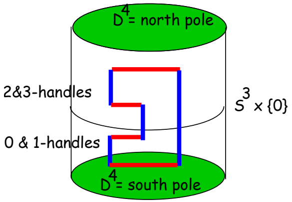
Conjecture (Schoenflies Conjecture)

X and $Y \cong_{PL/DIFF} D^4$.

Observations:

- $X \cong_{PL/DIFF} D^4 \iff Y \cong_{PL/DIFF} D^4$.
- Handles of P are level and appear with non-decreasing index
- A handle into X only affects topology of Y (and vice versa).
- Sample lemma: If all all $0, 1$-handles on same side, then X and $Y \cong_{PL/DIFF} D^4$.

P in Kearton-Lickorish position $\Longrightarrow$ $S^3_0 \cap P$ Heegaard splits P.



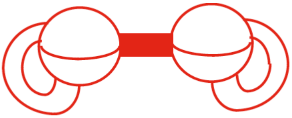


Figure : $S^3_0 \cap P \subset P$

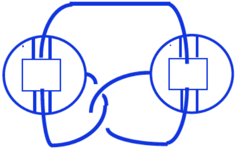
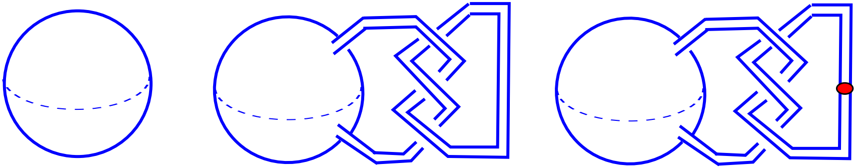


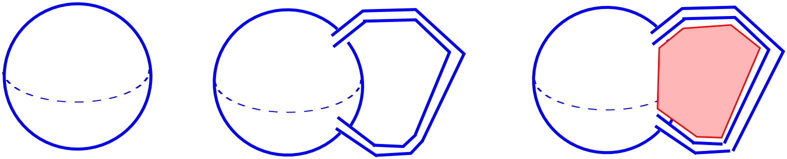
Figure : $S^3_0 \cap P \subset S^3_0$

The genus of the embedding is the genus of this Heegaard surface.

Example: Schoenflies Conjecture obviously true for genus 1:
If 1-handle knotted, where can 2-handle go?



This would give non-separating 2-sphere $\implies$ not S^3.
If 1-handle unknotted, where can 2-handle go?



This leaves X standard. (All handles lie in X.)

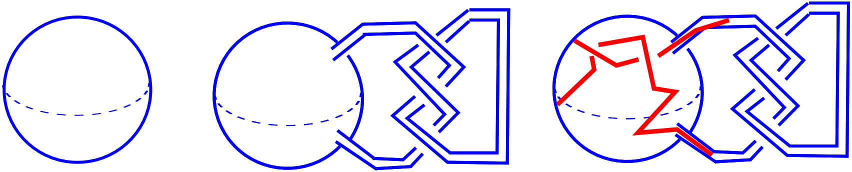
Some philosophical observations/questions:

- Argument hardest when $S^3_0 \cap P$ Heegaard splits S^3_0 also.
- Is focusing on genus a good idea? Eaton (1980's), Agol-Freedman (2013) think not.
- Agol-Freedman: if allowed to stabilize (increase genus) then any embedding with one max can be isotoped to Heegaard embedding: every generic $S^3_t \cap P$ Heegaard splits S^3_t.
- So focus on special case of Heegaard embeddings of P?

Suppose $P \subset S^4$ is genus 2 embedding.
Possibility 1: One 0-handle, two 1-handles, all lying in X, say.
Then all handles on same side $\implies$ done.

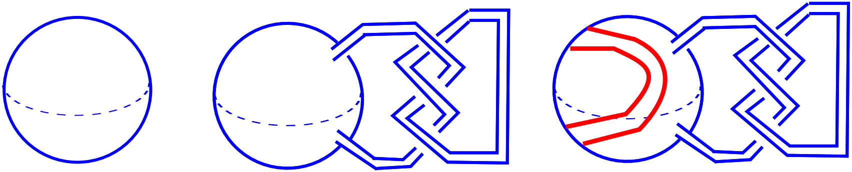
Possibility 2: One 0-handle, two 1-handles, one each in X and Y:

Generic case:



Solution: where's the next 2-handle?

Special case for second 1-handle:



Many possibilities for next 2-handle...

Regardless, know X will have only one 1-handle and one 2-handle (as will Y).

Lucky break - can invoke Gabai's celebrated Property R:

Theorem (Property R Gabai 1987)

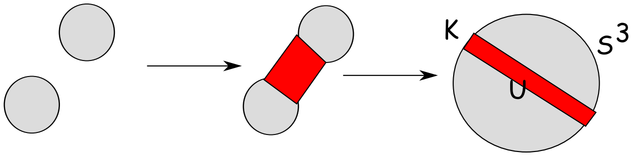
If surgery on knot $K \subset S^3$ gives $S^1 \times S^2$, then K is the unknot.

Relevance: Consider trace of this surgery, i. e. associated 4-manifold cobordism between S^3 and $S^1 \times S^2$.

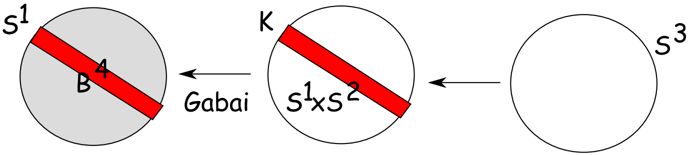
Corollary

Suppose U^4 is homology ball, with $\partial U \cong S^3$.
If U has a single 1-handle and a single 2-handle, then $U \cong B^4$.

Proof: (0-handle $\cup$ 1-handle) is $S^1 \times D^3$, so boundary is $S^1 \times S^2$. Adding 2-handle surgers boundary to S^3.

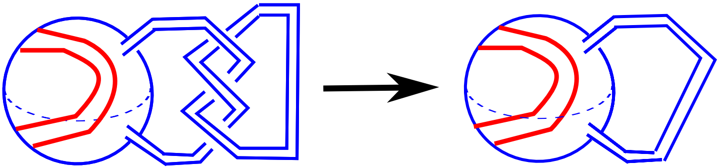


In reverse direction: have surgered S^3 to get $S^1 \times S^2$. Gabai says $K \subset S^3$ is just $S^1 \subset S^3$.



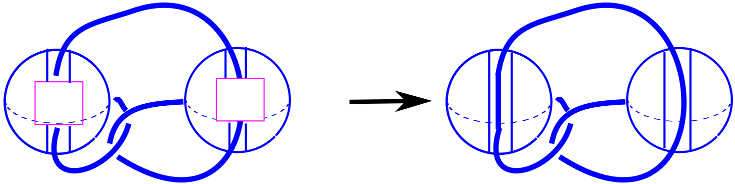
More general point: big 3-manifold theorem is useful for this 4-dimensional question.

Side observation: may as well assume first 1-handle is unknotted, by reimbedding:



- All 2-handles will be trapped in knotted torus, so reimbedding extends thru 2-handles.
- Reimbedding changes X (becomes X' say) but Y unchanged.
- If can show $X' \cong D^4$ then: $\implies Y \cong D^4 \implies X \cong D^4$.
- After the reimbedding, $S^2_0 \cap P$ Heegaard splits S^2_0.

Possibility 3: Two 0-handles, three 1-handles, one connecting:



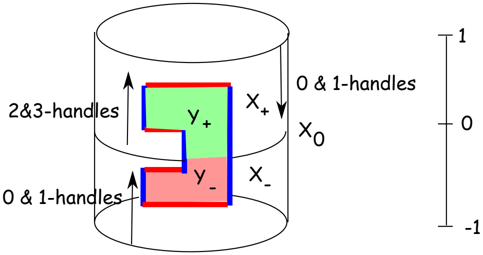
Use reimbedding trick.
Not a Heegaard surface, but at least one side Y is a handlebody.

For genus two, reimbedding always can make one side a handle body; here's why:

- $S_0^t \cap P$ compresses in either X or Y, say Y
- If compresses completely $\implies$ Y a handlebody.
- If doesn't compress then get knotted companion tori, which describe reimbedding.

Let

- $X_0 = X \cap S^3_0$
- $X_- = X \cap S^3 \times [-1, 0]$
- $X_+ = X \cap S^3 \times [0, 1]$



Similarly denote Y_0, Y_-, Y_+, with $Y_0 = Y_- \cap Y_+$

Is it possible to reimbed Y so X_0 is a handlebody (or vice versa)? Would such a reimbedding be useful?

Reimbedding possibility: Observation 1

Theorem (Fox 1948)

Any compact connected 3-dimensional submanifold of S^3 can be re-imbedded as the complement of handlebodies.

Hence always Y_0 can be reimbedded so X_0 becomes a handlebody.

But how to do this so that $Y_t, t \neq 0$ can also be reimbedded? Need reimbedding to extend over 2-handles. (Companion tori guaranteed this in genus 2 case above.)

Observation 2:

Suffices to construct a sequence of reimbeddings:

1. Reimbed Y so X_0 is more compressible than it was
2. Reimbed new X so Y_0 is more compressible than it was
3. Reimbed new Y...

If eventually get X_0 (or Y_0) a handlebody, and if this property can be used to show X or Y is now D^4, then done.

Observation 3:

Deep theorems in 3-manifold theory (using Gabai's sutured manifold theory) shows that reimbedding sequence to handlebody works in genus 3 case.

Observation 4:

Theorem (Agol-Freedman 2013)

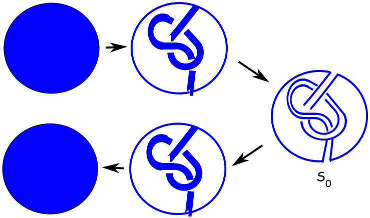
If P has single maximum then can isotope $P \subset S^4$ so that each X_t and Y_t are both handlebodies. (Does not require $P = S^3$.)

Each P_t is Heegaard surface in S^3_t, so it's a Heegaard embedding. (This may have been known to Eaton.)

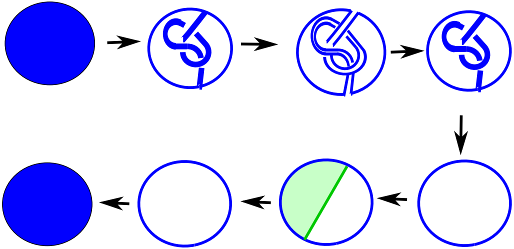
Sketch:

- For knotted 1-handle in Y_t (say), introduce extra 1-handles ('intervention arcs' = IA) so Y_t remains handlebody.
- With even more IA's, the IA's can be added or deleted one-by-one.
- Delete IA's by handle-cancellation. Then result isotopic (not level-preserving) with original

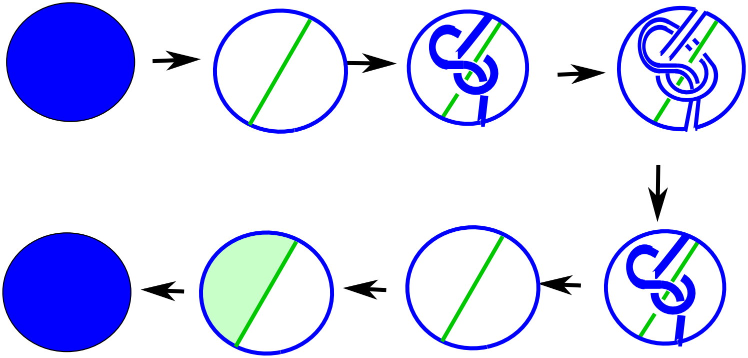
Toy example: $X = S^1 \times D^3$, $Y = D^2 \times S^2$, each X_t a handlebody



Clearly isotopic if, near end, just add canceling 1 & 2-handle

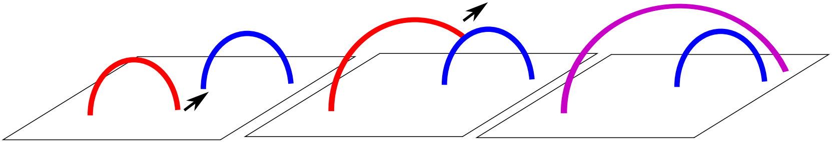


Introduce 1-handle earlier



Now each Y_t also a handlebody.
Problem? Requires massive stabilization of the embedding - no control on genus.

Another source of intervention arcs: Handle-slides in Heegaard theory...



are replaced with arc additions and deletions: