Geometry Questions

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The same questions seem to come up all the time in geometry. Here are solutions to some of the most common ones. I have listed what has come up in the past years up to 2006. At the moment, this is for the schols paper but I might update this to the whole exam when schols is over. If you spot any mistakes, please tell me in person or by e-mail.

1 Chain Rule(2010,2008,2007)

State and prove the chain rule for functions on finite-dimensional real vector spaces.

Let U, V, W be open subsets of finite-dimensional real normed spaces M, N, L respectively. Let the function $g : U \to N$ be differentiable at $a \in U$ and let $f : V \to L$ be differentiable at $g(a)$. Then $f \circ g$ is differentiable at a and is equal to $f'(g(a))g'(a)$.

We want to find $f \circ g(a + h)$. By assumption,

$$g(a + h) = g(a) + g'(a)h + \phi(h)$$

where $\frac{\|\phi(h)\|}{\|h\|} \to 0$ as $\|h\| \to 0$ since g is differentiable and

$$f(g(a) + y) = f(g(a)) + f'(g(a))y + \psi(y)$$

where $\frac{\|\psi(y)\|}{\|y\|} \to 0$ as $\|y\| \to 0$ since f is differentiable. Let $y = g'(a)h + \phi(h)$.

$$f(g(a) + g'(a)h + \phi(h)) = f(g(a) + f'(g(a)[g'(a)h + \phi(h)] + \|y\|\theta(y)$$

$$\text{with } \theta(y) = \begin{cases} \frac{\psi(y)}{\|y\|} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

This needs to be done in case $y = 0$. Then

$$f(g(a + h)) = f(g(a)) + \underbrace{f'(g(a))g'(a)h}_{\text{linear operator}} + \underbrace{f'(g(a))\phi(h) + \|y\|\theta(y)}_{\text{remainder term}}$$

Now we need to show that the remainder term divided by $\|h\|$ tends to 0 when $\|h\|$ tends to 0.

$$\frac{\|f'(g(a))\phi(h) + \|y\|\theta(y)\|}{\|h\|} \leq \frac{\|f'g(a)\|\|\phi(h)\|}{\|h\|} + \frac{\|y\|\|\theta(y)\|}{\|h\|}$$

But we have:

$$y = g'(a)h + \phi(h)$$
$$\|y\| \leq \|g'(a)\|\|h\| + \|\phi(h)\|$$
$$\frac{\|y\|}{\|h\|} \leq \|g'(a)\| + \frac{\|\phi(h)\|}{\|h\|}$$

so $\|y\|$ tends to 0 as $\|h\|$ tends to 0 and $\frac{\|y\|}{\|h\|}$ is bounded. Therefore $\frac{\|f'g(a)\|\|\phi(h)\|}{\|h\|} + \frac{\|y\|\|\theta(y)\|}{\|h\|}$ tends to 0 when $\|h\|$ tends to 0. Thus the remainder term tends to 0 so $f \circ g$ is differentiable and is equal to $f'(g(a))g'(a)$.

□

2 Chain Rule Applications(2010,2009)

These questions go something like this: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x,y) = \frac{xy^2}{x^2 + y^3}$$

if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (0,0) and $\frac{d}{dt}f(ta,tb)$ at $t = 0$. Show that f is not differentiable at $(0,0)$

The first thing to do is assume f is differentiable at $(0,0)$. f is 0 at all points along the x and y axes. By evaluating the partial derivatives along the x and y axis, $\frac{\partial f}{\partial x}(x,0)$ and $\frac{\partial f}{\partial y}(0,y)$ are zero. More importantly, setting $x, y = 0$ gives $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$ From the chain rule, we also know that

$$\left.\frac{d}{dt}f(ta,tb)\right|_{t=0} = \frac{\partial f}{\partial x}(0,0)a + \frac{\partial f}{\partial y}(0,0)b$$

But this is 0 from the previous result. If we assume f is differentiable at $(0,0)$, then this limit exists:

$$\begin{aligned}\left.\frac{d}{dt}f(ta,tb)\right|_{t=0} &= \lim_{t\to 0}\frac{f(ta,tb) - f(0,0)}{t} \\ &= \lim_{t\to 0}\frac{t^3ab^2}{t(t^2a^2 + t^3b^3)} \\ &= \lim_{t\to 0}\frac{t^3ab^2}{t^3(a^2 + tb^3)} \\ &= \frac{b^2}{a}\end{aligned}$$

which is a contradiction.

3 Eigenfunctions(2008,2006)

A C^1 function is homogenous of degree r if:

$$F(tx_1, \ldots, tx_n) = t^r F(x_1, \ldots, x_n)$$

Show that F is an eigenfunction of the operator:

$$x^1\frac{\partial}{\partial x^1} + \ldots x^n\frac{\partial}{\partial x^n}$$

LHS:

$$\begin{aligned}\frac{d}{dt}F(tx_1, \ldots, tx_n) &= \frac{\partial F}{\partial x^1}\frac{dtx_1}{dt} + \cdots + \frac{\partial F}{\partial x^n}\frac{dtx_n}{dt} \\ &= \frac{\partial F}{\partial x^1}x_1 + \cdots + \frac{\partial F}{\partial x^n}x_n \qquad \text{be aware of where } F \text{ is evaluated at}\end{aligned}$$

RHS:

$$\frac{d}{dt}t^r F(x_1,\dots,x_n) = rt^{r-1}F(x_1,\dots,x_n)$$
$$\Rightarrow \frac{\partial F}{\partial x^1}x_1 + \cdots + \frac{\partial F}{\partial x^n}x_n = rt^{r-1}F(x_1,\dots,x_n)$$

This is true for all t. Set t = 1.

$$(\frac{\partial}{\partial x^1}x_1 + \cdots + \frac{\partial}{\partial x^n}x_n)F = rF$$

Note again where F is evaluated at. This completes the question and the eigenvalue is r.

4 Matrix Derivative(2009,2007)

Let V be the space of non-singular matrices. Let $f : \mathbb{R}^{n\times n} \supset V \to \mathbb{R}^{n\times n}$ where $f(A) = A^{-1}$. Show it is differentiable and find the derivative.

Suppose:

$$f(A+H) = f(A) + f'(A)H + \phi(H)$$

Then to find $\phi(H)$:

$$\begin{aligned}(A+H)^{-1} - A^{-1} &= (A+H)^{-1}(I - (A+H)A^{-1})\\ &= (A+H)^{-1}(-HA^{-1})\\ &= A^{-1}(-HA^{-1}) + \phi(H)\end{aligned}$$

This is just a guess when the H is dropped. To see if you're right, go through the whole process. The remainder term is given as:

$$\begin{aligned}\phi(H) &= (A+H)^{-1} - A^{-1} + A^{-1}HA^{-1}\\ &= (A+H)^{-1}(I - (A+H)A^{-1} + (A+H)(A^{-1}HA^{-1}))\\ &= (A+H)^{-1}(I - I - HA^{-1} + HA^{-1} + HA^{-1}HA^{-1})\\ &= (A+H)^{-1}(HA^{-1}HA^{-1})\end{aligned}$$

Now to show that $\frac{\|\phi(H)\|}{\|H\|} \to 0$ when $\|H\| \to 0$.

$$\begin{aligned}\frac{\|\phi(H)\|}{\|H\|} &= \frac{\|(A+H)^{-1}(HA^{-1}HA^{-1})\|}{\|H\|}\\ &\leq \frac{\|(A+H)^{-1}\|\|H\|\|A^{-1}\|\|H\|\|A^{-1}\|}{\|H\|}\\ &= \|(A+H)^{-1}\|\|A^{-1}\|\|H\|\end{aligned}$$

This goes to zero as required. Therefore, f is differentiable and since $f(A+H) = f(A) - A^{-1}HA^{-1} + \phi(H)$, $f'(A)H = -A^{-1}HA^{-1}$. Aha indeed.

5 Partial Derivatives(2009)

Let f be a real valued function on an open set in $\mathbb{R}^2$. f is $C^1 \Leftrightarrow \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous.

" $\Rightarrow$ "

f is differentiable so,

$$f(x+h) = f(x) + f'(x)h + \phi(h)$$

We know that $f'(x)$ can be represented as a 1x2 matrix(Jacobian). In fact, the elements of the matrix are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. This completes the proof. This is the proof shown in the lectures. To be more rigorous, I would assume f is differentiable. Then you know that limit thing exists. Evaluate along the x and y axis and by definition, these are the partial derivatives.

"$\Leftarrow$"

Let the partials exist and be continuous. Now suppose that $h = (h_1, h_2)$ such that,

$$f(a+h) = f(a) + \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(a)h_2 + \phi(h)$$

$$\phi(h) = f(a+h) - f(a+h_2e_2) - \frac{\partial f}{\partial x}(a)h_1 + f(a+h_2e_2) - f(a) - \frac{\partial f}{\partial y}(a)h_2$$

By the mean value theorem, for some m, n,(draw a graph for this, it makes it easier to understand. It goes along direction of the x or y axis),

$$= \frac{\partial f}{\partial x}(m)h_1 - \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(n)h_2 - \frac{\partial f}{\partial y}(a)h_2$$

Now to show that this remainder term goes to 0.

$$\frac{\|\frac{\partial f}{\partial x}(m)h_1 - \frac{\partial f}{\partial x}(a)h_1 + \frac{\partial f}{\partial y}(n)h_2 - \frac{\partial f}{\partial y}(a)h_2\|}{\|h\|} \leq \left(\|\frac{\partial f}{\partial x}(m) - \frac{\partial f}{\partial x}(a)\|\right)\frac{\|h_1\|}{\|h\|} + \left(\|\frac{\partial f}{\partial y}(n) - \frac{\partial f}{\partial y}(a)\|\right)\frac{\|h_2\|}{\|h\|}$$

which tends to 0 as $\|h\| \to 0$ since $\|h_i\| \leq \|h\|$. So f is differentiable and is given by the partials which are continuous.

$\square$

6 Symmetry of second derivatives(2009,2006)

Let f be a C^2 function on an open set V in $\mathbb{R}^2$. Prove that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Take the points $(a, b), (a+h, b), (a, b+k), (a+h, b+k) \in V$ such that they form a rectangle in the open set V with $h, k > 0$. Define T as:

$$T = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$

Define $g(x) = f(x, b+k) - f(x, b)$. Then

$$T = g(a+h) - g(a)$$

By the mean value theorem,

$$T = g'(c)h$$

for $c \in (a, a+h)$. Using $g(x)$ to evaluate $g'(x)$:

$$\begin{aligned} T &= \left[\frac{\partial f}{\partial x}(c, b+k) - \frac{\partial f}{\partial x}(c, b)\right]h \\ &= \left[\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(c, d)\right]hk \end{aligned}$$

for some $d \in (b, b+k)$ by the MVT. This time define $u(x) = f(a+h, x) - f(a, x)$. With a similar argument,

$$T = \left[\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(c', d')\right]hk$$

for some $c' \in (a, a+h), d' \in (b, b+k)$. Equating,

$$\Rightarrow \left[\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(c, d)\right]hk = \left[\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(c', d')\right]hk$$
$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(c, d) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(c', d')$$

Now let $h, k \to 0$. Then $(c, d) = (c', d') = (a, b)$.

$\square$

7 Inverse Function Theorem(2010,2009,2008)

State and prove the inverse function theorem. This isn't fun to prove. Sometimes, he only asks you to prove about half of the theorem but that doesn't make it any more exciting. This year, I'm predicting that he's asking the full question.

Let M, N be finite dimensional vector spaces with V open in M and let $f : M \supset V \to N$ be a C^r function. If $f' : M \to N$ is invertible at $a \in V$, then there exists an open neighbourhood W of a in M such that f is a C^r-diffeomorphism mapping W onto $f(W)$ which is open in N.

Let T be the inverse of $f'(a)$ and let $F(x) = Tf(x+a) - Tf(a)$. Then by showing that if there exists an open neighbourhood U of 0 for which F maps onto $F(U)$ by a C^r-diffeomorphism, then$f$ maps $U + a$ onto open $f(U + a)$ by a C^r-diffeomorphism. Drawing a diagram will make this clear. The proof is split into three parts.

7.1 Homeomorphism

Firstly,

$$F(0) = Tf(0 + a) - Tf(a) = 0$$

$$F'(x) = Tf'(x + a)$$
$$F'(0) = 1_M$$

1_M is the identity. By continuity of F', $\forall \epsilon > 0, \exists \delta > 0$ st. $\|x\| < \delta \Rightarrow \|F'(0) - F'(x)\| < \epsilon$. Then a closed ball, B with radius $r > 0$, $B = \{x \in M : \|x\| \leq r\}$ can be chosen such that $\forall x \in B$,

$$\|1_M - F'(x)\| \leq \frac{1}{2}$$
$$\det F'(x) \neq 0$$

Now take $x, y \in B$ such that by the reverse triangle inequality,

$$\begin{aligned}\|x - y\| - \|F(x) - F(y)\| &\leq \|(1_M - F)x - (1_M - F)y\| \\ &\leq \frac{1}{2}\|x - y\|\end{aligned}$$

by the mean value theorem. Therefore

$$\|F(x) - F(y)\| \geq \frac{1}{2}\|x - y\|$$

When $F(x) = F(y), 0 \geq \frac{1}{2}\|x - y\| \Leftrightarrow x = y$. Thus, F is injective. Also, $F(x) \to F(y) \Rightarrow x \to y$(easily shown using $\epsilon\delta$). F and F^{-1} are continuous so $F : B \to F(B)$ is a homeomorphism.

7.2 $\frac{1}{2}B \subset F(B)$

$\frac{1}{2}B$ is defined as the closed ball with centre 0 and radius r/2 > 0. Let $g(x) = x - F(x) + a$ with $a \in \frac{1}{2}B$ for $x \in B$.

$$g'(x) = 1 - F'(x)$$

$$\|g'(x)\| = \|1 - F'(x)\| \leq \frac{1}{2}$$

from above for all $x \in B$. By the mean value theorem,

$$\|g(x) - g(y)\| \leq \frac{1}{2}\|x - y\|$$

for all $x, y \in B$. Also, since $g(0) = a$,

$$\begin{aligned}\|g(x)\| = \|g(x) - g(0) + a\| &\leq \|g(x) - g(0)\| + \|a\| \\ &\leq \frac{1}{2}\|x - 0\| + \|a - 0\| \\ &\leq \frac{1}{2}r + \frac{1}{2}r \\ &= r\end{aligned}$$

This means that $g(x) \in B, \forall x \in B$ ie. $g : B \to B$. And from $\|g(x) - g(y)\| \leq \frac{1}{2}\|x - y\|$, $g(x)$ is a contraction mapping. There exists a fixed point. Take $x_1 = g(x_0)$, $x_2 = g(x_1), \dots, x_k = g(x_{k-1})$ etc. Then taking k going to infinity, $\exists z \in B$ s.t. $g(z) = z$. So,

$$g(z) = z - F(z) + a$$

$$\Rightarrow F(z) = a$$

So $a \in F(B)$ and we have $\frac{1}{2}B \subset F(B)$.

7.3 Diffeomorphism

Let B_0 be the interior of B. Then let $U = B_0 \cap F^{-1}(\frac{1}{2}B_0)$. U is open as it is the union of two open sets. We know that F is a homeomorphism and C^r. All that is left is to show that F^{-1} is C^r. Denote F^{-1} as G. Take points x and $x + l$ in $F(U)$ such that $G(x + h) = y + l$ and $G(x) = y$. F is differentiable so we have,

$$F(y + l) = F(y) + F'(y)l + \phi(l)$$

with $\frac{\|\phi(l)\|}{\|l\|} \to 0$ as $\|l\| \to 0$. Denote $S = F'(y)$. Then,

$$\begin{aligned}F(y + l) &= F(y) + Sl + \phi(l) \\ x + h &= x + Sl + \phi(l) \\ l &= S^{-1}h - S^{-1}\phi(l)\end{aligned}$$

Now looking at G,

$$\begin{aligned} G(x+h) &= y+l \\ &= y + S^{-1}h - \underbrace{S^{-1}\phi(l)}_{\text{remainder term}} \end{aligned}$$

Check does the remainder term go to zero.

$$\frac{\|S^{-1}\phi(l)\|}{\|h\|} \leq \|S^{-1}\| \frac{\|\phi(l)\|}{\|l\|} \frac{\|l\|}{\|h\|}$$

From the previous result, $\|F(x) - F(y)\| \geq \frac{1}{2}\|x-y\|$ so $\|h\| \geq \frac{1}{2}\|l\|$ or $\frac{\|l\|}{\|h\|} \leq \frac{1}{2}$. This means that $\|l\| \to 0$ when $\|h\| \to 0$. So the remainder term goes to zero when $\|h\|$ does. Therefore, G is differentiable. The derivative is equal to:

$$G'(x) = S^{-1} = [F'(y)]^{-1} = [F'(G(x))]^{-1}$$

Now assume that G is C^s for $0 \leq s < r$. Since $G'(x)$ is the composition of C^s functions, $G'(x)$ is $C^s \Rightarrow G(x)$ is C^{s+1}. By induction, G is C^r. To summarise, on U, we have F is bijective and F and F^{-1} are C^r making it a C^r-diffeomorphism.

$\square$

8 Implicit Function Theorem(2010,2009,2008)

State and prove the implicit function theorem

Let $f = (f^1, \ldots, f^l)$ be C^r real valued functions on an open set V in $\mathbb{R}^n \to \mathbb{R}^l$. Let $X = \{x \in M : f(x) = 0\}$ be the solution space. Take $a \in X$ such that rank$f'(a) = l$ say where the first l columns are linearly independent. Then there exists an open neighbourhood U of a in X such that the coordinate functions $x^{l+1}, \ldots, x^n$ map U onto open set $F(U)$ in $\mathbb{R}^{n-l}$ homeomorphically and the functions $x^1, \ldots, x^l$ are C^r functions of $x^{l+1}, \ldots, x^n$.

Set $F = f(f^1, \ldots, f^l, x^{l+1}, \ldots, x^n)$. Noting that $\frac{\partial f^i}{\partial x^k} = 0$ for $1 \leq i \leq l, k \geq l+1$ and $\frac{\partial x^i}{\partial x^k} = \delta_{ik}$ for $k \geq l+1$, then,

$$F'(a) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \ldots & \frac{\partial f^1}{\partial x^l} & \frac{\partial f^1}{\partial x^{l+1}} & \ldots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f^l}{\partial x^1} & \ldots & \frac{\partial f^l}{\partial x^l} & \frac{\partial f^l}{\partial x^{l+1}} & \ldots & \frac{\partial f^l}{\partial x^n} \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix}$$

Now splitting a matrix into blocks with 0 in the bottom corner, $\det\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) = \det A \det D$. The determinant of the identity is 1. This gives $\det F'(a) = \det\left(\frac{\partial(f^1, \ldots, f^l)}{\partial(x^1, \ldots, x^l)}\right)$. This is nonzero because the columns are linearly independent. By the inverse function theorem, there exists an open neighbourhood W of a in V such that F maps W onto an open set $F(W)$ by a C^r diffeomorphism. Let $U = X \cap W$. F maps the first l coordinates to 0 because X is the solution space. So $F_U(x) = (0, \ldots, 0, x^{l+1}, \ldots, x^n)$. Hence, the coordinate functions $x^{l+1}, \ldots, x^n$ map U onto open $F(U)$ which is open in $\mathbb{R}^{n-l}$ by a C^r diffeomorphism.

Let $G = F^{-1} = G(G^1, \ldots, G^n)$. Then,

$$\begin{aligned} x^1 &= G^1(0, \ldots, 0, x^{l+1}, \ldots, x^n) \\ &\vdots \\ x^l &= G^l(0, \ldots, 0, x^{l+1}, \ldots, x^n) \end{aligned}$$

F is a C^r diffeomorphism so these are C^r functions on U.

$\square$

9 Pull-Back(2010,2009,2008,2006)

Prove that the pull-back commutes with the differential.

$$\phi^* df = d\phi^* f$$

Let be defined in $V \subset Y$. Let $v \in T_x X, x \in \phi^{-1} V$.

$$\begin{aligned} \langle (\phi^* df)_x, v \rangle &= \langle (df)_{\phi(x)}, \phi_* v \rangle && \text{by definition of the pullback of the differential} \\ &= [\phi_* v] f && \text{by definition of the differential} \\ &= v[\phi^* f] && \text{by definition of the pushforward} \\ &= \langle (d\phi^* f)_x, v \rangle && \text{by definition of the differential} \end{aligned}$$

Therefore, $(\phi^* df)_x = (d\phi^* f)_x \forall x \in X$ so $\phi^* df = d\phi^* f$.

10 Chain Rule(2010,2006)

Prove the chain rule for maps on manifolds.

The theorem states that for maps on manifolds $X \overset{\phi}{\to} Y \overset{\psi}{\to} Z$, $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. Let f be a scalar field and $v = \dot{\alpha}(t) \in TX$.

$$[(\psi \circ \phi)_* v] f = v[(\psi \circ \phi)^* f] = v[f \circ \psi \circ \phi] = v[\phi^*(f \circ \psi)]$$

$$= \phi_* v[f \circ \psi] = \phi_* v[\psi^* f] = [(\psi_* \circ \phi_*) v] f$$

$\square$

In words, recall that,

$$\begin{aligned} [\phi_* v] f = v[\phi^* f] = (\phi \dot{\circ} \alpha)(t) f &= \text{velocity vector of } \phi \circ \alpha \text{ at } t \\ &= \phi_*[\text{velocity vector of } \alpha \text{ at } t] \end{aligned}$$

$$\begin{aligned} (\psi \circ \phi)_* \dot{\alpha}(t) &= \text{velocity vector of } (\psi \circ \phi)\alpha \text{ at } t \\ &= \psi_*(\text{velocity vector of } (\phi)\alpha \text{ at } t) \\ &= (\psi_* \circ \phi_*) \dot{\alpha}(t) \end{aligned}$$

v is arbitrary so the proof is complete.

$\square$

11 Definitions

n-dimensional coordinate system
Let V be an open set in X, a topological space. An n-dimensional coordinate system on X with domain V, $y = (y^1, \ldots, y^n)$, is a homeomorphism mapping V onto an open set $y(V)$ in $\mathbb{R}^n$.

C^r compatible
Let $y = (y^1, \ldots, y^n)$ with domain V and $z = (z^1, \ldots, z^n)$ with domain W be n-dimensional coordinate systems on topological space X. y and z are C^r compatible if y^i is a C^r function of $z^1, \ldots, z^n$ and z^i is a C^r function of $y^1, \ldots, y^n$ on $V \cap W$.

Manifold
Let X be a topological space. X is an n-dimensional smooth manifold if given a collection of mutually C^∞ compatible n-dimensional coordinate systems, the domains of the coordinate systems cover X.

Paramaterised Path
A paramaterised path is a continuous map $\alpha : \mathbb{R} \supset U \to X$, $t \mapsto \alpha(t)$ where U is an open set and X is a topological space with coordinates $y^i(\alpha(t))$ being C^∞.

Scalar Field
A scalar field, f, is a real-valued smooth function on an open set V in manifold X, $f : V \to \mathbb{R}$.

Partial Derivative
Let f be a scalar field and y^j a coordinate system. The partial derivative $\frac{\partial}{\partial y^j_a}$ of f at $a \in X$ is defined as,

$$\frac{\partial f}{\partial y^j_a} = \frac{\partial}{\partial x^j} F(y(a))$$

Velocity Vector
If $\alpha(t)$ is a path in X, the velocity vector at parameter t of a parametrised path in X is defined as the operator $\dot{\alpha}(t)$ that acts on a scalar field f,

$$\dot{\alpha}(t) f = \frac{d}{dt} f(\alpha(t)) = \text{rate of change of } f \text{ along } \alpha$$

More importantly,

$$\begin{aligned}
&= \frac{d}{dt} F(y^1(\alpha(t)), \ldots, y^n(\alpha(t))) \\
&= \frac{\partial F}{\partial x^1}(y(\alpha(t)) \frac{d}{dt} y^1(\alpha(t)) + \cdots + \frac{\partial F}{\partial x^n}(y(\alpha(t)) \frac{d}{dt} y^n(\alpha(t)) \\
&= \frac{\partial f}{\partial y^1_{\alpha(t)}} \frac{d}{dt} y^1(\alpha(t)) + \cdots + \frac{\partial f}{\partial y^n_{\alpha(t)}} \frac{d}{dt} y^n(\alpha(t)) \\
&= \left[\frac{d}{dt} y^1(\alpha(t)) \frac{\partial}{\partial y^1_{\alpha(t)}} + \cdots + \frac{d}{dt} y^n(\alpha(t)) \frac{\partial}{\partial y^n_{\alpha(t)}} \right] f
\end{aligned}$$

So the velocity vector is a linear combination of partial derivatives. This leads to the next definition.

Tangent Space
The tangent space to X at a is the set of all velocity vectors at the point a. It is denoted $T_a X$.

Differential of f at a
The differential of f at a, df_a, is the linear form, $T_a X \to \mathbb{R}$ given by

$$\langle df_a, v \rangle = vf = \dot{\alpha}(t) f$$

df_a belongs to the dual space $(T_aX)^*$.

Vector Field
A vector field, v with domain V, is a function on open V of manifold X, $a \mapsto v_a$ where $a \in V \subset X$ and $v_a \in T_aX$, $\forall a \in V$.

Differential 1-form
A differential 1-form with domain V, ω, is a function, $a \mapsto \omega_a$ where $\omega_a \in (T_aX)^*$.

Contraction
Let ω be a differential 1-form and v be a vector field both with domain W on manifold X. The contraction,$\langle \omega, v \rangle$, is defined as the scalar field,

$$\langle \omega, v \rangle(a) = \langle \omega_a, v_a \rangle$$

$\forall a \in W$ and where $\omega_a \in (T_aX)^*$, $v_a \in T_aX$.

Differential of f
The differential of f, df, is the differential 1-form whose value at a is df_a. df measures the rate of change of f.

Pull-back
A function $\phi : X \to Y$ is called a map of manifolds if X and Y are manifolds. Let f be a scalar field with domain V on Y.The pullback $\phi^* f$ is defined as,

$$\phi^* f = f \circ \phi$$

and it has domain $\phi^{-1}V$.

Push-forward
Let $v \in T_aX$. Then the push-forward $\phi_* v$ belonging to $T_{\phi(a)}Y$ is defined as,

$$[\phi_* v]f = v[\phi^* f]$$

Pull-back of ω under ϕ
Let $\omega \in (T_{\phi(a)}Y)^*$ be a differential 1-form with domain V. Then the pull-back of ω under ϕ is the differential 1-form on X with domain $\phi^{-1}V$ defined as,

$$\langle (\phi^* \omega)_a, v \rangle = \langle \omega_{\phi(a)}, \phi_* v \rangle$$

$\forall v \in T_aX, a \in \phi^{-1}V$.

Tangent Bundle
The tangle bundle of X, TX is the set of all tangent vectors to X.

Metric Tensor
A metric tensor $(.|.)$ on a manifold X is a function on X defined as

$$a \mapsto (.|.)_a$$

$(.|.)_a$ is a scalar product on T_aX. ie $(.|.)_a$ takes two vectors from T_aX and outputs a real number with the property of being bilinear, symmetric and nonsingular. If y is a coordinate system on V, then denote,

$$g_{ij} = (\frac{\partial}{\partial y_a^i} | \frac{\partial}{\partial y_a^j})_a$$

as the components of the metric tensor w.r.t. the coordinate system.

Line Element
The line element, $(ds)^2_a$ associated to the metric tensor $(.|.)$ is the associated quadratic form,

$$(ds)^2_a(v) = (v|v)_a$$

$\forall v \in T_aX$. If v is a vector field with domain W, then $(ds)^2(v) = (v|v)$ is a scalar field with domain W.