PY4A04 Cosmology Continuous Assessment

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1 Question 1

From the definition of the Ω parameter:

$$\Omega = \frac{8\pi G\rho}{3H^2} \tag{1}$$

where ρ represents the total density of all material in the Universe, show that the quantity:

$$\rho a^2(1 - \Omega^{-1}) = constant \tag{2}$$

I will ignore the contribution of dark energy when writing the Friedmann equation;

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}. \tag{3}$$

In this equation, H is the Hubble parameter, G is the gravitational constant, ρ is the density of the Universe, k is the curvature of the Universe and c is the speed of light. a is a length parameter, which, by convention, is often set to 1 at the present time.

The **density parameter** can also be defined as

$$\Omega = \frac{\rho}{\rho_c}, \tag{4}$$

where the critical density, ρ_c, is the value of density required to make $k = 0$ (producing a Universe with flat geometry). We can use this equation to substitute a value for ρ in equation 3, which becomes

$$H^2 = \frac{8\pi G}{3}(\rho_c\Omega) - \frac{kc^2}{a^2}. \tag{5}$$

From the equation for Ω given in the question (equation 1), we can derive an expression for ρ_c:

$$\begin{aligned} \Omega &= \frac{8\pi G\rho}{3H^2}, \\ \therefore \rho_c &= \frac{3H^2}{8\pi G}. \end{aligned} \tag{6}$$

We can now substitute the expression for ρ_c from equation 6 into equation 5. We can rearrange the resultant relation as follows:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\left(\frac{3H^2}{8\pi G}\Omega\right) - \frac{kc^2}{a^2} \\ &= H^2\Omega - \frac{kc^2}{a^2} \\ \therefore H^2a^2(\Omega - 1) &= kc^2. \end{aligned} \tag{7}$$

The curvature of the Universe, k, is constant with respect to time. The speed of light c is also a constant. Therefore, the first time derivative of equation 7 = 0.Equivalently, we can write

$$H^2a^2(\Omega - 1) = constant. \tag{8}$$

We were asked to show that the left hand side (LHS) of equation 2 is constant with respect to time. Equation 2 can be rearranged to give

$$\rho a^2(1 - \Omega^{-1}) = \rho a^2 \frac{1}{\Omega}(\Omega - 1)$$

At this point we can use equation 1 to substitute for the $1/\Omega$ term above, giving us

$$\begin{aligned} \rho a^2(1 - \Omega^{-1}) &= \rho a^2\left(\frac{3H^2}{8\pi G\rho}\right)(\Omega - 1) \\ &= \left(\frac{3}{8\pi G}\right) \times H^2a^2(\Omega - 1). \end{aligned} \tag{9}$$

Note that this expression (equation 9) is simply equation 7, multiplied by the pre-factor

$$\frac{3}{8\pi G}. \tag{10}$$

We have shown that equation 7 is constant with respect to time. The pre-factor in expression 10 is comprised of constants, and is also invariant with respect to time.

$$\begin{aligned}\frac{d}{dt}\left(\rho a^2(1-\Omega^{-1})\right) &= \\ &= \frac{3}{8\pi G}\frac{d}{dt}\left(H^2a^2(\Omega-1)\right) \\ &= \frac{3}{8\pi G}\times 0 = 0. \\ \therefore \rho a^2(1-\Omega^{-1}) &= constant.\end{aligned}$$

By working from the present epoch, and assuming matter dominates now, show that, for a Universe with arbitrary spatial curvature that contains both radiation and matter components, the epoch of matter-radiation equality occurs when:

$$(1-\Omega^{-1})_{equality} = \frac{(1-\Omega^{-1})_0}{1+z_{eq}} \tag{11}$$

where z_{eq} is the redshift corresponding to that time.

The Newtonian fluid equation governs the evolution of density with respect to the pressure in a medium;

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{12}$$

In this equation, p is the pressure in the medium and c is the speed of light.

In considering the evolution of the Universe from the epoch of radiation-matter equality, I will apply the approximation that the evolution has been matter dominated since epoch of equality up to the present epoch. Matter, in this context, denotes non-relativistic, "dusty" material, which exerts a negligible pressure on itself. Hence, we can let $p = 0$ in equation 12 when considering the evolution from the epoch of matter-radiation equality. Equation 12 becomes

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0. \tag{13}$$

Now I will introduce another expression, which I will show to be equivalent to equation 13:

$$\begin{aligned}\frac{1}{a^3}\frac{d}{dt}(\rho a^3) &= \frac{1}{a^3}\left(\rho\frac{d(a^3)}{dt} + a^3\frac{d(\rho)}{dt}\right)\\ &= \frac{1}{a^3}\rho(3)a^2\frac{da}{dt} + \frac{d(\rho)}{dt}\\ &= \dot{\rho} + 3\frac{\dot{a}}{a}\rho.\end{aligned}$$

This is equivalent to the matter-only fluid equation, equation 13.

$$\therefore \frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0. \tag{14}$$

a is the scale factor for distance in the Universe and is non-zero in general. Hence, from equation 14, we can infer the following:

$$\frac{d}{dt}(\rho a^3) = 0.$$

In this equation, the first time derivative of ρa^3 is equal to zero. Hence, we can write

$$\begin{aligned}\rho a^3 &= constant,\\ \therefore \rho &\propto \frac{1}{a^3}.\end{aligned} \tag{15}$$

We can use this result to relate the current density to the density at any value of a;

$$\begin{aligned}\rho &\propto \frac{1}{a^3},\\ \text{and } \rho_0 &\propto \frac{1}{a_0^3},\\ \therefore \frac{\rho}{\rho_0} &= \left(\frac{a_0}{a}\right)^3.\end{aligned} \tag{16}$$

At this point, it is important to note that the absolute value of a is arbitrary, as only the relation $(\dot{a}/a)$ is included in the equation of state and the Friedmann equation. Therefore, for convenience, we can set $a_0 = 1$. When this convention is applied, equation 16 becomes

$$\frac{\rho}{\rho_0} = \left(\frac{1}{a}\right)^3. \tag{17}$$

I have shown previously that the expression in equation 2 is a constant with respect to time in the Universe.

Therefore we can write

$$\rho_{eq}a_{eq}^2(1-\Omega_{eq}^{-1}) = \rho_0 a_0^2(1-\Omega_0^{-1}), \tag{18}$$

where ρ_{eq}, a_{eq}^2 and Ω_{eq} denote the density, scale factor and density parameter at the epoch of matter-radiation equality. ρ_0, a_0^2 and Ω_0 denote the density, scale factor and density parameter at the present time.
The redshift of a source of light is given by the following relation

$$1+z = \frac{\lambda_{recieved}}{\lambda_{emitted}} = \frac{a(recieved)}{a(emitted)}, \tag{19}$$

where z is the redshift of the source; $\lambda_{recieved}$ and $\lambda_{emitted}$ are the wavelengths of light received by an observer and emitted by the source, respectively; $a(recieved)$ is the scale factor of the Universe when the light is detected and $a(emitted)$ is the scale factor of the Universe when the light was emitted.
For the case of the redshift of the matter-equality epoch relative to the present time, we can write

$$1+z_{eq} = \frac{a_0}{a_{eq}}, \tag{20}$$

which, under the convention$a_0 = 1$, becomes

$$1+z_{eq} = \frac{1}{a_{eq}} \tag{21}$$

We can rearrange equation 18 to give

$$\begin{aligned}(1-\Omega_{eq}^{-1}) &= \frac{\rho_0}{\rho_{eq}}\left(\frac{a_0}{a_{eq}}\right)^2(1-\Omega_0^{-1}),\\ &= \frac{\rho_0}{\rho_{eq}}\left(\frac{1}{a_{eq}}\right)^2(1-\Omega_0^{-1}).\end{aligned} \tag{22}$$

We can make substitutions from equations 17 and 21 into equation 22 to obtain the following:

$$\begin{aligned}(1-\Omega_{eq}^{-1}) &= \left(\frac{1}{a_eq}\right)^{-3}\times\left(\frac{1}{a_{eq}}\right)^2\times(1-\Omega_0^{-1})\\ &= a_{eq}\times(1-\Omega_0^{-1})\\ \therefore (1-\Omega_{eq}^{-1}) &= \frac{1}{z_{eq}+1}\times(1-\Omega_0^{-1}).\end{aligned} \tag{23}$$

Here I have shown that equation 23 is a valid for a matter dominated evolution from the epoch of matter-radiation equality.

Given that matter-radiation equality occurs when $z_{eq} = 6000$, and that the density of matter

today is $\approx 30\%$ that of the critical density, estimate the density of matter relative to the critical density at the epoch of matter-radiation equality. Is this result in agreement with what you would expect?

We can treat Ω_0 as 30%, as matter dominates the density at the present time. In other words, I'm choosing to ignore the present-day contribution of radiation density to our calculation. We can substitute this value, along with the value provided for redshift, into equation 23 to estimate the density parameter at the epoch of matter-radiation equality.

$$\begin{aligned}(1-\Omega_{eq}^{-1}) &= \frac{1-\Omega_0^{-1}}{1+z_{eq}}\\ (1-\Omega_{eq}^{-1}) &= \frac{1-(0.3)^{-1}}{1+6000}\\ \therefore \Omega_{eq} &= \left(\frac{1}{6001}\left(\frac{1}{0.3}-1+6001\right)\right)^{-1}\\ &= 0.9996.\end{aligned} \tag{24}$$

Note that by definition, at the epoch of matter-radiation equality, the density in matter form was equal to the density in the form of radiation. Hence, we can argue that at this epoch, the density parameter for matter was

$$\Omega_M = \frac{0.9996}{2} \approx 0.5. \tag{25}$$

One of the problems with traditional big bang scenarios is that for the density parameter to have the value it does now, it would have to have been either very close to, or exactly, one at the big bang. To account for this, the curvature parameter, k was introduced.

However, for our purposes, a value of $\Omega_{eq} \approx 1$ is in agreement with an evolution from a big bang in which the initial value of Ω was almost exactly unity. Therefore our equations have produced a value of Ω_{eq} that we would would otherwise expect for this epoch.

What is the temperature at this time?

We can assert that temperature of the CMBR is related to the scale parameter according to $T \propto 1/a$

(see equation 39) . Considering this, we can write the following:

$$\frac{T_{eq}}{T_0} = \frac{1}{a_{eq}} \frac{a_0}{1} = \frac{a_0}{a_{eq}}$$

$$\therefore T_{eq} = (2.725 \text{ Kelvin}) \frac{a_0}{a_{eq}}$$

$$\text{Note that} \frac{a_0}{a_{eq}} = z_{eq} + 1 \text{ (equation 20)}$$

$$\therefore T_{eq} = (2.725 \text{ Kelvin}) \times (6000 + 1) = 16352.725 \text{ Kelvin} \tag{26}$$

2 Question 2

The temperature of matter evolves differently to radiation after decoupling. How do the temperatures evolve prior to decoupling? Justify your answer.

Prior to decoupling, the path-length for radiation is very low as it undergoes Thompson scattering from matter in a plasma state. Therefore the matter and radiation are kept in a state of thermal equilibrium and their temperatures are equal; $T_M = Trad$. Therefore, equation 39 would also have applied to matter prior to decoupling and $T_M \propto 1/a$.

Explain, in your own words, the principles of how observations of Type Ia SNe are used to deduce the expansion rate of the Universe (from detection to production of luminosity, and including any corrections or caveats required) and whether this approach can address quintessence.

If we work under the (seemingly correct) assumption that the nature of type Ia SNe are consistent across all generations of stars, we can plot the apparent brightness of them as a function of apparent redshift (z) in order to obtain an estimate of the rate of expansion of the Universe. As we measure brightness of these events to increasingly high z, deviation from a consistent rate of expansion is observed. In cosmological models, this acceleration is contributed to the combined influence of the curvature of space-time (k) and dark energy (Λ).

Difficulty lies in the detection of suitably large samples of these SNe, as only a small number occur per galaxy every thousand years. In order to facilitate the acquisition of a statistically significant data set, Perlmutter (1999) employed a batch surveying scheme for the initial detection of type Ia SNe, in collaboration with other cosmologists.

Although all type Ia SNe have very similar peak intensities and photometric characteristics, there is some deviation from the mean values. However, the brightness deviation is strongly correlated to the timescale over which the event takes place. Measurement of the time for a given SN to reach its peak intensity and decline again allows astronomers to calibrate measurements of the peak intensity luminosity (the star's power output) accordingly.

Scattering of light by dust in the interstellar medium (ISM) is wavelength dependant; the detected spectra of SNe, as measured through ISM dust, are subsequently "reddened". This must also be taken into account when calibrating apparent magnitude. The dispersion of peak magnitude values is $\sim 10-12\%$, making type Ia SNe consistent standard candles for cosmological measurement.

Quintessence is expansion with an as yet undetected, slow-varying Λ term. In theoretical descriptions, it is associated with a vacuum energy density which exerts a large, negative pressure on the Universe, driving expansion. The ratio of this exerted pressure to the density is represented by ω. In a quintessence model ω is variable, while for the case of a "cosmological constant", $\omega = -1$. A number of theoretical quintessence models predict less negative values of ω than our current estimate, although the margin of error associated with current measurements is very large. Perlmutter (1999) suggested employing a satellite based study of type Ia SNe, to furthur redshifts ($z = 1.7$) in order to obtain more accurate estimates of the cosmological parameters. Such a study would reduce error in our calculation of ω to the order of 0.01, allowing a number of quintessence models to be tested and possibly ruled out.

Explain, in your own words, how slow-roll inflation can explain some of the problems with the simple Big Bang model. What is the argument in favour of the Higgs particle being associated with the inflaton?

Big Bang theories describe the expansion and cooling of the Universe; the "bang" itself is treated as a singularity. Inflationary Universe theory provides a mechanism by which the initial expansion could have occurred.

The CMBR is observed to be homogeneous to one part in 10^5. From this we can conclude that the Universe was at an effective thermal equilibrium at the epoch of decoupling, $\sim 3 \times 10^5$ years after the big bang. This poses a problem, as wee observe uniformity between points that are greater than a "Hubble distance" from one another; travelling at the speed of light, information could not have moved between these regions in order for them to reach equilibrium. This shortcoming of the big bang theory is known as the "Horizon problem".

Estimates of the current value of the density parameter (Ω_0) provide a value 1 $\pm 5\%$. An Ω_0 of order unity is an unstable equilibrium point; it necessitates that 1 second after the big bang, Ω would have to deviated from 1 by less than 1×10^{-15}. If we extrapolate further, we can show that on the order of Planck time after the big bang, Ω would have to have been within $\sim 2 \times 10^{-59}$ of 1.

Inflationary theory relies on the proposition a region of space came to be in a false vacuum, in which the energy density is greater than that of a true vacuum, but cannot be rapidly lowered. When this region eventually does relax to a true vacuum state, it will exert a negative pressure which will in turn exert a repulsive gravitational field, driving inflation. In one traditional inflationary theory, the radius of this region would have increased by a factor of 10^{25} in the inflationary process.

Prior to inflation, the region that became the Universe would have had sufficient time to reach thermal equilibrium with itself; this uniformity was subsequently maintained under the inflationary process, providing an explanation for the "Horizon problem".

A short rapid inflation would also drive $k \to 0$, and therefore drive $\Omega \to 1$. Hence an initial value of $\Omega \approx 1$, as necessitated in a big bang model, could be provided.

Ellis and Uzan (2014) suggested Higgs particle would make a convenient candidate for the particle that seeded this inflation. They argue that such a link would unify two seemingly distinct physical phenomena, i.e., the inflationary mechanism that produced the big bang and the Higgs mechanism. In a Higgs inflation model, there is only one free parameter to augment, the coupling of the Higgs particle to the curvature of space (k). A small number of free parameters is an appealing feature of a model; it constrains the model's behaviour and makes it easy to test. While appealing in its simplicity, thus far there has been no evidence to link these phenomena, making any discussion of Higgs inflationary models highly speculative.

3 Question 3

Calculate the contribution of starlight to the radiation energy-density of the Universe, assuming that there are 10^{11} galaxies in the Universe, each containing 10^{11} stars similar to our own Sun, and that the photons are conserved.

In the question above, I shall consider "Universe" to mean the observable region of space within the horizon distance. This volume forms a sphere, whose volume given by

$$V = \frac{4}{3}\pi\left(\frac{c}{H_0}\right)^3. \tag{27}$$

Note that the radius of this sphere is the maximum distance that light could have travelled since the beginning of the Universe (the Hubble distance). The Hubble distance has been calculated to be $\sim$ 4300 Mpc. Using the volume equation above and values provided in the question, I will calculate the number density of stars in the Universe (N).

$$\begin{aligned} N &= \frac{10^{11} \times 10^{11}}{V} \\ &= 10^{22}\frac{3}{4}\frac{1}{\pi}(4300)^{-3}\text{Mpc}^{-3} \\ &= 3 \times 10^{10}\text{Mpc}^{-3}. \end{aligned} \tag{28}$$

Under the assumption that each of these stars is like our sun, we can suggest that they have the same luminosity as our sun, $L_\odot = 3.839 \times 10^{26} W$.
Hence the total luminosity density of all stars is

$$N \times L_\odot = (3 \times 10^{10})(3.839 \times 10^{26}) = 1.1517 \times 10^{37}\text{W Mpc}^{-3}. \tag{29}$$

We can multiply the value obtained in equation 29 by the age of the Universe in order to estimate the energy density of starlight in the Universe.
Note that the age of the Universe will be obtained from the present-epoch value of the Hubble parameter

(H_0), which was calculated by the Planck mission to be

$$H_0 = 67.8\text{km s}^{-1}\text{Mpc}^{-1} = 2.197 \times 10^{-18}\text{s}^{-1}. \tag{30}$$

Our starlight energy density calculation also relies on the following approximations:

1. The number density of stars has remained the same over the lifetime of the Universe.
2. Each generation of stars has had the same mean luminosity.
3. Only a negligible amount of photons radiated by stars are subsequently destroyed (photons are conserved).

$$\begin{aligned}
\epsilon_{starlight,0} &= N \times L_{\odot} \times t_0 & (31)\\
&= N \times L_{\odot} \times \frac{1}{H_0} \\
&= 1.1517 \times 10^{37}\ \text{W Mpc}^{-3} \times 4.55 \times 10^{17}\text{s}^{-1} \\
&= 5.264 \times 10^{54}\ \text{J Mpc}^{-3} \\
&= 1.792 \times 10^{-13}\ \text{J m}^{-3} \\
&\approx 1.1\ \text{MeV m}^{-3} & (32)
\end{aligned}$$

What fraction of the closure density is this?

The current value of the closure density $\rho_{c,0}$ is defined as

$$\rho_{c,0} = \frac{3}{8\pi G} H_0^2. = 8.638 \times 10^{-27}\text{kg m}^{-3}. \tag{33}$$

We can write the energy density of star light as an equivalent mass density through the relation $E = mc^2$;

$$\rho_{starlight,0} = \frac{\epsilon_{starlight,0}}{c^2} = \frac{1.792 \times 10^{-13}}{(3 \times 10^8)^2} = 1.99 \times 10^{-30}\text{kg m}^{-3}. \tag{34}$$

Therefore the ratio of the mass-equivalent density of starlight to the critical mass density is given by

$$\frac{\rho_{starlight,0}}{\rho_{c,0}} = \frac{1.99 \times 10^{-30}}{8.638 \times 10^{-27}} = 2.31 \times 10^{-4}. \tag{35}$$

The density of starlight represents less that 1 part in a thousand of the mass density necessary for a closed Universe (neglecting the contributions of Λ and k).

4 Question 4

One non-Big Bang explanation for the CMB radiation is that it is due to "tired light" in which the original photons lose energy on their journey towards us, but are otherwise unaffected. Outline why this cannot be true.

The assertion of constancy of the speed of light is attributed to Einstein's relativity and in order to consider a variable light speed, a propagation medium must be introduced. No evidence for a suitable medium has ever been found that would cause photons to lose energy in this way.

The tired light mechanism would rely on interactions between photons and the medium. Such interactions would detract energy from the light (changing its wavelength), without altering its direction of propagation. Such a scattering process is not observed in nature.

As part of their work with the Supernova Cosmology Project, Goldhaber et al. (2001) observed stretching in B band light curves of type Ia SNe explosions. This is considered to me a direct observation of the effect of relativistic time dilation. Such a stretching effect cannot be produced by tired light hypotheses and is instead in good agreement with a "homogeneous and isotropic expanding Universe". This result allows tired light mechanisms to be ruled out as cause for the observed redshift of distant galaxies.

Recalling that the Planck spectrum is given by:

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{d\nu}{exp(h\nu/k_BT)} \tag{36}$$

and assuming that the total number of photons is conserved under expansion, show that the spectrum of the CMB remains blackbody (BB) in character under the transformation:

$$\nu' = \frac{\nu}{1+z} \tag{37}$$

Given relation 37, it is trivial to show

$$\begin{aligned} \nu &= \nu'(1+z), \\ \therefore \frac{d\nu}{d\nu'} &= (1+z), \\ d\nu' &= \frac{d\nu}{1+z}. \end{aligned} \tag{38}$$

A similar relation can be derived between the temperature of the emitting blackbody (T) and the apparent temperature of the redshifted signal (T').

The radiation density of a blackbody is related to the scale factor by

$$\rho_{Rad} \propto \frac{1}{a^4}.$$

From thermodynamics, we know that

$$\rho_{Rad} \propto T_{BB}^4,$$

$$\therefore T_{BB} \propto \frac{1}{a}, \tag{39}$$

$$\therefore \frac{T'}{T} = \frac{1}{a'}\frac{a}{1} \tag{40}$$

In the equations above a' is the scale factor at the time of detection of the BB spectrum and a is the scale factor at the time of emission of the radiation.

Note, if we impose the condition $a' = 1$, a can be related to its redshift;

$$\frac{1}{a} = z + 1 \tag{41}$$

Therefore, equation 40 can be written in terms of redshift;

$$\begin{aligned} \frac{T'}{T} &= \frac{1}{z+1} \\ \therefore T' &= T \times \frac{1}{z+1} \end{aligned} \tag{42}$$

The expression for the redshifted BB spectrum is

$$B_{\nu'} = \frac{2h\nu'^3}{c^2}\frac{d\nu'}{exp(h\nu'/k_BT')}. \tag{43}$$

With substitution from equations 37, **??** and 42, we can rewrite equation 43 as follows:

$$\begin{aligned}
B_{\nu'} &= \frac{2h}{c^2}\left(\frac{\nu}{1+z}\right)^3 \frac{d\nu}{1+z}\frac{1}{exp\left(\frac{h}{k_B}\frac{1+z}{T}\frac{\nu}{1+z}\right)} \\
&= \left(\frac{1}{1+z}\right)^4 \times \frac{2h\nu^3}{c^2}\frac{d\nu}{exp(\frac{h\nu}{k_B T})} \\
&= \left(\frac{1}{1+z}\right)^4 \times B_\nu. \qquad (44)
\end{aligned}$$

Note that this is simply equation 36 multiplied by a constant pre-factor. Therefore the blackbody emission spectrum retains its shape under redshift.

5 Question 5

Recall the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \qquad (45)$$

Using the chain rule, show that for a matter-only Universe with k ¿ 0 the following are parametric solutions to the Friedmann equation, if θ is a variable that runs from 0 to 2 π:

$$a(\theta) = \frac{4\pi G\rho_0}{3kc^2}(1 - cos\theta) \qquad (46)$$

$$t(\theta) = \frac{4\pi G\rho_0}{3(kc^2)^{3/2}}(\theta - sin\theta) \qquad (47)$$

First I will use the chain rule to demonstrate that

$$\dot{a} = \frac{da}{dt} = \frac{da}{d\theta}\frac{d\theta}{dt} = \frac{da}{d\theta}\left(\frac{dt}{d\theta}\right)^{-1}. \qquad (48)$$

We can differentiate equations 46 and 47 with respect to θ to obtain the following:

$$\begin{aligned}
\frac{da}{d\theta} &= \frac{4\pi G\rho_0}{3kc^2} sin\theta \\
\frac{dt}{d\theta} &= \frac{4\pi G\rho_0}{3(kc^2)^{3/2}}(1 - cos\theta)
\end{aligned} \qquad (49)$$

we can substitute these expressions into equation 48 to obtain expressions for $\dot{a}$ and $\dot{a}^2$;

$$
\begin{aligned}
\dot{a} &= \left(\frac{4\pi G\rho_0}{3kc^2} sin\theta\right) \times \left(\frac{4\pi G\rho_0}{3(kc^2)^{3/2}}(1-cos\theta)\right)^{-1} \\
&= (kc^2)^{1/2} \times \frac{sin\theta}{1-cos\theta} \\
\therefore \dot{a}^2 &= kc^2 \times \frac{sin^2\theta}{(1-cos\theta)^2} = kc^2 \times \frac{(1-cos\theta)(1+cos\theta)}{1-cos\theta} \\
\therefore \dot{a} &= kc^2 \frac{1+cos\theta}{1-cos\theta} \qquad (50)
\end{aligned}
$$

We are told in the question that solutions 46 and 47 should be tested for the case of a matter only Universe (i.e. $\rho = \rho_M$, $\Theta = 0$ and the medium's pressure, $p = 0$). We are also told that the Universe has positive curvature ($k > 1$).

For the case of a matter only Universe, we have shown previously (see equation 16) that $\rho = \rho_0/a^3$. The current value of the density parameter is given by

$$
\Omega_0 = \frac{8\pi G}{3} \frac{\rho_0}{H_0^2}; \qquad (51)
$$

$$
\therefore \rho_0 = \frac{3}{8\pi G} \times H_0^2 \times \Omega_0 \qquad (52)
$$

We can now use equations 16 and 52 to obtain an expression for ρ;

$$
\rho = \frac{1}{a^3}\rho_0 = \frac{1}{a^3} \times \frac{3}{8\pi G} \times H_0^2 \times \Omega_0. \qquad (53)
$$

By substitution from equation 53 into equation 45, the Friedmann equation for our matter-only Universe can be rearranged;

$$
\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3} \frac{1}{a^3} \times \frac{3}{8\pi G} \times H_0^2 \times \Omega_0 - \frac{kc^2}{a^2} \\
&= \frac{1}{a^3} \times H_0^2 \times \Omega_0 - \frac{kc^2}{a^2} \\
\dot{a}^2 &= \frac{1}{a} \times H_0^2 \times \Omega_0 - kc^2 \qquad (54)
\end{aligned}
$$

Into the form of the Friedmann expression obtained in equation 54, I will substitute the expression for Ω_0 from equation 51, and I will substitute an expression for a from the solutions we are trying to evaluate (equation 46).

The Friedmann equation becomes

$$\dot{a}^2 = \left(\frac{4\pi G\rho_0}{3kc^2}(1 - cos\theta)\right)^{-1} \times \frac{8\pi G}{3}\frac{\rho_0}{H_0^2} \times H_0^2 - kc^2$$
$$= kc^2 \left. \frac{1}{1 - cos\theta} \times 2 - 1 \right)$$
$$\dot{a}^2 = kc^2 \left(\frac{1 + cos\theta}{1 - cos\theta}\right) \tag{55}$$

Equations 50 and 55 are equivalent. Therefore the solutions provided in equations 46 and 47 satisfy the Friedmann equation for the scenario of a matter-only Universe with positive curvature.

Graph the variation of a vs. θ and t vs. θ and describe the behaviour of this Universe. Graph a vs. t.

In order to obtain the plots required, I made the following estimations:

1. I used the number density of stars calculated in equation 28 in order to obtain a value for ρ_0. By multiplication of this value by the mass of our sun (1.99×10^{30} kg), I acquired a value of $\rho_0 \approx 2 \times 10^{-9}$ kg m^{-3}.

2. I used a curvature of $k = 1$, corresponding to positive curvature and a closed Universe.

A plot of a vs. θ is provided in figure 1.

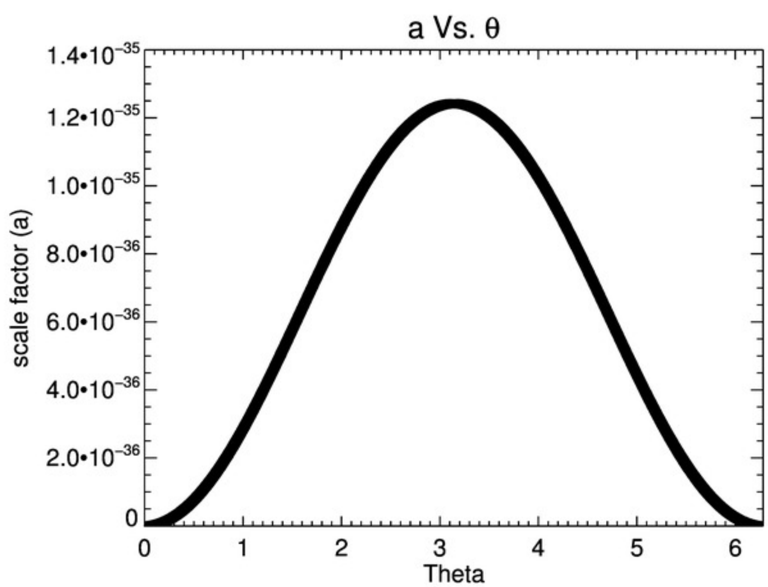

Figure 1: Evolution of the scale factor (a) as a function of the periodic variable θ.

A plot of t vs. θ is provided in figure 2.

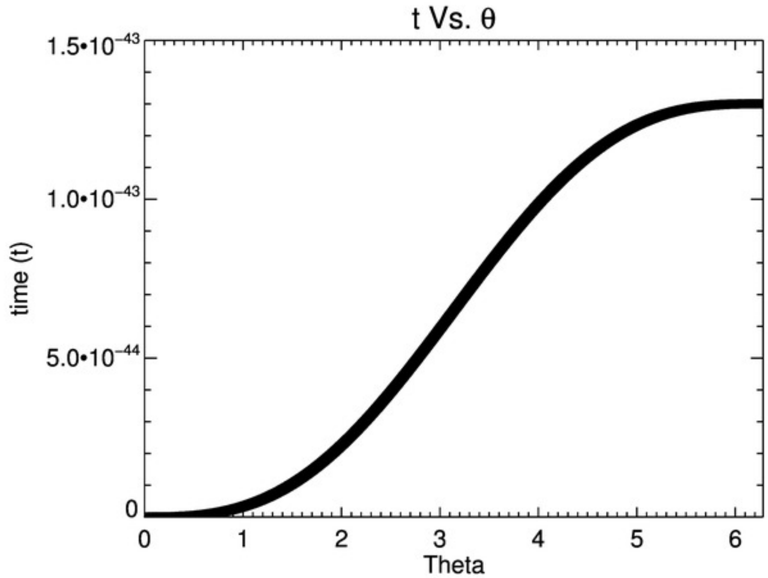

Figure 2: Evolution of time (t) as a function of the periodic variable θ.

This Universe is an unstable one. Friedmann realised that the formal solution to this equation in this scenario corresponds to an oscillatory Universe, sinusoidally expanding and collapsing with time. The plot of a vs. t (figure 3) provides some insight into this oscillation. The scale factor falls back to zero in a time-frame comparable with Planck time. Qualitatively, we can choose to view this Universe as a manifestation of a "quantum foam" of matter cycling in and out of existence.

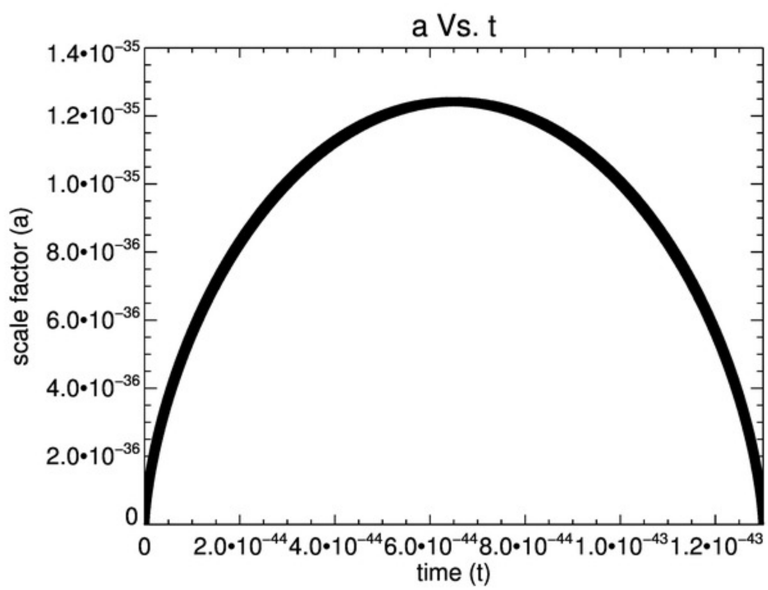

Figure 3: Evolution of the scale factor (a) as a function of time (t).

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