The Pendulum

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Abstract

The trajectory of a linear pendulum is found using the trapezoid algorithm for several initial conditions. The trajectory is then found for the non-linear case so that a comparison can be made. The Runge-Kutta method is then used on the non-linear pendulum and the two algorithms are compared with the Runge-Kutta method showing better accuracy. The damped pendulum is then examined using the Runge-Kutta method. Finally, the trajectory of the damped driven pendulum is found with different driving amplitudes and the phase portrait is plotted. It was found that with increasing amplitude, the trajectory became more complex and eventually reached the chaotic regime.

⋆ Introduction

The equation of motion for the simple pendulum with length l under gravity is

$$\ddot{\theta} + \frac{g}{l} sin\theta = 0$$

This does not have an analytic solution. If we restrict the motion to small angles, ie $\theta << 1$, then $\sin\theta \approx \theta$. This gives,

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This is a linear second order differential equation with solution $\theta = A\cos(\beta t + \phi_0)$ where $\beta^2 = \frac{g}{l}$. However, this is merely a simplification and in reality, we need to be able to deal with non-linear motion. The simple pendulum is generalised by the non-linear damped driven pendulum which has equation of motion,

$$\ddot{\theta} + k\dot{\theta} + \beta^2 \sin\theta = A\cos(\Omega t)$$

This can be solved numerically.

Simple Euler Method

Firstly, the second order differential equation is converted to a pair of first order differential equations by the substitution $\frac{d\theta}{dt} = \omega$. This gives,

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\beta^2 \sin\theta - k\omega + A\cos(\Omega t) = f(\theta, \omega, t)$$

Taking the Taylor expansion about intial time t,

$$\theta(t + \Delta t) = \theta(t) + \dot{\theta}(t)\Delta t + O(\Delta t^2)$$

$$\omega(t + \Delta t) = \theta(t) + \dot{\omega}(t)\Delta t + O(\Delta t^2)$$

For small time increments, we can neglect the higher order terms. Writing this as an iteration,

$$\theta_{n+1} = \theta_n + \omega_n \Delta t$$

$$\omega_{n+1} = \omega_n - \theta_n \Delta t$$

Hence, if we know the initial conditions at time t, we can work out the trajectory. The problem with this is that there is a high amount of inaccuracy.

Trapezoid Rule

The trapezoid rule refines the simple Euler method and improves the accuracy. The trapezoid rule approximates the area under a curve as a trapezoid. This can be expressed as,

$$\theta_{n+1} - \theta_n = \int_t^{t+\Delta t} \frac{d\theta}{dt} dt \approx \frac{\Delta t}{2}\left(\frac{d\theta(t)}{dt} + \frac{d\theta(t+\Delta t)}{dt}\right)$$

$$\Rightarrow \theta_{n+1} = \theta_n + \frac{\Delta t}{2}(\omega_n + \omega_{n+1})$$

Taylor expanding the second term and neglecting second order terms gives,

$$\theta_{n+1} = \theta_n + \frac{\Delta t}{2}(\omega_n + \Delta t(\omega_n + f(\theta_n, \omega_n, t)))$$

This can be written as,

$$\theta_{n+1} = \theta_n + \frac{k_{1a} + k_{2a}}{2}$$

where $k_{1a} = \omega_n \Delta t$ and $k_{2a} = (\omega_n + f(\theta_n, \omega_n, t)\Delta t)\Delta t$. Similarly,

$$\omega_{n+1} = \omega_n + \frac{k_{1b} + k_{2a}}{2}$$

where $k_{1b} = f(\theta_n, \omega_n, t)\Delta t$ and $k_{2b} = f(\theta_{n+1}, \omega_n + k_{1b}, t_{n+1})\Delta t$.

Runge Kutta Method

The trapezoid rule is mostly sufficient when solving the non-linear pendulum. To solve the damped driven pendulum, it is required to use the Runge-Kutta method to improve the accuracy. The Runge-Kutta Method is a similar iterative method with,

$$\theta(t + h) = \theta(t) + \frac{1}{6}(k_{1a} + 2k_{2a} + 2k_{3a} + k_{4a})$$

$$\omega(t+h) = \omega(t) + \frac{1}{6}(k_{1b} + 2k_{2b} + 2k_{3b} + k_{4b})$$

where

$$k_{1a} = h\omega \qquad k_{1b} = hf(\theta, \omega, t)$$

$$k_{2a} = h(\omega + \frac{k_{1b}}{2}) \qquad k_{2b} = hf(\theta + \frac{k_{1a}}{2}, \omega + \frac{k_{1b}}{2}, t + \frac{h}{2})$$

$$k_{3a} = h(\omega + \frac{k_{2b}}{2}) \qquad k_{3b} = hf(\theta + \frac{k_{2a}}{2}, \omega + \frac{k_{2b}}{2}, t + \frac{h}{2})$$

$$k_{4a} = h(\omega + k_{3b}) \qquad k_{4b} = hf(\theta + k_{3a}, \omega + k_{3b}, t + h)$$

⋆ Experimental Method

Linear Pendulum

The trapezoid rule is used to solve the linear pendulum. β^2 is set to 1. Δt is set as 0.1 and 100 steps are taken. Initial values are taken to be $\theta(0) = 0$, $\omega(0) = 1$. Other initial conditions taken are $\theta(0) = 0.2, 1.0, 3.124$ with $\omega(0) = 0$ for all cases.

Non-Linear Pendulum

The same procedure is done for the non-linear case with the same initial conditions. The results are then compared.

The Runge Kutta method is then implemented with the same initial conditions and the results of the trapezoid rule and Runge Kutta are compared.

Damped Pendulum

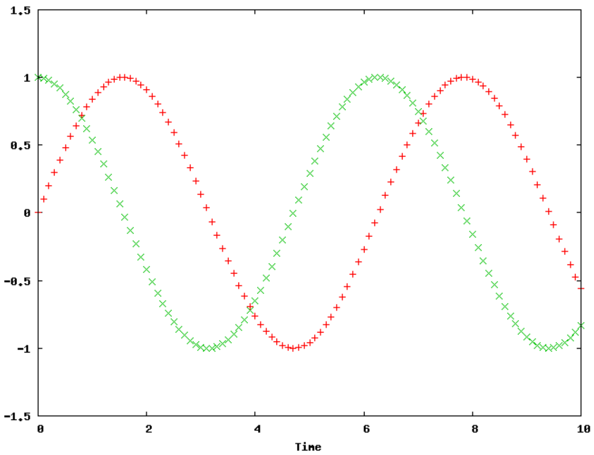
k is set as 0.5. The number of steps is now 500. The same initial conditions are used again for the nonlinear and linear case such that a comparison can be made.

Damped Driven Pendulum

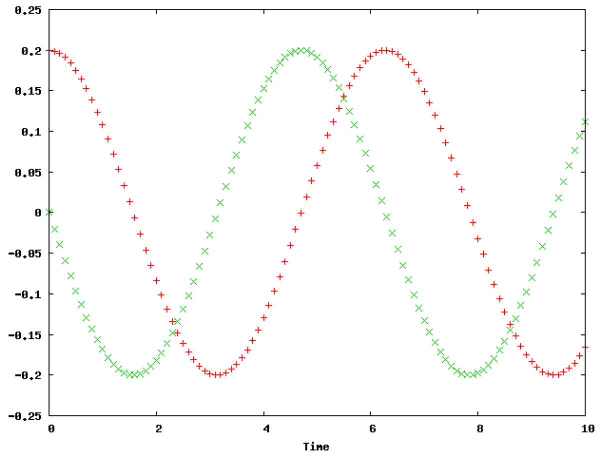
k is set as 0.5 and Ω is set as 0.6667. The number of steps is increased to 10000. θ is restricted to $[-\pi, \pi]$ since the pendulum can overshoot. The program is then run for values $A = 0.9, 1.07, 1.35, 1.47, 1.5$ with the intial conditions $\theta(0) = -2$ and $\omega(0) = 0$. An if statement is implemented to discard the first 500 points so that the trajectory settles down and the phase portrait of the trajectory is then plotted.

⋆ Results and Analysis

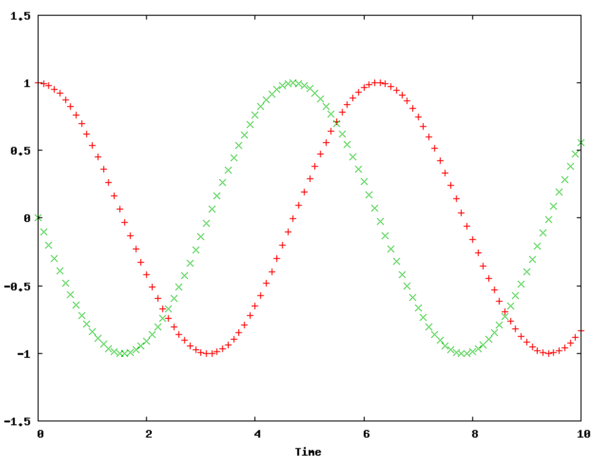
Linear Pendulum



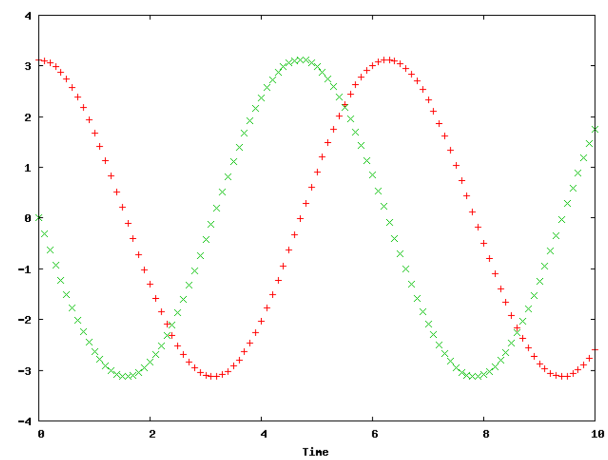
(a) $\theta = 0$ $\omega = 1$



(b) $\theta = 0.2$ $\omega = 0$



(c) $\theta = 1$ $\omega = 0$

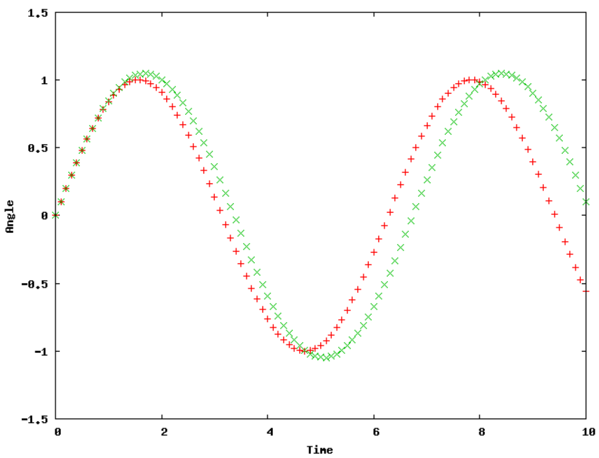


(d) $\theta = 3.124$ $\omega = 0$

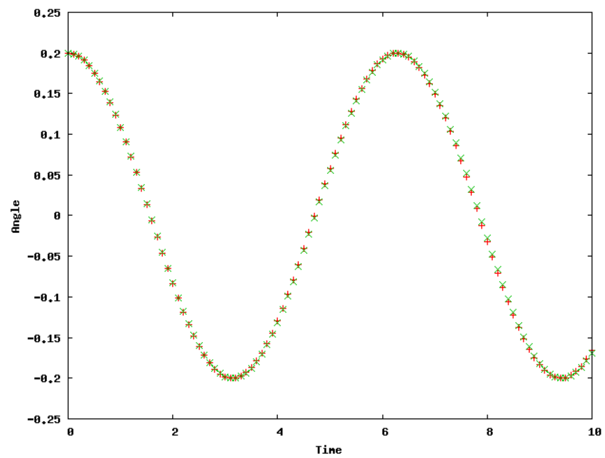
Figure 1: Linear Pendulum with position in red and velocity in green

Figure 1 shows the motion for the linear pendulum. As expected, it undergoes simple harmonic motion. The velocity reaches a maximum when the position is at 0 and vice versa.

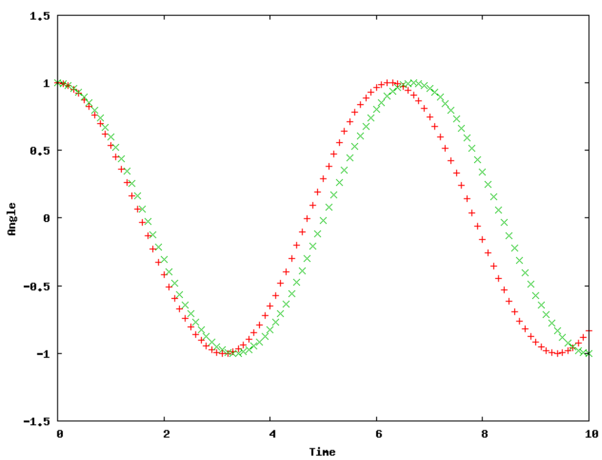
Non-Linear Pendulum



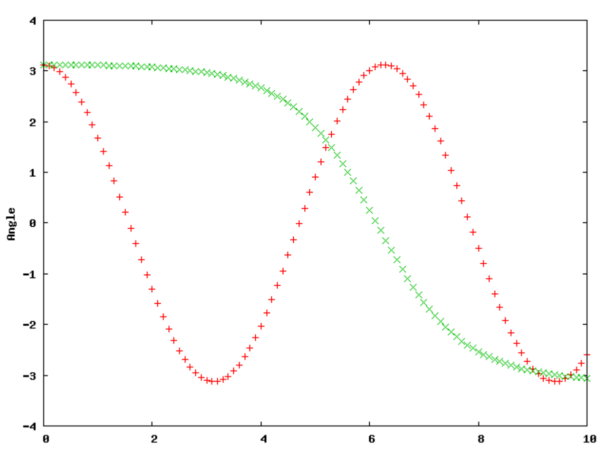
(a) $\theta = 0$ $\omega = 1$



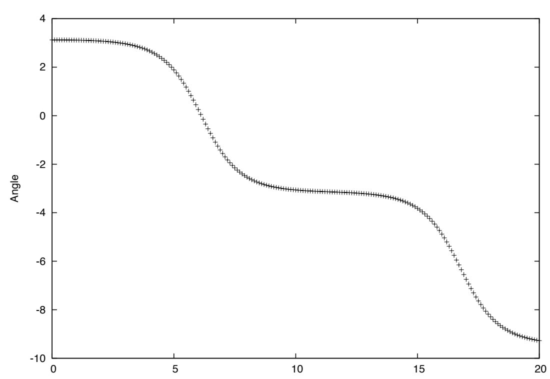
(b) $\theta = 0.2$ $\omega = 0$



(c) $\theta = 1$ $\omega = 0$



(d) $\theta = 3.124$ $\omega = 0$

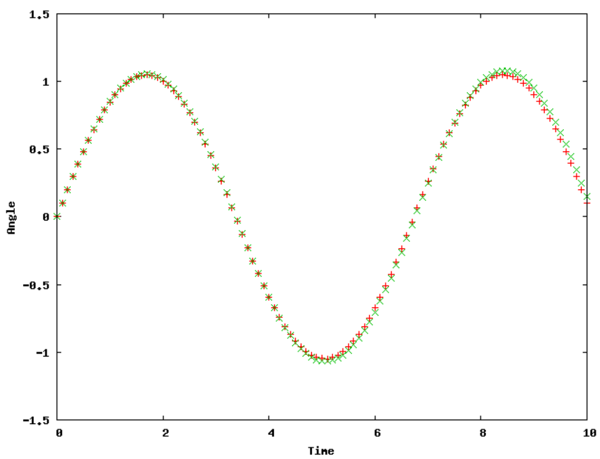


(e) $\theta = 3.124$ $\omega = 0$ with 2000 steps

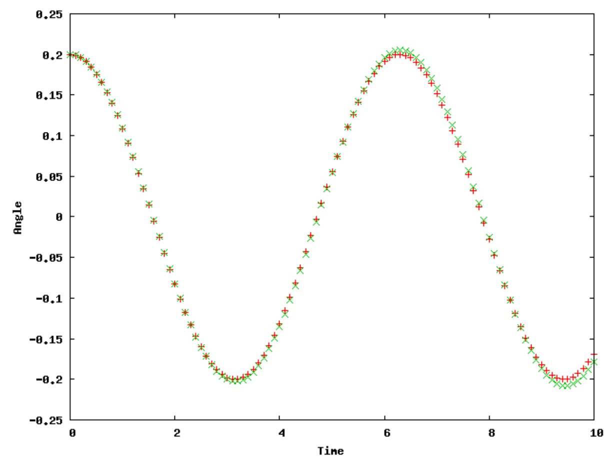
Figure 2: Comparison between Linear(red) and Non-Linear(green) Pendulum

Clearly, from (b), the linear pendulum is a good approximation for small angles. However, it is clear that when the angle is big, the linear pendulum no longer describes what happens in reality accurately. This is most evident in (d) where the pendulum is at the very top and the angle is significantly large. The two graphs look completely different. Another thing to note is in (e) where the pendulum is actually rotating instead of oscillating. This shows the errors emerging from the algorithm since by the conservation of energy, the pendulum should not have enough energy to overshoot and rotate. It is clear from this that the algorithm is not sufficient to predict the motion when the angle is large for this time increment.

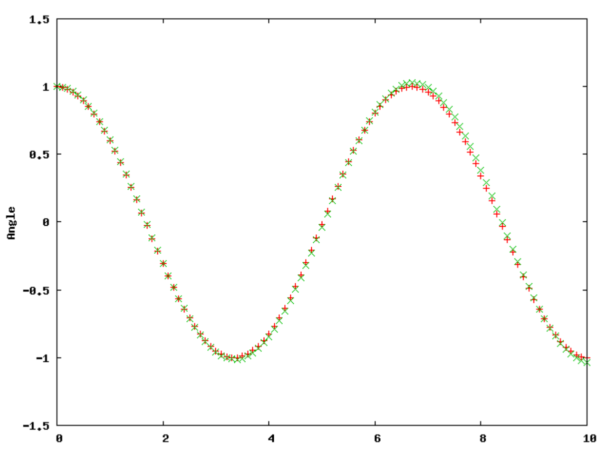
Non-Linear Pendulum: Trapezoid Method vs Runge-Kutta



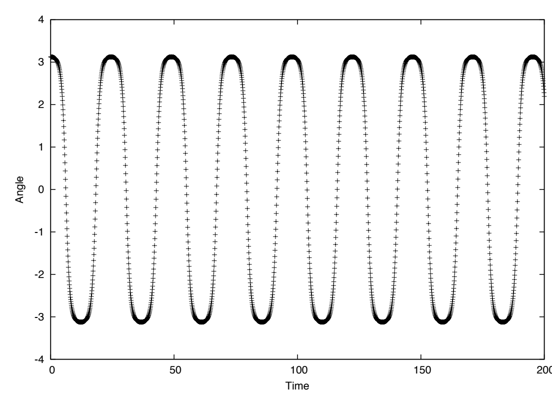
(a) $\theta = 0$ $\omega = 1$



(b) $\theta = 0.2$ $\omega = 0$



(c) $\theta = 1$ $\omega = 0$

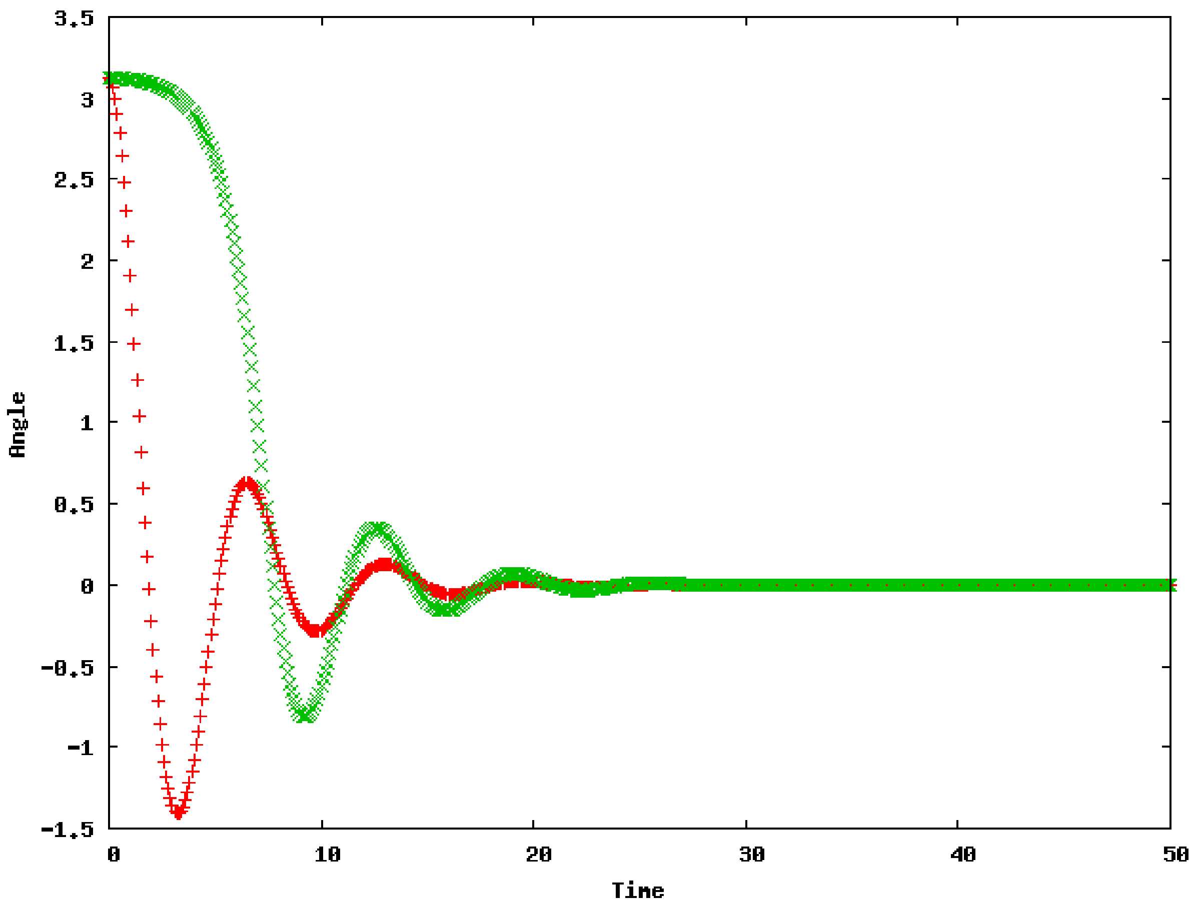


(d) $\theta = 3.124$ $\omega = 0$

Figure 3: Comparison between Trapezoid Method(red) and Runge-Kutta(green)

Figure 3 shows the difference between the trapezoid method and the Runge-Kutta method. Although it is slightly different, it is apparent that the trapezoid method is a sufficient method in determining the motion of the non-linear pendulum. Figure 2 showed the errors in the trapezoid method manifesting but the Runge-Kutta method is able to correctly predict that the pendulum will oscillate in Figure 3(d). This shows that the Runge-Kutta method is more accurate when using the same time increment but the trapezoid method is still mostly sufficient.

Damped Pendulum

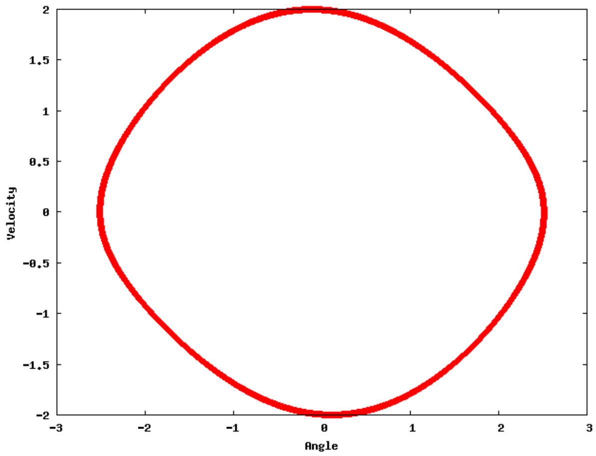


(a) $\theta = 3.124$ $\omega = 0$

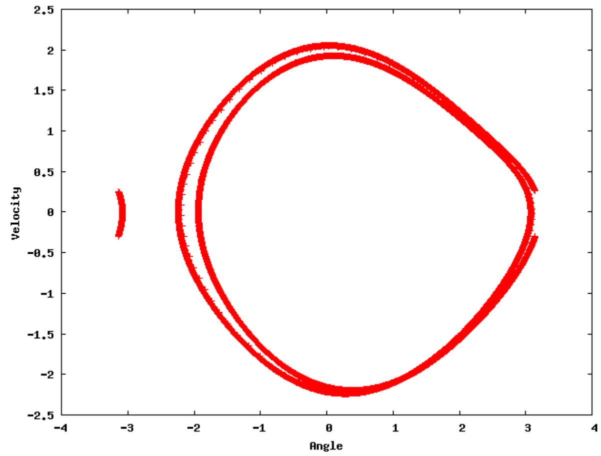
Figure 4: Damped Linear(red) and Non-Linear(green) Pendulum

Figure 4 shows the damped linear and damped non-linear pendulum. A comparison shows that the linear case becomes very good after a sufficiently long time. This is because the amplitude of oscillation decreases and eventually, the small angle limit is reached so the linear and non-linear cases coincide. The amplitude will reach 0 as $t \to \infty$.

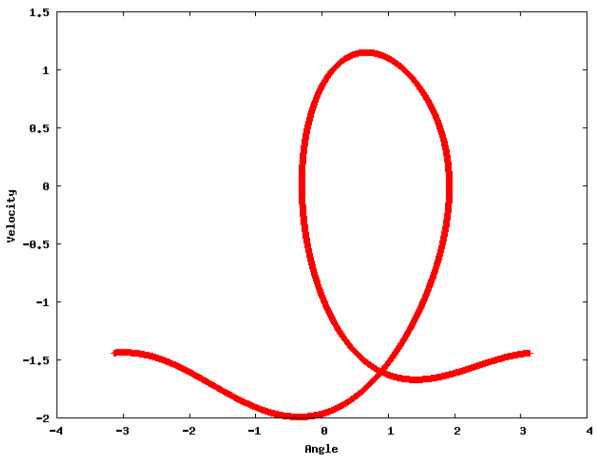
Damped Driven Pendulum



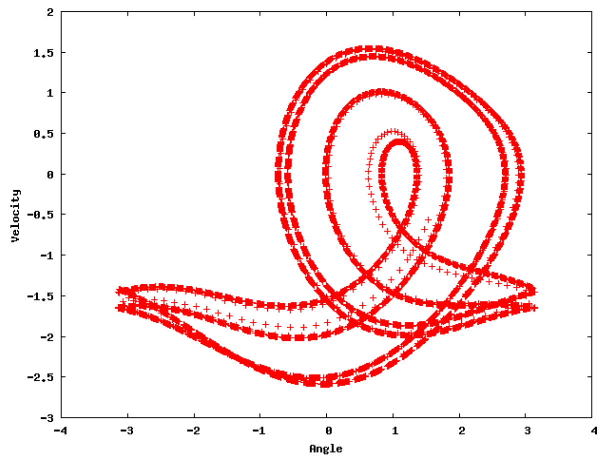
(a) $A = 0.9$



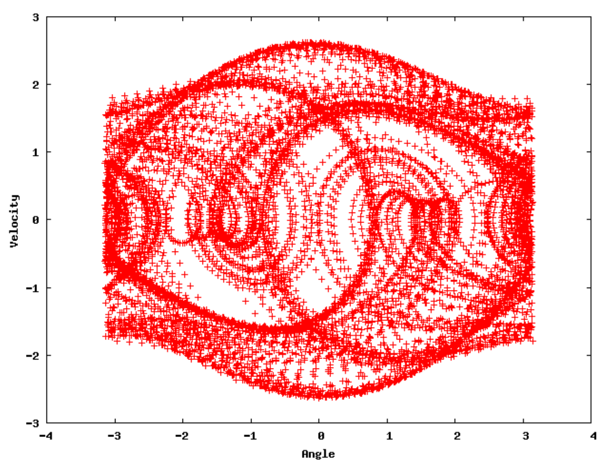
(b) $A = 1.07$



(c) $A = 1.35$



(d) $A = 1.47$



(e) $A = 1.5$

Figure 5: Phase Portraits of Damped Driven Pendulum

Figure 5 shows the phase portraits of the damped driven pendulum with different driving amplitudes. (a) shows the pendulum has settled into a periodic oscillation. This is a good example of a limit cycle. (b) shows period doubling. The motion is still periodic but it will switch periods as it oscillates. As A increases, the motion becomes more complex. Although the motion is more complicated, (c) and (d) seem to be following fixed paths. (e) quite clearly shows the chaotic regime. The motion is extremely complex and there is no periodic behaviour.

⋆ Conclusion

For small angle displacement, the linear pendulum is a good approximation of the pendulum. However, it quickly breaks down when the angle is larger than 1. The trapezoid method is a powerful algorithm in determining the trajectory of the linear pendulum. Unfortunately, it is not sufficient for larger angles and the Runge-Kutta method is more accurate. The damped pendulum showed once again that the linear model is good for small angles.

The phase portraits of the damped driven pendulum were good examples of chaos emerging from a somewhat simple system. With increasing driving amplitude, the motion became more complex and eventually became chaotic.