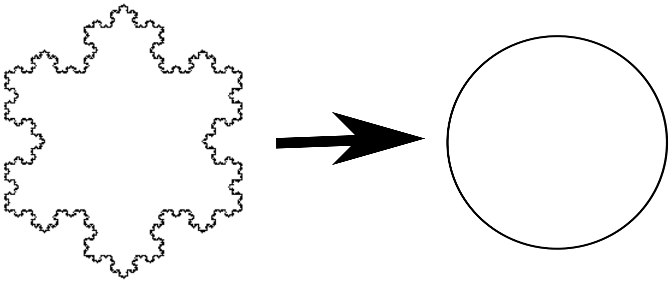
Codimension one embeddings: history and handle theory

Dublin

August 2015

Theorem (Jordan-Schoenflies)

Complement of simple closed curve $c \subset \mathbb{R}^2$ has unique bounded component. For Y its closure, $(Y, c) \cong (D^2, \partial D^2)$.



Corollary

Curve $c \subset S^2$ has two complementary components, X[terior], Y[nterior], each with this property.

Schoenflies: Does any $S^n \subset S^{n+1}$ split S^{n+1} into $(n+1)$ balls?
Alexander: For $S^2 \subset S^3$ answer is 'Yes', then (1924) 'No':

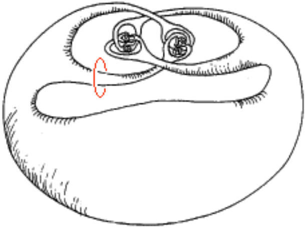


Figure : Alexander horned sphere

So sphere must be 'locally flat', e. g. smooth or PL sub-manifold.

Theorem (M. Brown 1960)

Schoenflies conjecture true for locally flat $S^n \subset S^{n+1}$, all n.

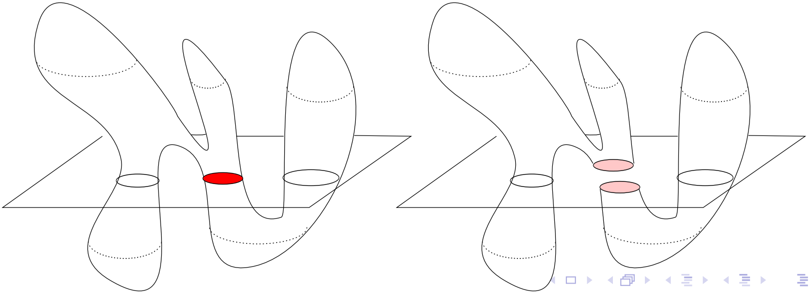
Proof requires infinite construction.

Theorem (Alexander 1924)

If $S^2 \cong P \subset \mathbb{R}^3$ is smooth submanifold and Y is region it bounds, then (Y, P) is diffeomorphic to (D^3, S^2).

Idea behind proof:

- Projection $h : \mathbb{R}^3 \to \mathbb{R}$ cuts P by planes $h^{-1}(t) = \mathbb{R}^2_t, t \in \mathbb{R}$.
- $P \cap \mathbb{R}^2_t =$ circles; innermost circle in $\mathbb{R}^2_t$ bounds disk D.
- Cut P along D to get two simpler spheres in $\mathbb{R}^3$. Then induct.



Theorem (Smale 1960)

For $n \geq 4$, if $S^n \cong P \subset \mathbb{R}^{n+1}$ is smooth submanifold and Y is region it bounds, then (Y, P) is diffeomorphic to (D^{n+1}, S^n).

Proof via h-cobordism theorem (smooth or PL)

Theorem (h-cobordism)

If $\partial W^{n+1} = V_0 \cup V_1$ and

- *W, V_i are simply connected*
- *$H_*(W, V_0) = 0$ (equivalently $H_*(W, V_1) = 0$)*
- *$n \geq 5$*

Then $W = V_0 \times I$.

Application: For $n \geq 5$, remove (smooth or PL) disk D^{n+1} from X. h-cobordism theorem says $X - D^{n+1}$ is just collar of ∂D^{n+1}.

Case $n = 4$ a bit more complicated: Consider $X^+ = X \cup D^5$, a homotopy 5-sphere.

Kervaire-Milnor say X^+ then bounds contractible W^6.

h-cobordism theorem says $W^6 - D^6 = X^+ \times I$
Hence $X^+ = \partial D^6 = S^5$.

Exercise: Show that the complement of any smooth D^m in S^m is smooth D^m

Apply exercise to $X^+ - D^5 = X$.

Only one case remains:

Conjecture (Schoenflies Conjecture)

If $S^3 \cong P \subset \mathbb{R}^4$ is smooth submanifold and Y is region it bounds, then (Y, P) is diffeomorphic to (D^4, S^3).

Freedman (1986) stretches Smale's handlebody approach to dimension 4, but uses infinite process $\implies$ not smooth.

Idea: Try to modify Alexander's approach:

- Clarifying thought: $h : P \subset S^4 \to [-1, 1]$ with $h^{-1}(t) = S^3_t$.
- Bad news: $P \cap S^3_t$ is a surface in S^3. These are much more complicated than circles in the plane.
- Good news: Surfaces morphing in S^3 are fun to think about!

Some useful mathematical points:

- DIFF and PL are the same in this dimension.
- If either X or Y is diffeomorphic to D^4, then so is the other, since smooth $D^m \subset R^m$ deforms to standard D^m via

$$f_t(x) = \frac{f(tx)}{t}, t > 0 \qquad f_0(x) = Df(x)$$

- If the Schoenflies Conjecture is false, then

$$S^4 \cong X^{capped} \# Y^{capped}.$$

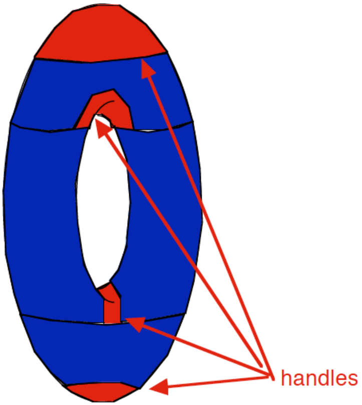
$\implies$ Prime decomposition of 4-manifolds is hopeless.

- Smooth Poincaré Conjecture $\implies$ Schoenflies Conjecture. (Since X^{capped} is a homotopy 4-sphere.)

Theorem (Gospel of Morse)

Any smooth manifold M^m has a handlebody structure: $M \cong (0 - handles) \cup (collar) \cup (1 - handles) \cup (collar) \ldots \cup (m - handles)$.

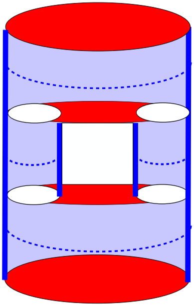
Here collar is $Q^{m-1} \times I$; i-handle is $D^i \times D^{m-i}$.



Theorem (Addendum of Kearton-Lickorish)

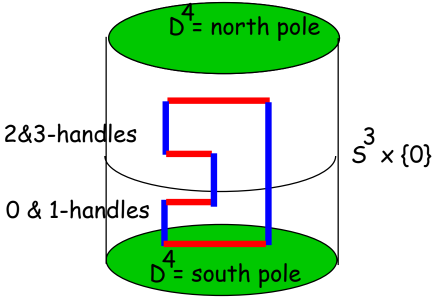
For M^m imbedded in $N^m \times I$, can isotope M so that, for some handle structure:

- *All the handles are horizontal, i. e. each lies in some $N \times \{t\}$.*
- *All the collars are vertical, i. e. each looks like $Q^{m-1} \times [s, t]$.*
- *The handles appear in increasing order of index*



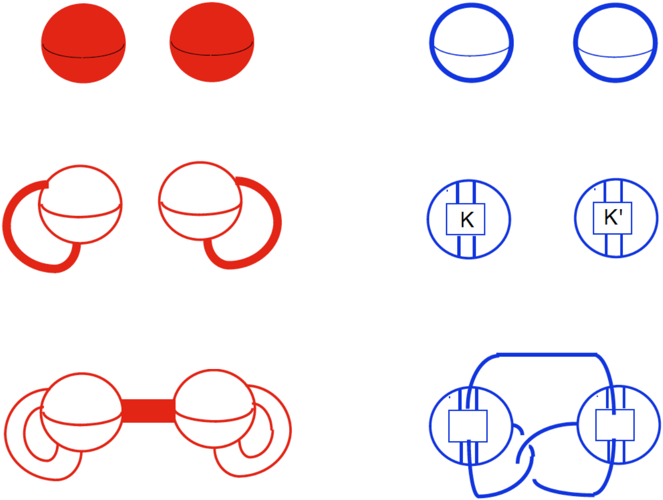
How to apply Kearton-Lickorish to arbitrary $M^3 \subset S^4$

- Think of S^4 as $S^3 \times (-1, 1)$ with N & S poles attached.
- Put 0 and 1-handles of M below $S^3 \times \{0\}$
- Put 1 and 2-handles of M above $S^3 \times \{0\}$



Then $S^3_0 \cap M$ is Heegaard surface, splitting M into handlebodies.

Knotted $M = \#_2(S^2 \times S^1)$ in S^4: first viewed in M then in S^3_t



Then do the reverse!

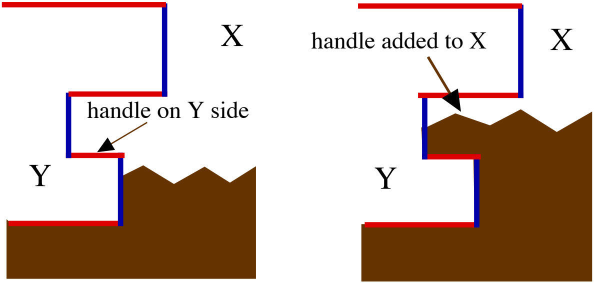
What is the handle-structure of X and Y?

Theorem (Rising Water Principle)

Adding a 3-dimensional i-handle lying in $Y \cap S^3_t$

- *has no effect on the topology of $Y \cap (S^3 \times [-1, t])$*
- *adds a 4-dimensional i-handle to $X \cap (S^3 \times [-1, t])$*

And symmetrically.

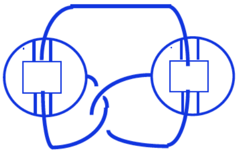


Tracking the handle structure for our example:

Two 0-handles that lie in X added $\implies$ X still $\cong D^4$ but $Y \cong D^4 \cup D^4$

1-handles in Y added $\implies$
$X \cong S^1 \times D^3 \natural S^1 \times D^3$
Y still $\cong D^4 \cup D^4$

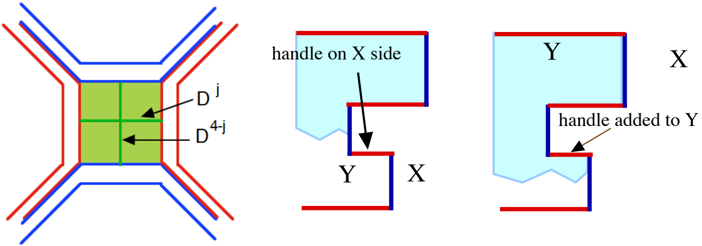
1-handle in X added $\implies$
X still $\natural_2 S^1 \times D^3$, $Y \cong D^4$



When reverse: outside 1-handle dual to inside 2-handle...
$\implies$ $X \cong S^1 \times D^3 \natural S^1 \times D^3$

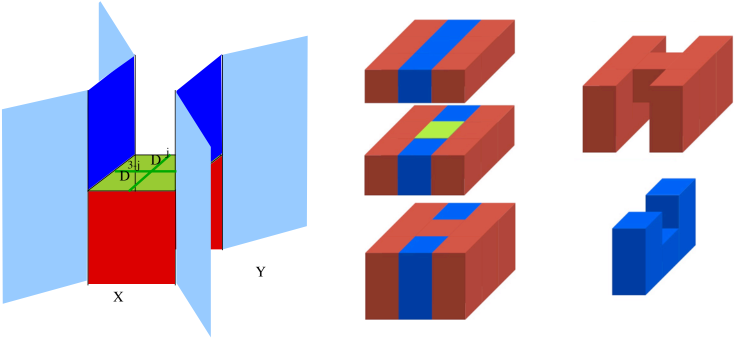
There are then two types of duality on X and Y:
Consider a j-handle added to X:

- As a j-handle in a 4-dimensional manifold X it is dual to a $(4-j)$-handle in X



- Since it's represented by j-handle on Y-side as water rises $\implies (3-j)$-handle lying on X side as 'hydrogen descends'.

The picture:



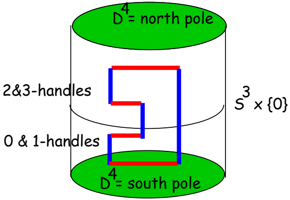
Example: Suppose $S^3 \cong P \subset S^4$ and learn that (ascending from below) all 0- and 1-handles are attached on exterior (X-side).

- Then (from below) X has only 2- and 3-handles $\{h\}$.
- 4-dimensional duality $\implies$ X needs only 1- and 2-handles. Not known to be enough to conclude $X \cong D^4$.

Consider Y constructed from above:

- All 0- and 1-handles from below in X $\implies$ from above, all 2- and 3-handles lie in Y. These have no effect on Y.
- Then only handles of Y from above come from $\{h\}$; correspond to 0- and 1-handles (3-dimensional duality).
- Hence Y is homotopy ball made only from 0- and 1-handles $\implies$ $Y \cong D^4$ $\implies$ $X \cong D^4$. □

Recall: $S^3_0 \cap M$ is a Heegaard surface for M:



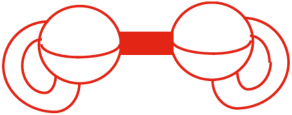


Figure : $S^3_0 \cap M \subset M$

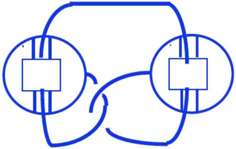


Figure : $S^3_0 \cap M \subset S^3_0$

The genus of the embedding is the genus of this Heegaard surface.